

Mechanics II

CHAPTER 3

Rigid Body Dynamics: Part 1

3.1–3.52

Introduction	3.1
Rigid Body	3.1
General Rigid Body Motion	3.1
Relative Angular Velocity in Case of a Rigid Body	3.2
Types of Rotational Motion	3.3
Rotation About a Fixed Axis or Pure Rotation	3.3
Combined Rotation and Translation	3.3
<i>Concept Application Exercise 3.1</i>	3.5
Instantaneous Axis of Rotation	3.5
Finding Instantaneous Axis of Rotation in Different Situations	3.6
Rolling Motion	3.7
Rolling as Pure Rotation About Point of Contact	3.8
Constraint Relations in Case of Rotation and Translation Combined	3.8
Constraints Due to Ropes and Pulleys	3.8
Constraints on a Rod Having Combined Translational and Rotational Motion	3.9
<i>Concept Application Exercise 3.2</i>	3.10
Moment of Inertia	3.10
Moment of Inertia of a Rigid Body	3.11
Moment of Inertia of a Straight Rod About an Axis Through Its Centre	3.11
Moment of Inertia of a Circular Ring About an Axis Passing Through Its Centre and Perpendicular to Its Plane	3.12
Moment of Inertia of a Disc	3.12
MI of a Triangular Lamina About Its Base	3.12
MI of a Square Lamina About Its Diagonal	3.12
Moment of Inertia About a General Axis of Rotation	3.13
Perpendicular Axis Theorem	3.13
Parallel Axis Theorem	3.14
Radius of Gyration	3.15
Mass Distribution and Moment of Inertia	3.15
<i>Concept Application Exercise 3.3</i>	3.17
Torque	3.18
Calculation of Torque	3.19
Couple	3.20

Equilibrium of Rigid Bodies	3.20
<i>Concept Application Exercise 3.4</i>	3.23
Toppling and Shifting of Normal Reaction	3.24
<i>Concept Application Exercise 3.5</i>	3.26
Newton's Second Law for Rotational Motion	3.27
<i>Concept Application Exercise 3.6</i>	3.30
<i>Exercises</i>	3.32
<i>Single Correct Answer Type</i>	3.32
<i>Multiple Correct Answers Type</i>	3.39
<i>Linked Comprehension Type</i>	3.42
<i>Matrix Match Type</i>	3.44
<i>Numerical Value Type</i>	3.47
<i>Archives</i>	3.49
<i>Answers Key</i>	3.52

3

Rigid Body Dynamics: Part 1

INTRODUCTION

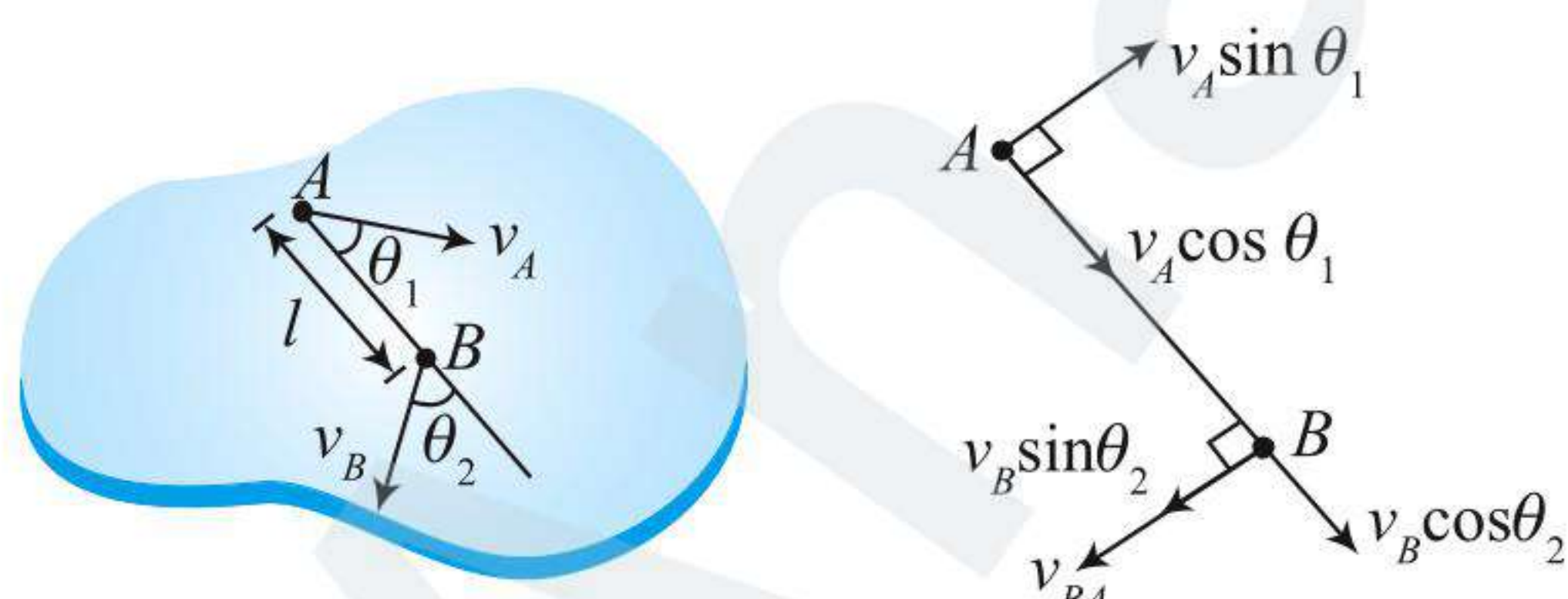
In previous sections we had learnt about particle dynamics and introduced some basic concepts of circular motion. In this chapter we will learn about rigid body dynamics. Rigid body dynamics studies the movement of systems of interconnected bodies under the action of external forces.

RIGID BODY

Generally, we call an object **rigid** when we cannot deform (squeeze and twist) it. This means, the shape and size of an ideal rigid body does not change. However, in practice, any object can be compressed or elongated by certain extent. In nature nothing is perfectly rigid. When the deformation of an object is insignificant compared to its dimension, practically we call it a **rigid body**. (Here, we have used the word 'deformation' for compression, elongation and shearing (twisting or bending).) We can define a rigid body as a body which undergoes no deformation compared to its size, under the action of an external force or torque. In consequence, the distance between any two particles of rigid body is fixed.

For every pair of particles in a rigid body, there is no velocity of separation or approach between the particles i.e., any relative motion of a point B on a rigid body with respect to another point A on the rigid body will be perpendicular to line joining A to B . Hence the motion of any particle B with respect to any other particle A of a rigid body is circular motion.

Let velocities of A and B with respect to ground be $\vec{v}_A$ and $\vec{v}_B$ respectively in the figure.



$$v_A \cos \theta_1 = v_B \cos \theta_2 \text{ (velocity of approach/separation is zero)}$$

$$v_{BA} = \text{Velocity of } B \text{ with respect to } A.$$

$$v_{BA} = v_A \sin \theta_1 + v_B \sin \theta_2 \text{ (which is perpendicular to line } AB)$$

B will appear to move in a circle to an observer fixed at A .

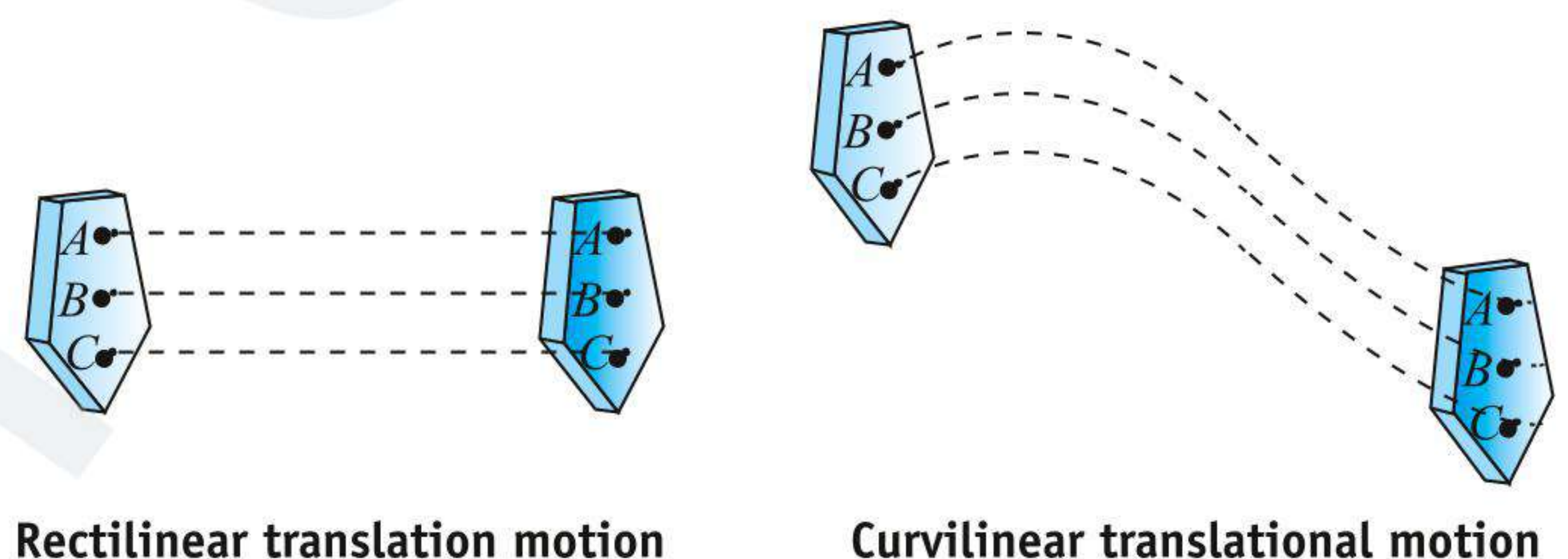
The angular velocity of B with respect to A , $\omega_{B,A} = \frac{v_{B,A}}{l}$, where l is distance between A and B .

This is also angular velocity of rigid body about its centre of mass.

GENERAL RIGID BODY MOTION

Translation

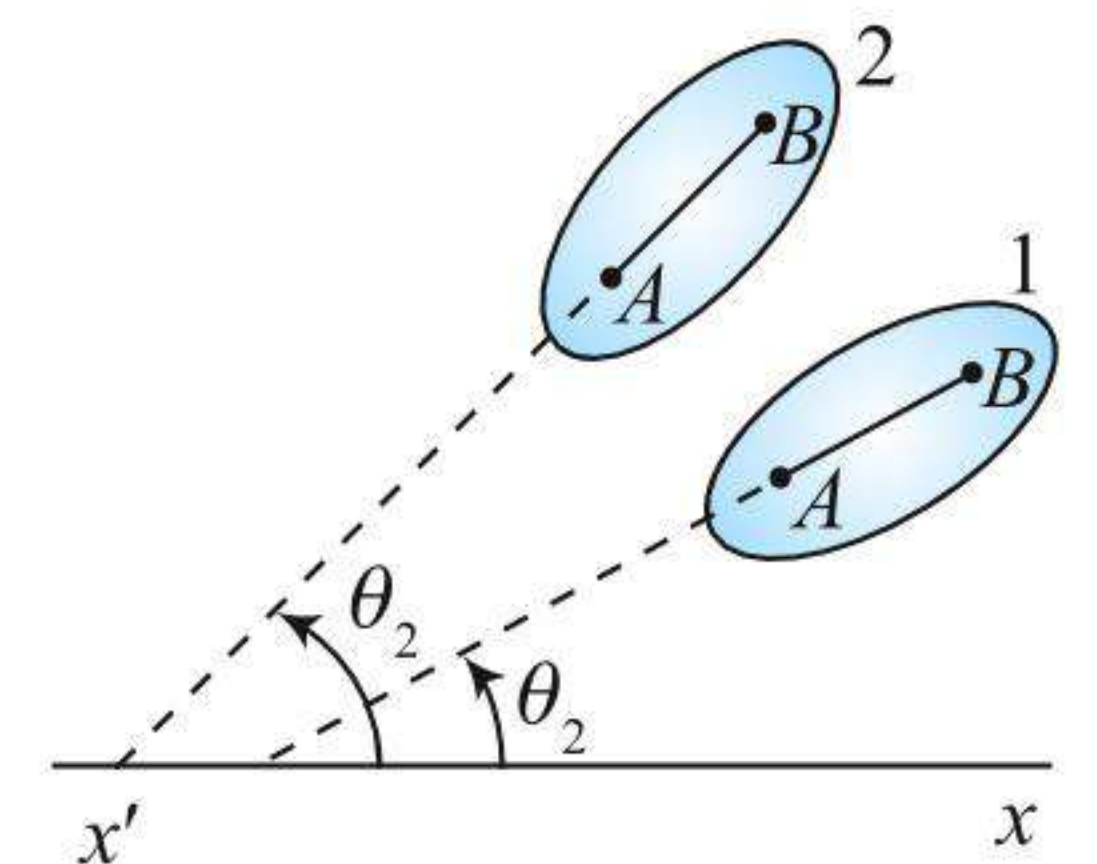
If a body is moving such that a line drawn between any two internal points remains parallel to itself, the motion is said to be translation. In translation, all the particles of the body move along parallel paths. If these parallel paths are straight lines, the motion is said to be a rectilinear; if the paths are along curved lines, the motion is a curvilinear. For a rigid body in translation, all the points of the body have the same velocity and same acceleration at any given instant.



Rotation

The simplest case of rotation is rotational motion in a plane, i.e., the two dimensional.

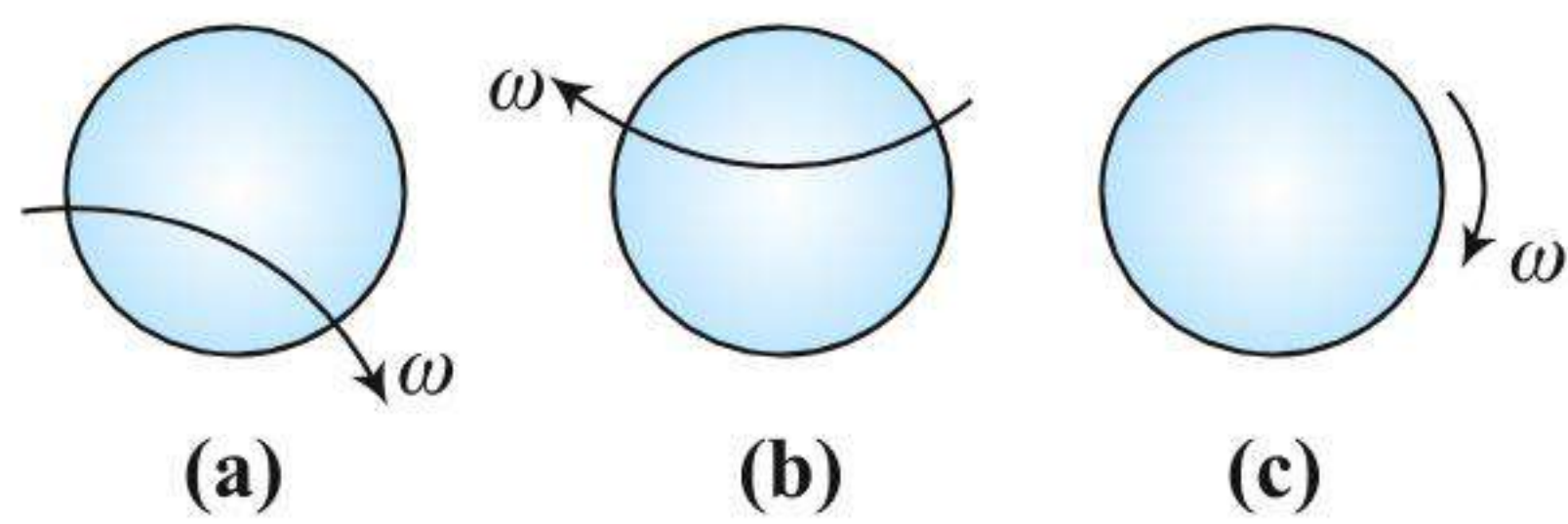
Let us consider a plate like (laminar) rigid body and draw a straight line connecting any two points A and B , say, on the plate. Then produce the line AB towards a fixed reference line $X'X$. Let the line AB make an angle θ_1 with $X'X$. We call this angle as 'angular position' of the straight-line AB . As the plate moves from position 1 to position 2, we can see that the angular position of the line AB changes from θ_1 to θ_2 . That is what we call **angular motion** of the straight line AB . More popularly angular motion is known as 'rotational motion or rotation'.



The straight-line AB changes its angle of orientation with $X'X$ line from θ_1 to θ_2 as the plate moves from position 1 to position 2. From the above discussions we conclude that:

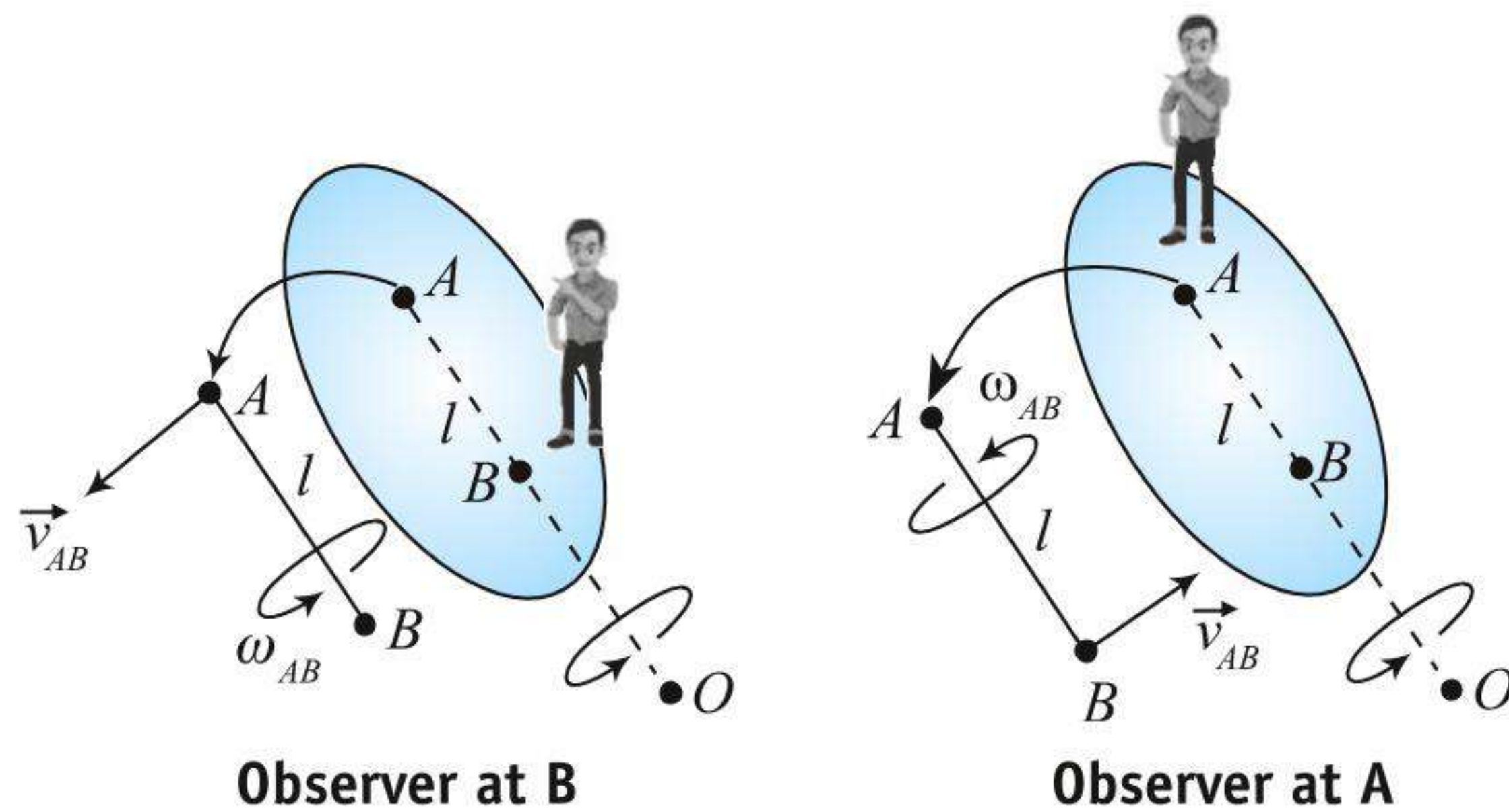
"Whenever the angle of orientation of a straight line drawn inside a rigid body changes with respect to a fixed reference line, we say that the rigid body undergoes angular motion or rotational motion or rotation."

A rigid body does not require any axis to rotate in an unconstructed rotation. In this case you can mention the angular velocity in any way you like, keeping its sense constant as shown in Figs. (a), (b) and (c).



RELATIVE ANGULAR VELOCITY IN CASE OF A RIGID BODY

In a rigid body the distance between two points remains constant. Let us assume two points A and B on a rotating plate.



If we observe point A from point B (see Fig. (a)), A appears to move perpendicular to the line AB with velocity $\vec{v}_{AB}$. Here we can say that point A rotates about point B with angular velocity ω_{AB} , which can be expressed as

$$\omega_{AB} = \frac{v_{AB}}{l} \quad (\text{in anticlockwise sense})$$

$$\text{In vector form, we can write } \vec{\omega}_{AB} = \frac{v_{AB}}{l} (\hat{k}) \quad \dots(i)$$

Now if we place our observer at A and observe point B [see Fig. (b)], we find B appear to move with velocity $\vec{v}_{BA}$ perpendicular to line AB . It means point B rotates about A with angular velocity ω_{BA} which can be expressed as in previous case.

$$\omega_{BA} = \frac{v_{BA}}{l} \quad (\text{in anticlockwise sense})$$

$$\text{In vector form } \vec{\omega}_{BA} = \frac{v_{BA}}{l} (\hat{k}) \quad \dots(ii)$$

Comparing Eqs. (i) and (ii), we find $\vec{\omega}_{AB} = \vec{\omega}_{BA}$

“Each point in a rigid body turns relative to the other points of the rigid body with same angular velocity at a given instant.” The angular velocity of any point of a rigid body at any instant relative to any other point of rigid body is constant which is defined as the angular velocity of the rigid body, whereas at any instant the angular velocities of different points of the rigid body relative to any point outside the rigid body are different.

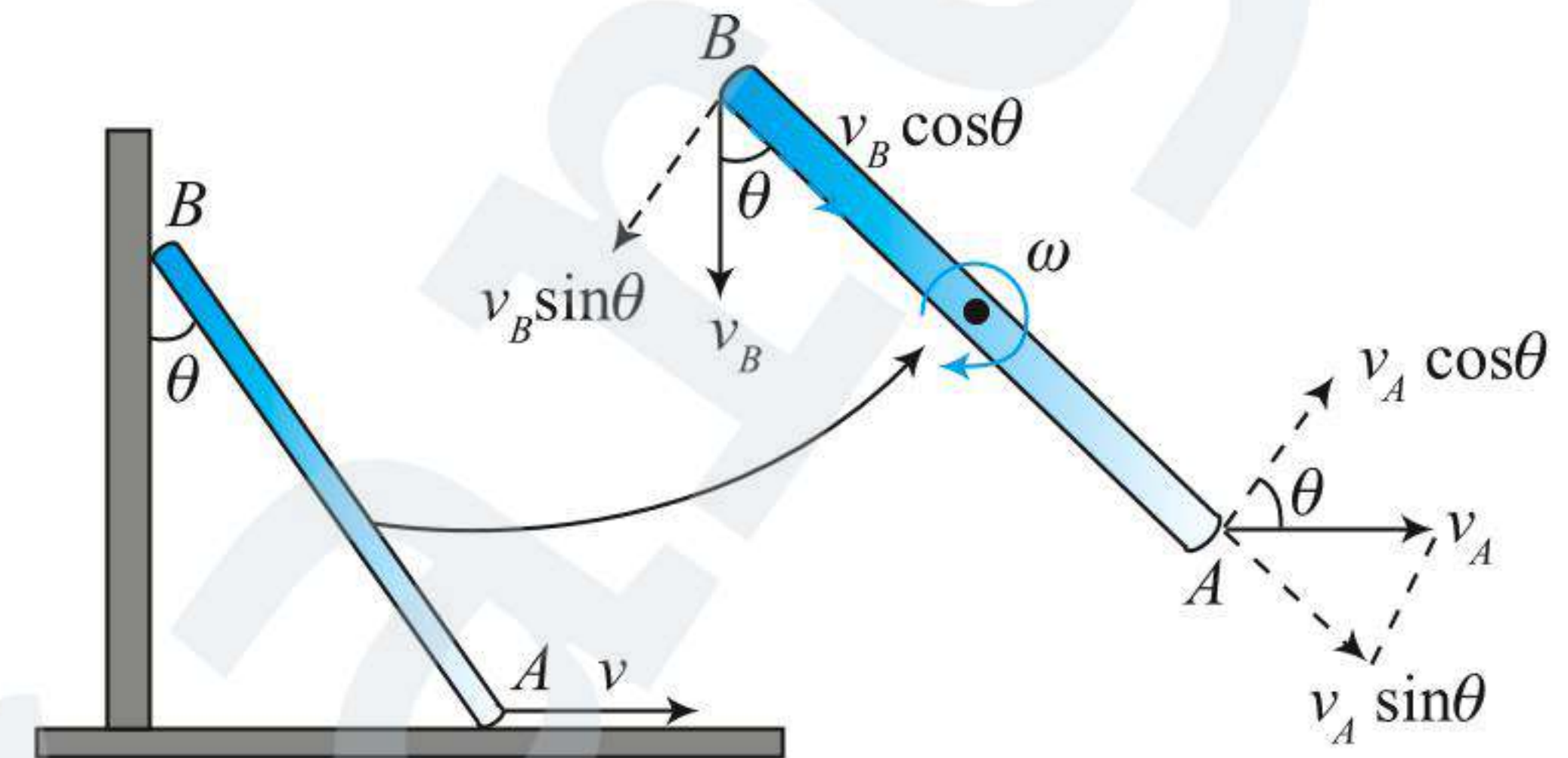
Important Points:

- With respect to any particle of the rigid body, the motion of any other particle of that rigid body is circular.
- In case of a rotating body we can say angular velocity of one point w.r.t. to other point of rigid body will be equal, i.e., $\vec{\omega}_{12} = \vec{\omega}_{21}$, but their relative linear velocities will not be equal $\vec{v}_{12} \neq \vec{v}_{21}$. Here we can say $\vec{v}_{12} = -\vec{v}_{21}$.
- In case of rotation of a rigid body, the statement that the relative angular velocity of point w.r.t. to other point is equal, i.e., $\vec{\omega}_{12} = \vec{\omega}_1 - \vec{\omega}_2$ is vague, but for relative linear velocity $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$ is valid.
- When any two points A and B (say) of a rigid body have velocities $\vec{v}_A$ and $\vec{v}_B$, respectively, the angular velocity of rigid body ω can be expressed as $\omega = \frac{|\vec{v}_A - \vec{v}_B|}{l}$.

ILLUSTRATION 3.1

A rod of length l resting on a wall and the floor. Its lower end A is pulled towards right with a constant velocity v . Find the velocity of the other end B downward when the rod makes an angle θ with the vertical. Also, find the angular velocity of the rod.

Sol. As rod is a rigid body, the distance between point A and point B should remain unchanged. Hence the relative velocity of ends of rod along its length should be zero.



It means the velocities of ends of rod along its length must be equal.

$$v_A \sin \theta = v_B \cos \theta \Rightarrow v_B = v_A \frac{\sin \theta}{\cos \theta} = v \tan \theta$$

$$\text{or } v_B = v \tan \theta \quad \dots(i)$$

$$\text{Then, } \omega_{BA} = \frac{v_{BA_{\perp}}}{l} = \frac{|v_A \cos \theta - (-v_B \sin \theta)|}{l}$$

$$\Rightarrow \omega_{BA} = \frac{v_A \cos \theta + v_B \sin \theta}{l} \quad \dots(ii)$$

Using Eqs. (i) and (ii),

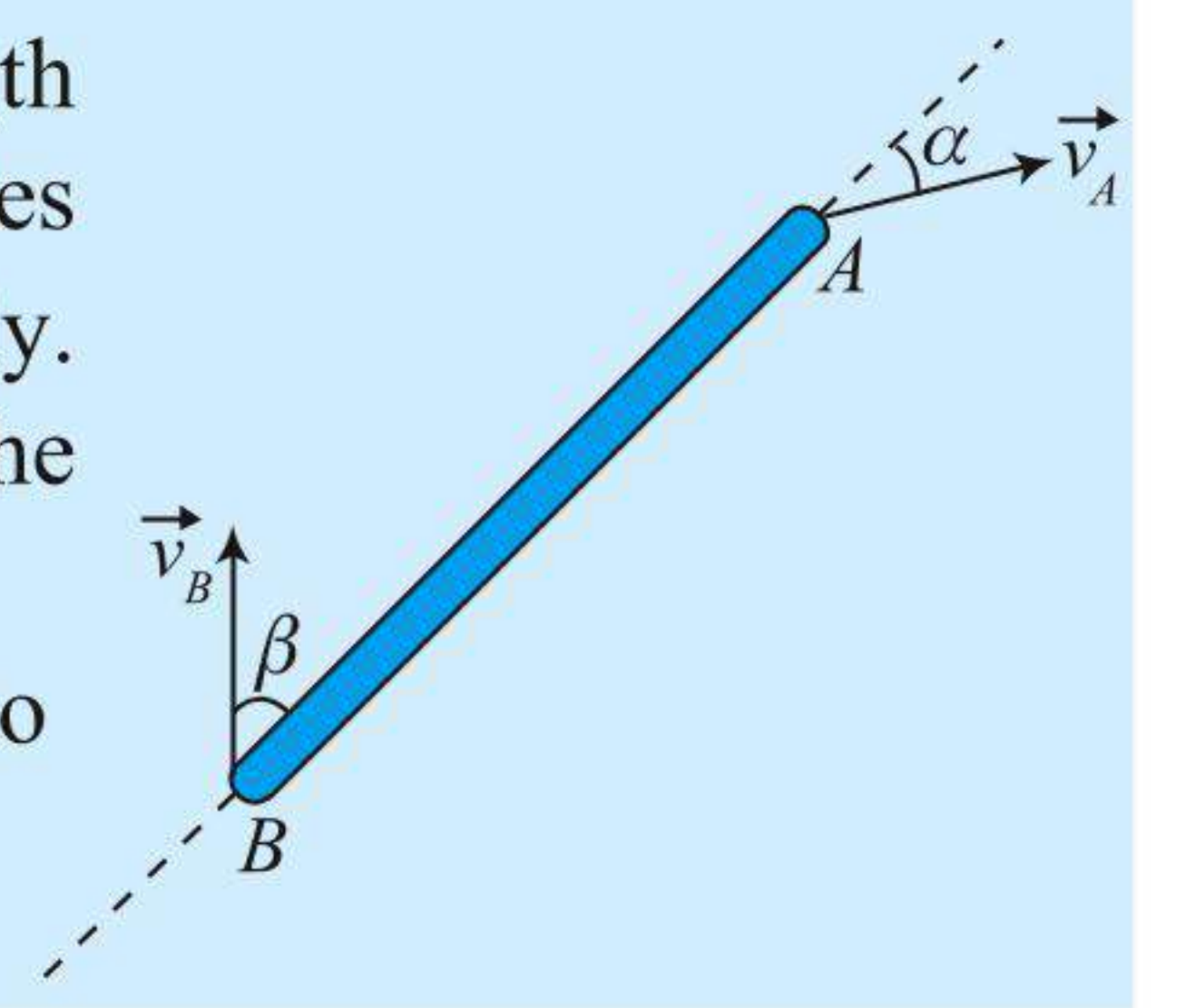
$$\omega_{BA} = \frac{v}{l \cos \theta}$$

This is also angular velocity of rigid body about its centre of mass.

ILLUSTRATION 3.2

The ends A and B of a rod of length l have velocities of magnitudes $|\vec{v}_A| = v$ and $|\vec{v}_B| = 2v$, respectively. If the inclination of $\vec{v}_A$ relative to the rod is α , find the:

- inclination β of $\vec{v}_B$ relative to the rod,
- angular velocity of the rod.



Sol.

- We know in a rigid body the velocities of two points along the line joining two should remain same.

$$\text{i.e., } (v_A)_{\parallel} = (v_B)_{\parallel}$$

$$\text{Hence } v \cos \alpha = 2v \cos \beta$$

$$\cos \beta = \frac{1}{2} \cos \alpha \quad \text{or} \quad \beta = \cos^{-1} \left(\frac{1}{2} \cos \alpha \right)$$

- Angular velocity of the rod, $\omega = \frac{|\vec{v}_{AB}_{\perp}|}{l} = \frac{|\vec{v}_A - \vec{v}_B|_{\perp}}{l}$

$$\begin{aligned} |\vec{v}_{AB}| &= |\vec{v}_A - \vec{v}_B| = v \sin \alpha - (-2v \sin \beta) \\ &= v(\sin \alpha + 2 \sin \beta) \end{aligned}$$

$$\Rightarrow \omega = \frac{(\sin \alpha + 2 \sin \beta) v}{l} \text{ (in clockwise direction)}$$

We have calculated $\cos \beta = \frac{\cos \alpha}{2}$. Thus,

$$\sin \beta = \sqrt{1 - \cos^2 \beta} \Rightarrow \sin \beta = \frac{\sqrt{3 + \sin^2 \alpha}}{2}$$

$$\text{Hence } \omega = \frac{(\sin \alpha + \sqrt{3 + \sin^2 \alpha}) v}{l} \text{ (in clockwise direction)}$$

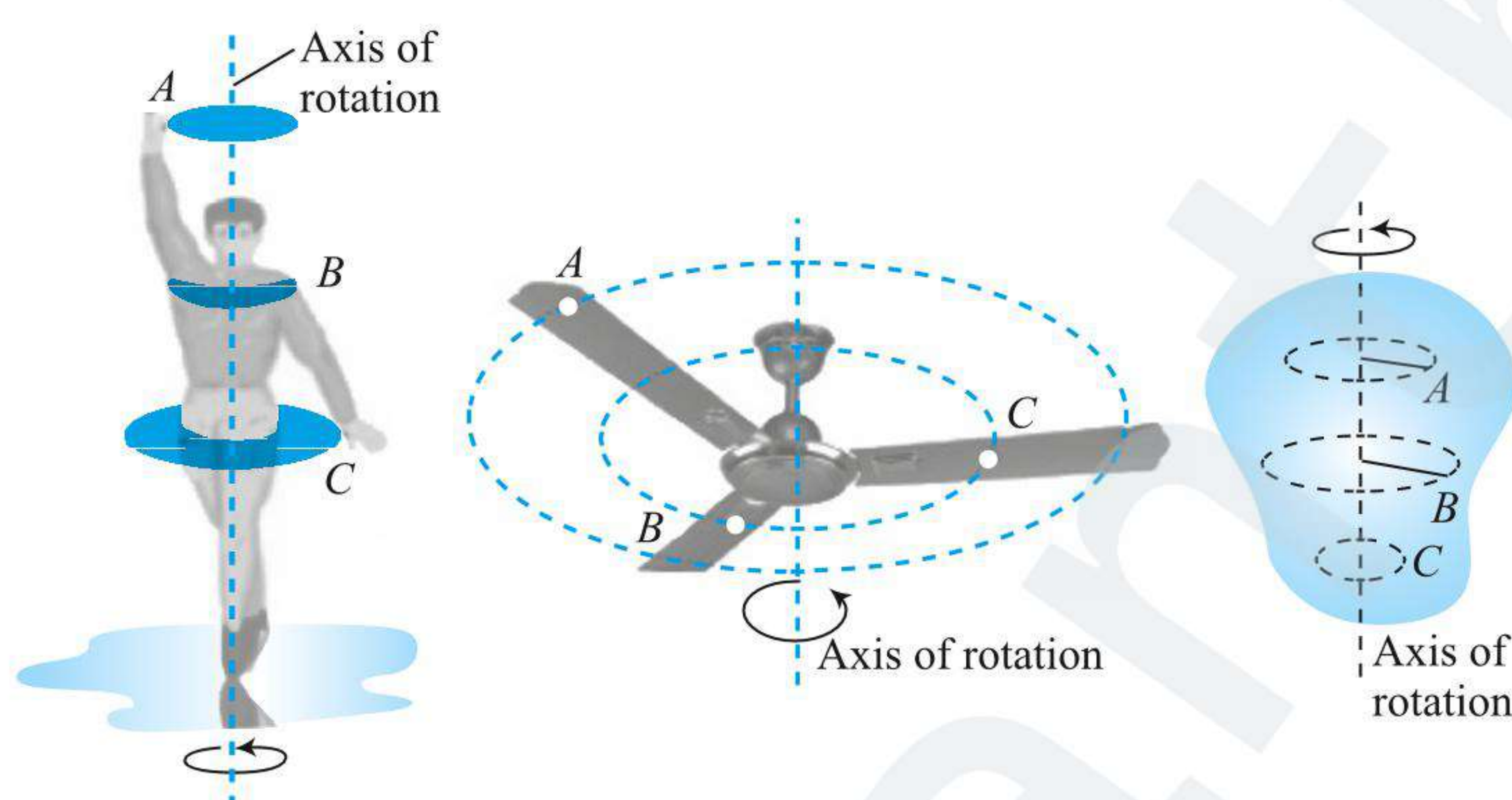
TYPES OF ROTATIONAL MOTION

Motion of body involving rotation can be classified into the following two categories.

1. Rotation about a fixed axis or pure rotation.
2. Rotation about an axis in translation or combined rotation and translation.

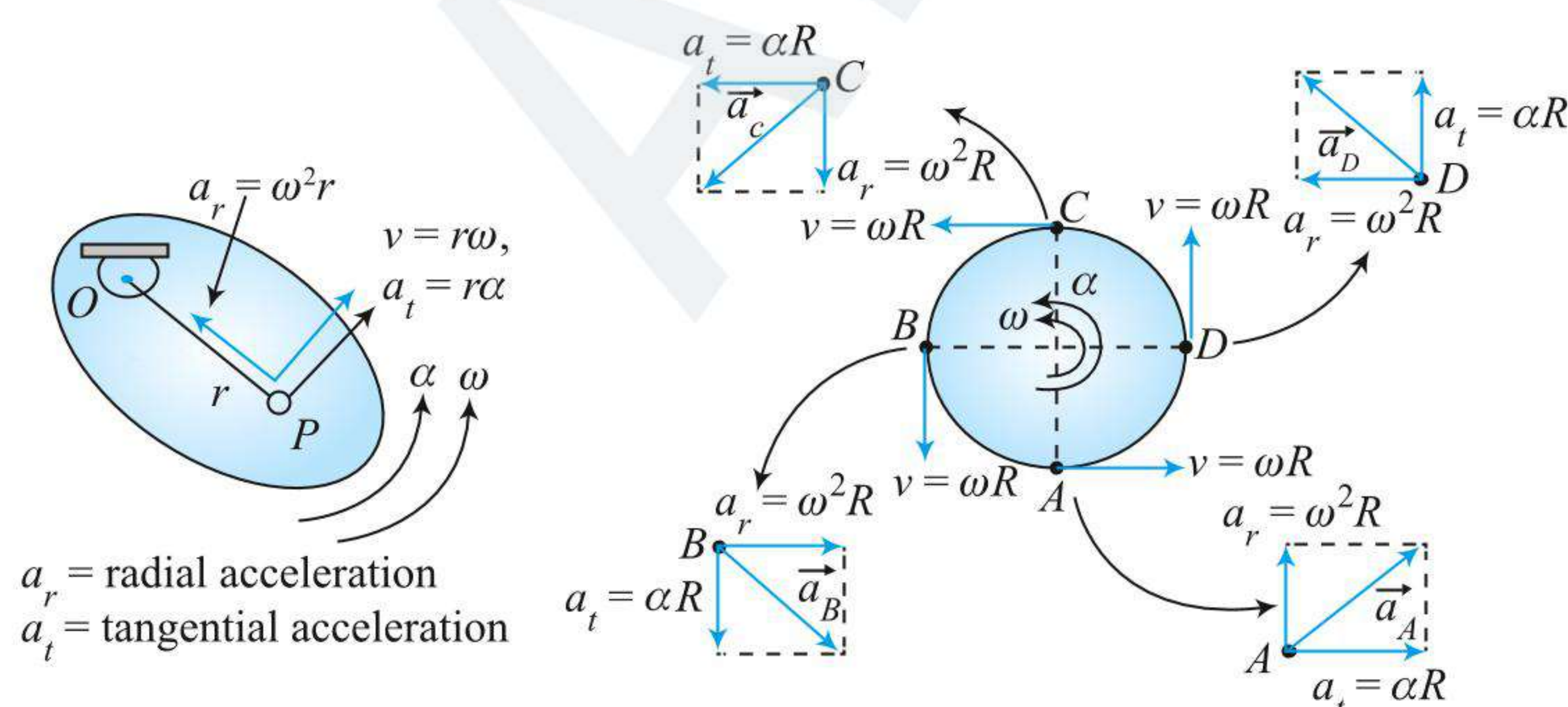
ROTATION ABOUT A FIXED AXIS OR PURE ROTATION

When a rigid object rotates, points on the object, such as A , B , or C , move in circular paths. An imaginary line perpendicular to plane of circular paths of particles of a rigid body in rotation and containing the centers of all these circular paths is known as **axis of rotation**. Rotation of ceiling fan, potter's wheel, opening and closing of doors and hands of a wall clock etc. come into this category.



Velocity and Acceleration of a Point in Rotating Rigid Body

For pure translation, angular velocity and angular acceleration is zero and linear velocity is constant for all points. For pure rotation (fixed axis rotation) the angular velocity is constant for all particles about the axis of rotation (see figure).



Velocity and Acceleration in case of pure rotation

We can write the speed of point P [see Fig. (a)] as $v = \omega r$. In vector form, $\vec{v} = \vec{\omega} \times \vec{r}$. In Fig. (b), we can observe the velocities and accelerations of points A , B , C and on the disc. We can make two components of acceleration of point P , the tangential acceleration and radial acceleration.

The tangential acceleration of point P , $a_t = \alpha r$.

In vector form, it can be expressed as $\vec{a}_t = \vec{\alpha} \times \vec{r}$

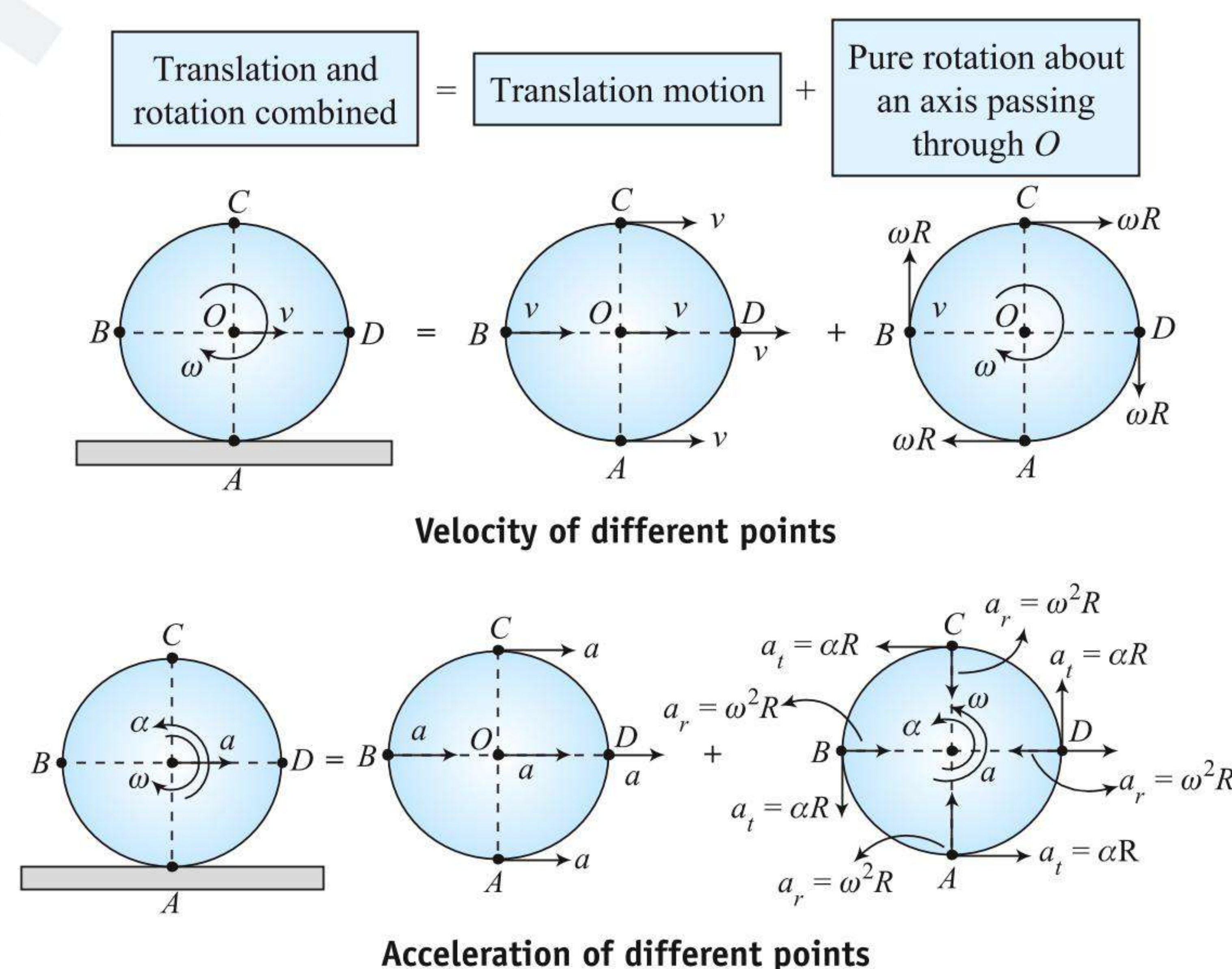
The radial acceleration of point P , $a_r = \omega^2 r$

Important Point:

The direction of $\vec{\omega}$ can be found by using right hand rule. In vector form it can be expressed as $\vec{v}_{AB} = \vec{\omega}_{AB} \times \vec{r}_{AB}$, where $\vec{r}_{AB}$ = position vector of A relative to B .

COMBINED ROTATION AND TRANSLATION

The most general motion of a rigid body is the combination of translational and rotational motions. We have studied pure rotation and pure translation motions but what happens when the two are combined? The most familiar example of combined rotational and translational motion is a rolling wheel. Consider rolling of wheels of a vehicle, moving on straight level road. Relative to a reference frame the axel fixed with the vehicle, the wheel appears rotating about its axel. The rotation of the wheel from this frame is pure rotation about fixed axis. But relative to a reference frame fixed with the ground, the wheel appears rotating about the moving axel, therefore, rolling of a wheel is superposition of two simultaneous but distinct motions—rotation about the fixed axel and translation of the wheel along the level road.



Velocity and Acceleration of different points in case translation and rotation combined

Point A	Velocity	$A \xrightarrow{v} + \xleftarrow{\omega R} A = A \xrightarrow{v - \omega R}$
	Acceleration	$A \xrightarrow{a} + \begin{matrix} \uparrow a_r = \omega^2 R \\ \rightarrow a_t = \alpha R \end{matrix} = \begin{matrix} \nearrow \omega^2 R \\ \nearrow a + \alpha R \end{matrix}$

Point B	Velocity	$B \xrightarrow{v} + B \xrightarrow{\omega R} = \xrightarrow{v + \omega R}$ $ \vec{v}_B = \sqrt{v^2 + \omega^2 R^2}$
	Acceleration	$B \xrightarrow{a} + B \xrightarrow{\omega^2 R} = \xrightarrow{a + \omega^2 R}$ αR
Point C	Velocity	$C \xrightarrow{v} + C \xrightarrow{\omega R} = \xrightarrow{v + \omega R}$
	Acceleration	$C \xrightarrow{a} + C \xrightarrow{\omega^2 R} = \xrightarrow{a + \omega^2 R}$ $\omega^2 R$
Point D	Velocity	$D \xrightarrow{v} + D \xrightarrow{\omega R} = \xrightarrow{v + \omega R}$ $ \vec{v}_D = \sqrt{v^2 + \omega^2 R^2}$
	Acceleration	$D \xrightarrow{a} + D \xrightarrow{\omega^2 R} = \xrightarrow{a + \omega^2 R}$ αR

Expressing Translation and Rotation Combined in Vector Form

In this type of motion, axis of rotation is not stationary. If a body rotates about an axis with an angular velocity ω , then with respect to the axis of rotation linear velocity of any particle 'P' in the body at a distance r from the axis of rotation is equal to

$$\vec{v} = \vec{\omega} \times \vec{r}$$

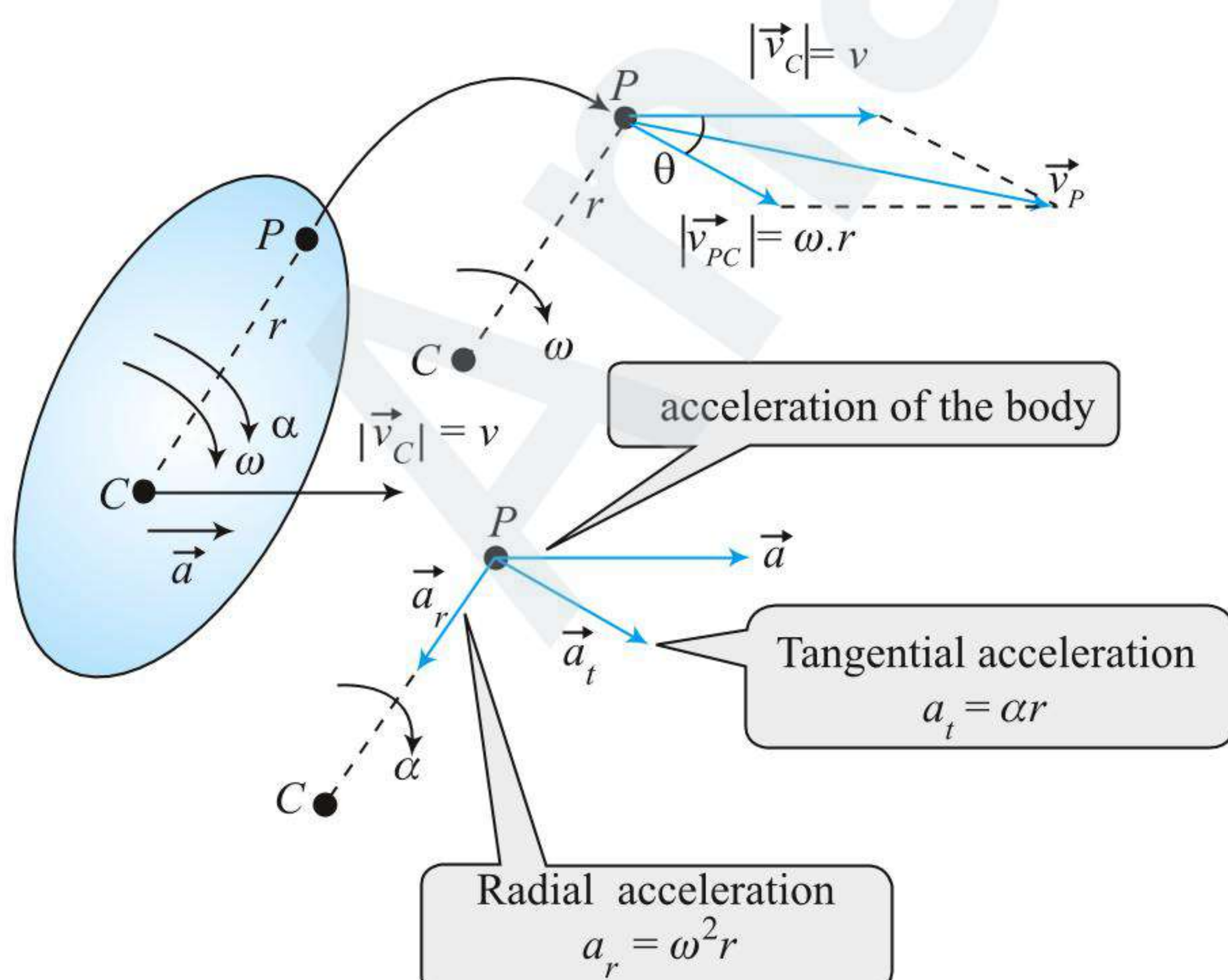
If the axis of rotation also moves with velocity $\vec{v}_0$, then the net velocity of the particle relative to stationary frame will be

$$\vec{v} = \vec{\omega} \times \vec{r} + \vec{v}_0$$

Now let us take a body which is moving with velocity $\vec{v}$ and rotating about center of mass with angular velocity ω . Let us analyze a point P on the body. The velocity of point P in the body is

$$\vec{v}_P = \vec{v}_{PC} + \vec{v}_C$$

$$v_P = \sqrt{v_{PC}^2 + v_C^2 + 2v_{PC} \cdot v_C \cdot \cos \theta}$$



We have and $v_{PC} = r\omega$ and $v_C = v$. Thus

$$v_P = \sqrt{(r\omega)^2 + v^2 + 2(r\omega)v \cdot \cos \theta}$$

In same way we can write the acceleration of point P

$$\vec{a}_P = \vec{a}_{P,C} + \vec{a}_C$$

$\vec{a}_{P,C}$ has both tangential as well as radial acceleration components. Hence, we can express, $\vec{a}_{P,C}$ as

$$\vec{a}_{P,C} = (\vec{a}_{P,C})_{\text{tangential}} + (\vec{a}_{P,C})_{\text{radial}}$$

$$(\vec{a}_{P,C})_{\text{tangential}} = r\alpha \text{ (acts perpendicular to line CP)}$$

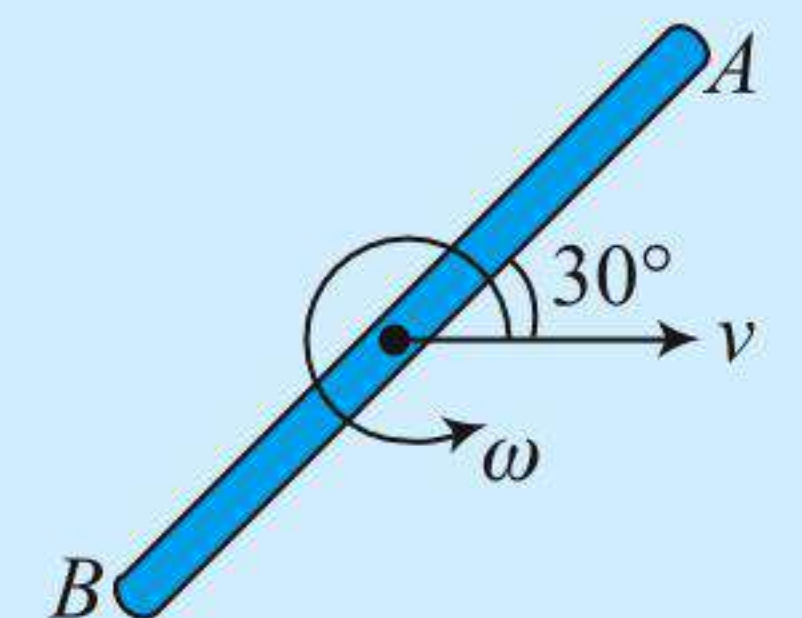
$$(\vec{a}_{P,C})_{\text{radial}} = \omega^2 r \text{ (acts along the line PC)}$$

Hence net acceleration of point P can be express as

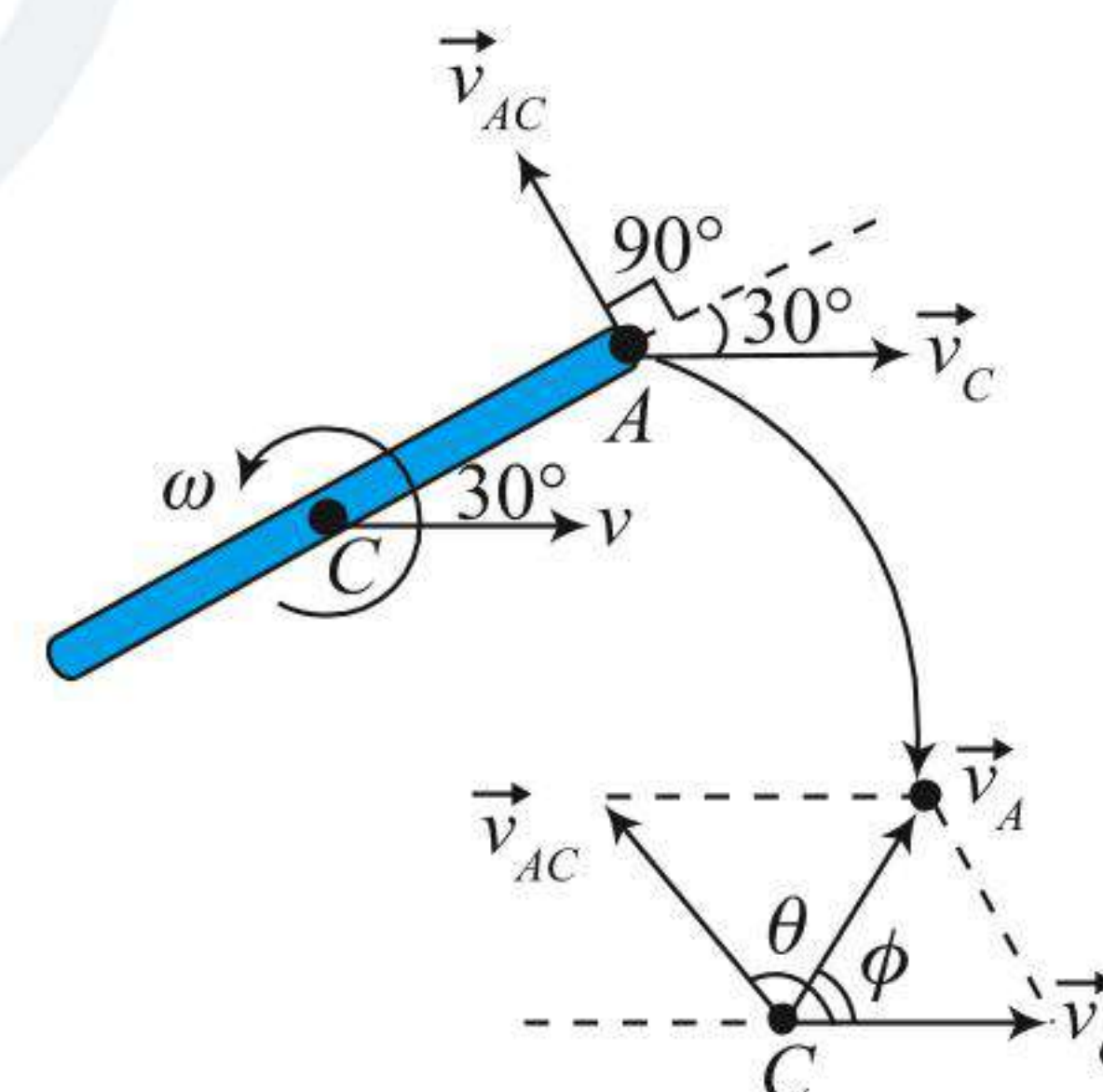
$$\vec{a}_P = [(\vec{a}_{P,C})_{\text{tangential}} + (\vec{a}_{P,C})_{\text{radial}}] + \vec{a}_C$$

ILLUSTRATION 3.3

A uniform rod of length l is spinning with an angular velocity $\omega = 2v/l$ while its centre of mass moves with a velocity v . Find the velocity of the end A of the rod.



Sol. The motion of point A is translation and rotation combined, the velocity of end A can be expressed by, $\vec{v}_A = \vec{v}_{A,C} + \vec{v}_C$



Hence the velocity of A is

$$v_A = \sqrt{v_{AC}^2 + v_C^2 + 2v_{AC}v_C \cos \theta}$$

We know, $v_C = v$, $v_{AC} = \frac{l}{2} \cdot \omega$

and $\theta = 90^\circ + 30^\circ = 120^\circ$

$$\text{Hence, } v_A = \sqrt{v^2 + \frac{l^2 \omega^2}{4} + lv \omega \cos 120^\circ}$$

$$\text{But, } \omega = \frac{2v}{l}, v_{AC} = \frac{l}{2} \omega = \frac{l}{2} \left(\frac{2v}{l} \right) = v$$

which gives $v_A = v$

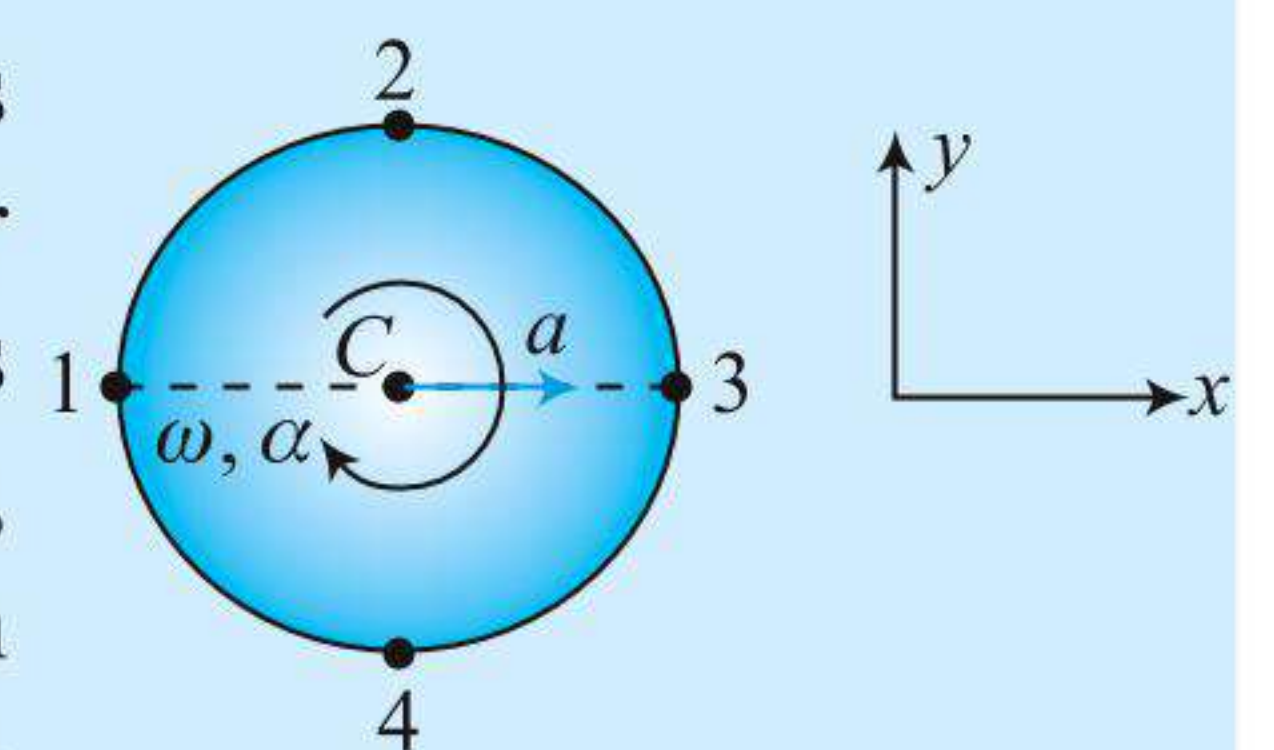
Let $\vec{v}_A$ make angle ϕ with the direction of $\vec{v}_C$

$$\text{Then } \phi = \tan^{-1} \left(\frac{v_{AC} \sin \theta}{v_{AC} \cos \theta + v_C} \right)$$

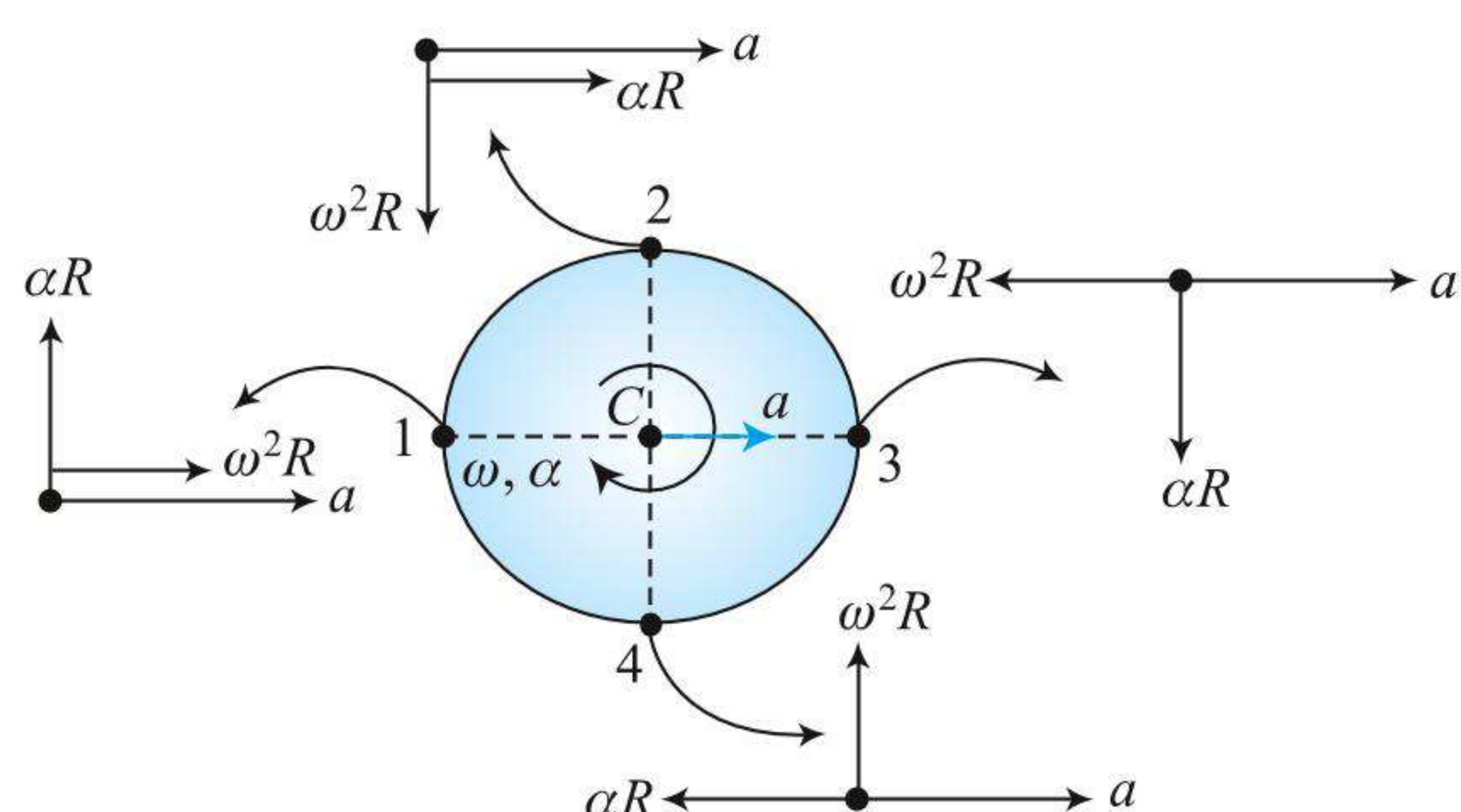
$$\text{We get } \phi = \tan^{-1} \left(\frac{v \sin 120^\circ}{v \cos 120^\circ + v} \right) = \frac{\pi}{3}$$

ILLUSTRATION 3.4

A uniform disc of radius r spins with angular velocity ω and angular acceleration α . If the centre of mass of the disc has linear acceleration a , find the magnitude and direction of acceleration of the points 1, 2, 3 and 4.



Sol. The motion of disc is translation and rotation combined, the acceleration of any point 'P' on the disc can be expressed by:



$$\vec{a}_P = \vec{a}_{PC} + \vec{a}_C$$

The acceleration of point '1':

$$\vec{a}_1 = \vec{a}_{1,C} + \vec{a}_C$$

But $\vec{a}_{1,C} = (\vec{a}_{1,C})_{\text{tangential}} + (\vec{a}_{1,C})_{\text{radial}}$

$$\Rightarrow \vec{a}_{1,C} = \alpha R \hat{j} + \omega^2 R \hat{i} \text{ and } \vec{a}_C = a \hat{i}$$

$$\text{Hence } \vec{a}_1 = (\omega^2 R + a) \hat{i} + \alpha R \hat{j}$$

Similarly, the acceleration of '2':

$$\vec{a}_2 = \vec{a}_{2,C} + \vec{a}_C$$

Here $\vec{a}_{2,C} = (\alpha R \hat{i} - \omega^2 R \hat{j})$ and $\vec{a}_C = a \hat{i}$

$$\text{Hence } \vec{a}_2 = (a + \alpha R) \hat{i} - \omega^2 R \hat{j}$$

Also, acceleration of '3':

$$\vec{a}_3 = \vec{a}_{3,C} + \vec{a}_C$$

Here $\vec{a}_{3,C} = -\omega^2 R \hat{i} - \alpha R \hat{j}$ and $\vec{a}_C = a \hat{i}$

$$\text{Hence } \vec{a}_3 = (a - \omega^2 R) \hat{i} - \alpha R \hat{j}$$

The acceleration of point '4':

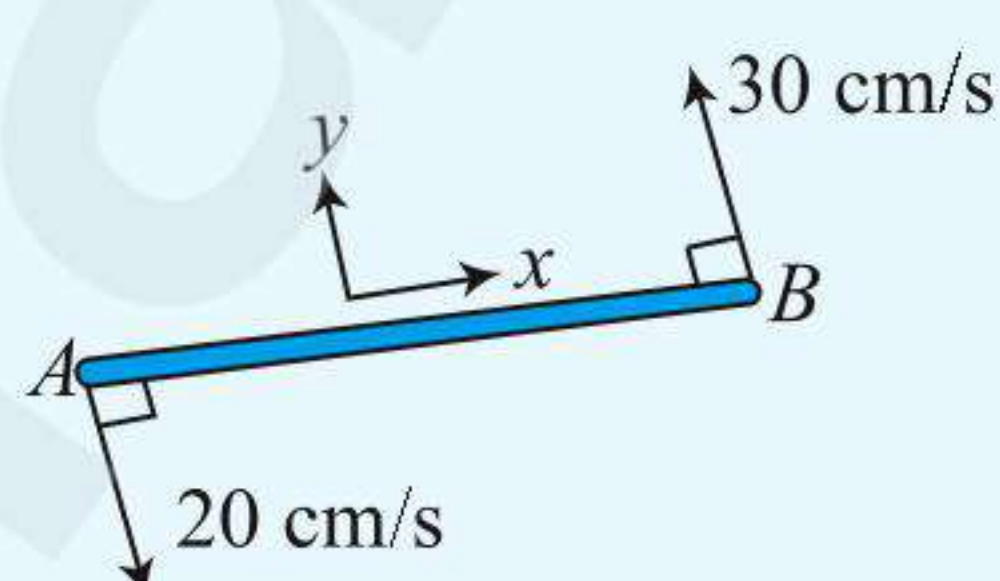
$$\vec{a}_4 = \vec{a}_{4,C} + \vec{a}_C$$

$\Rightarrow \vec{a}_{4,C} = -\alpha R \hat{i} + \omega^2 R \hat{j}$ and $\vec{a}_C = a \hat{i}$

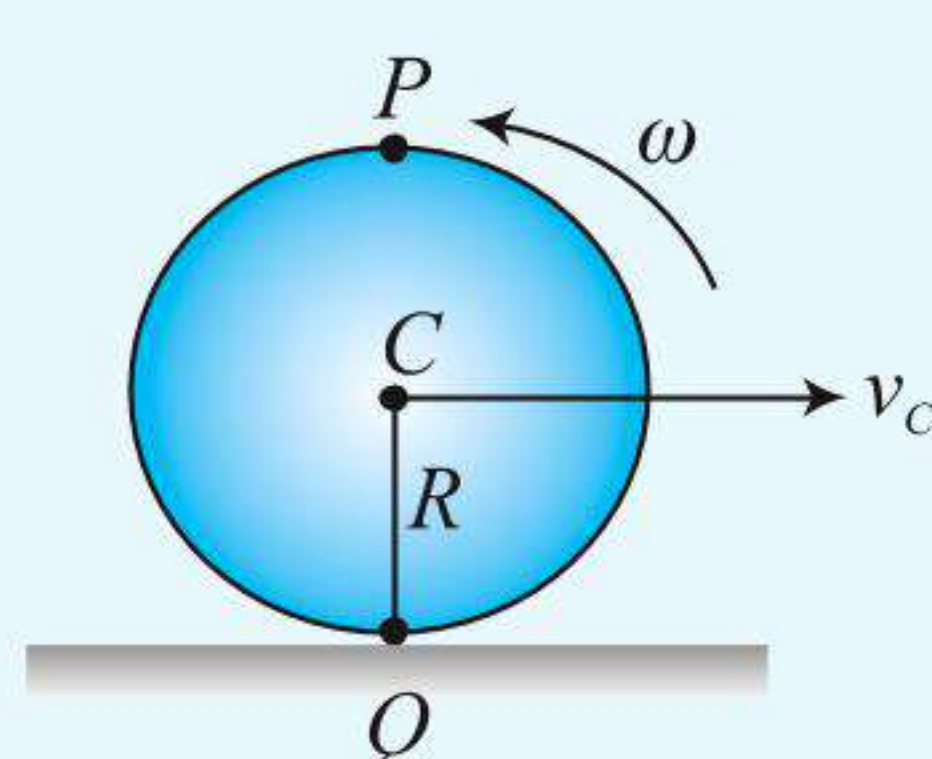
$$\text{Hence } \vec{a}_4 = (a - \alpha R) \hat{i} + \omega^2 R \hat{j}$$

CONCEPT APPLICATION EXERCISE 3.1

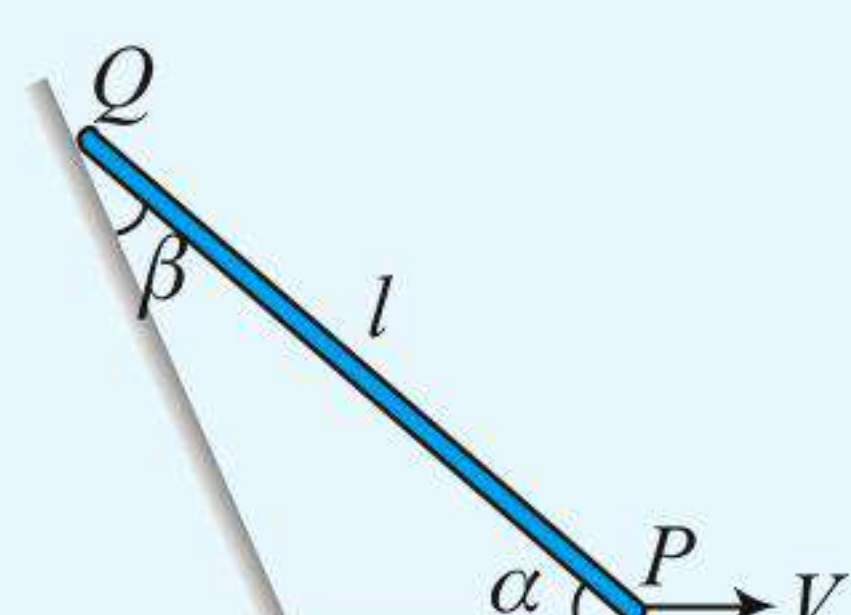
1. A 100 cm rod is moving on a horizontal surface. At an instant, when it is parallel to the x-axis its ends A and B have velocities 20 cm/s and 30 cm/s as shown in the figure. Find its angular velocity of the rod.



2. A ball of radius $R = 20$ cm is moving with combined translation and rotation. If the velocity of centre of mass of ball is $v_C = 10$ m/s and spins with an angular speed $\omega = 10$ rad/s, find the speeds of P and Q.

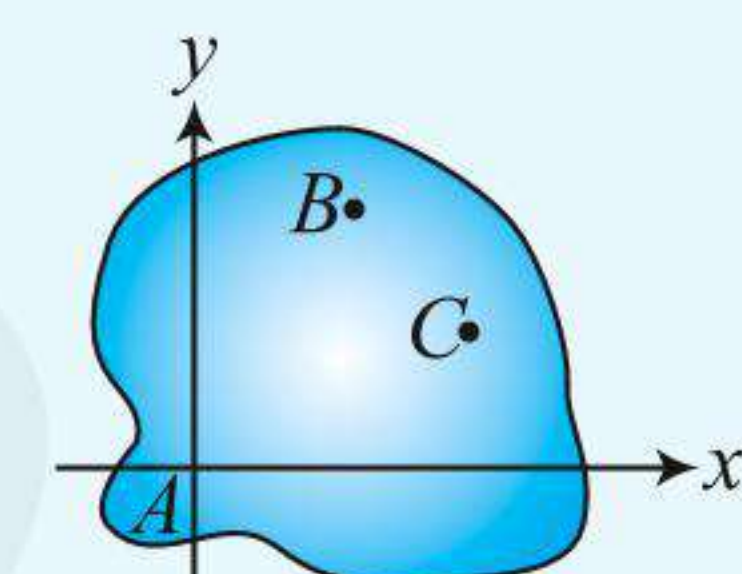


3. A rod of length l has a velocity v at P. Find the velocity at Q and the angular velocity of the rod.



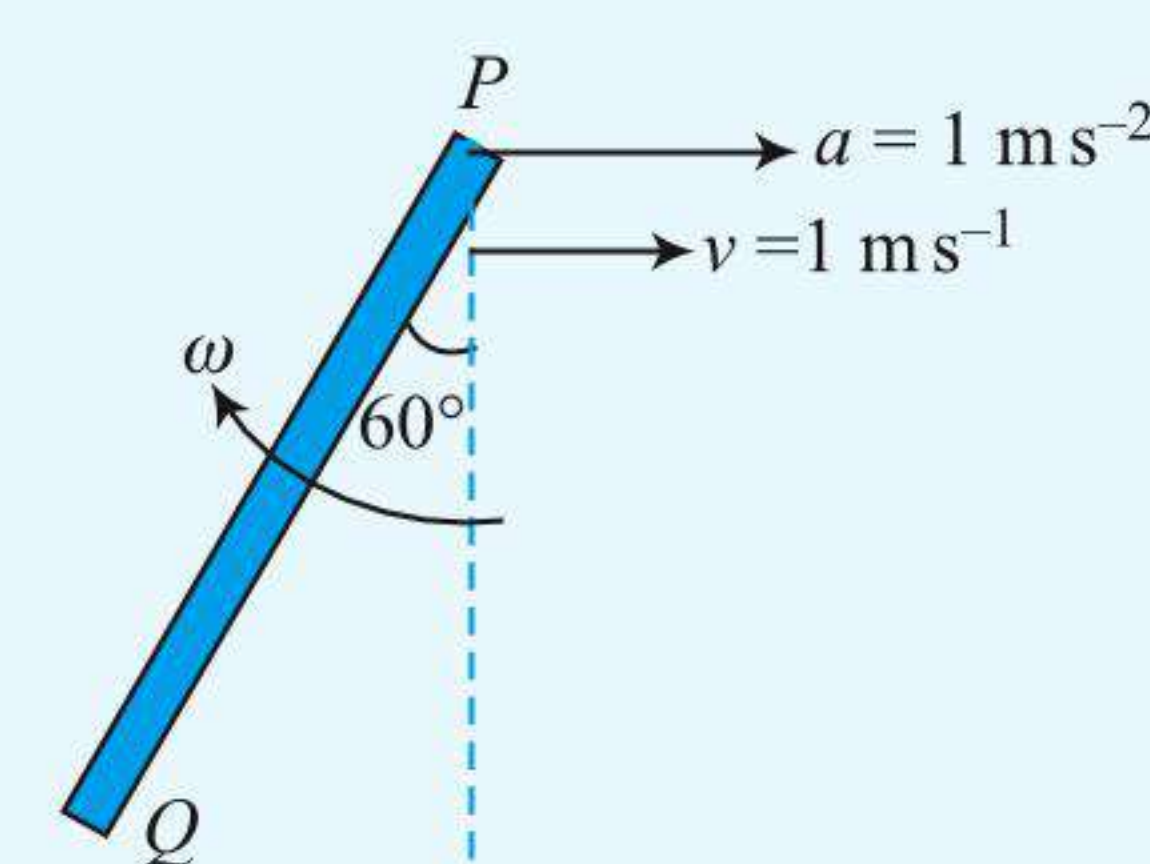
4. If the ends of a rod of length l have velocities $\vec{v}_A$ and $\vec{v}_B$, find the velocity of mid-point of the rod.

5. A flat rigid body is moving in x-y plane on a table. The body lies in the x-y plane. At an instant it was found that some of the velocity components of its three particles A, B and C were $v_{Ax} = 4$ m/s, $v_{Bx} = 3$ m/s and $v_{Cy} = -2$ m/s, respectively. At the instant the three particles A, B and C were located at (0,0), (3,4) and (4,3) (all in metre) respectively in a co-ordinate system attached to the table.

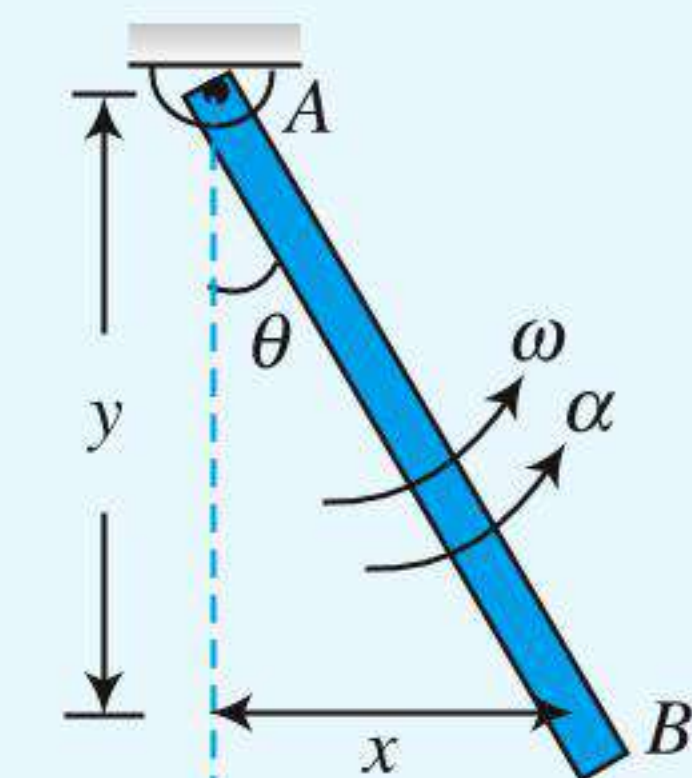


- (a) Find the velocity of A, B and C.
- (b) Find the angular velocity of the body.

6. The rod of length $l = 1$ m rotates with an angular velocity $\omega = 2$ rad s^{-1} and the point P moves with velocity $v = 1$ m s^{-1} and acceleration $a = 1$ m s^{-2} . Find the velocity and acceleration of Q.



7. The angular velocity and angular acceleration of the pivoted rod are given as ω and α respectively. Find the x and y components of acceleration of B.



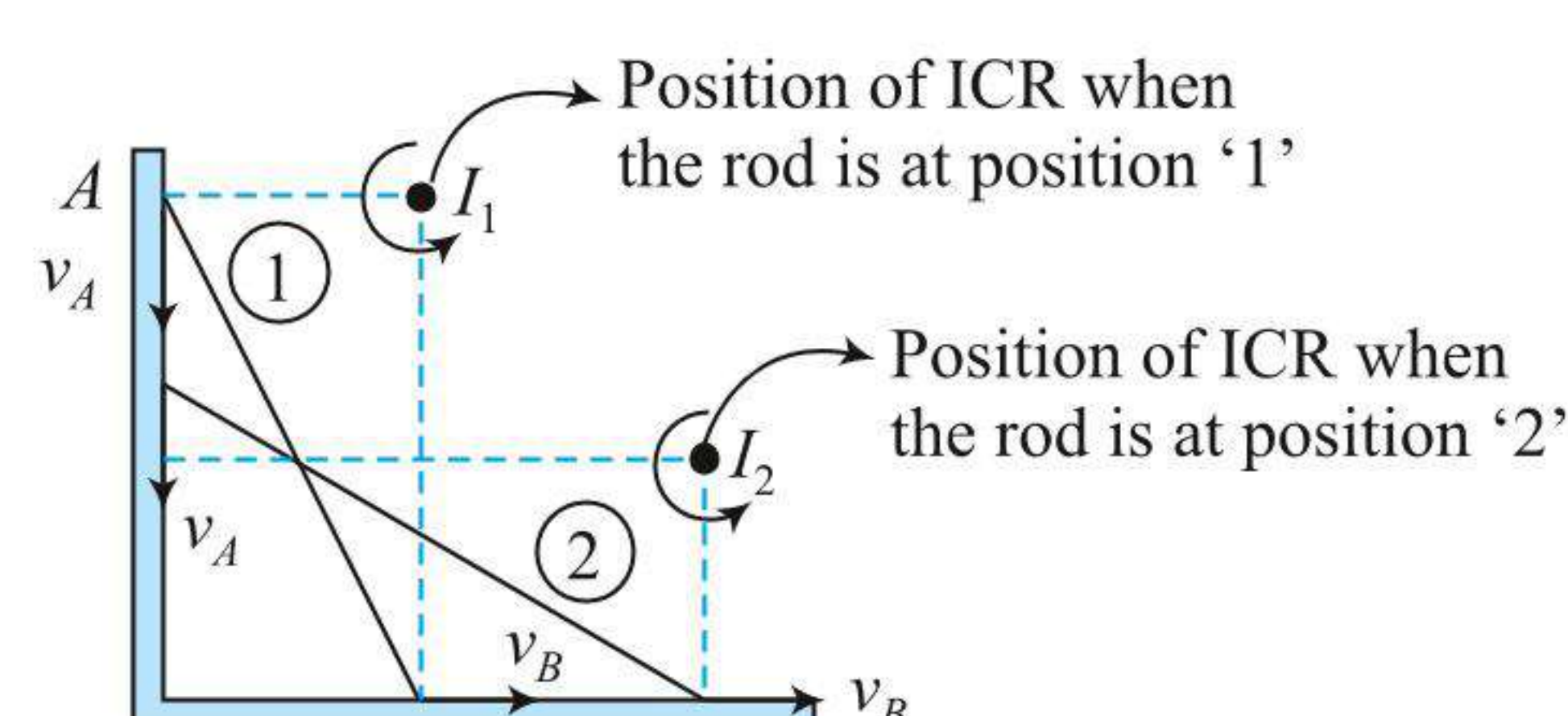
ANSWERS

1. $0.50 \hat{k}$ rad / s
2. 8 m/s, 12 m/s
3. $\frac{v \cos \alpha}{\cos \beta}, \frac{v \sin(\alpha + \beta)}{l \cos \beta}$
4. $\vec{v}_C = \frac{\vec{v}_A + \vec{v}_B}{2}$
6. $\sqrt{3}$ m/s, $\sqrt{17 + 4\sqrt{3}}$ m/s²

INSTANTANEOUS AXIS OF ROTATION

In combined translational and rotational motion, we cannot have a permanent axis of rotation. However, there may be a point inside the given body (or extended body) which can be momentarily at rest. This is what we call **instantaneous axis of rotation**.

Consider a rod which remains always in the same vertical plane with its ends in contact with smooth wall and floor as shown in figure. As the rod slides, its centre of mass translates and the rod rotates about a horizontal axis passing through centre of mass. The motion of such an object can be analyzed by making equations for translation and rotation separately, which involves lengthy calculations.



But the motion of such an object may be looked as pure rotation about a point that has zero velocity to simplify the study. Such a

point is called **instantaneous centre of rotation (ICR)** and the axis passing through this point and perpendicular to the plane of motion is called **instantaneous axis of rotation**.

In the figure shown I_1 and I_2 are the instantaneous centers of rotation of rod at two different instants.

Important Points:

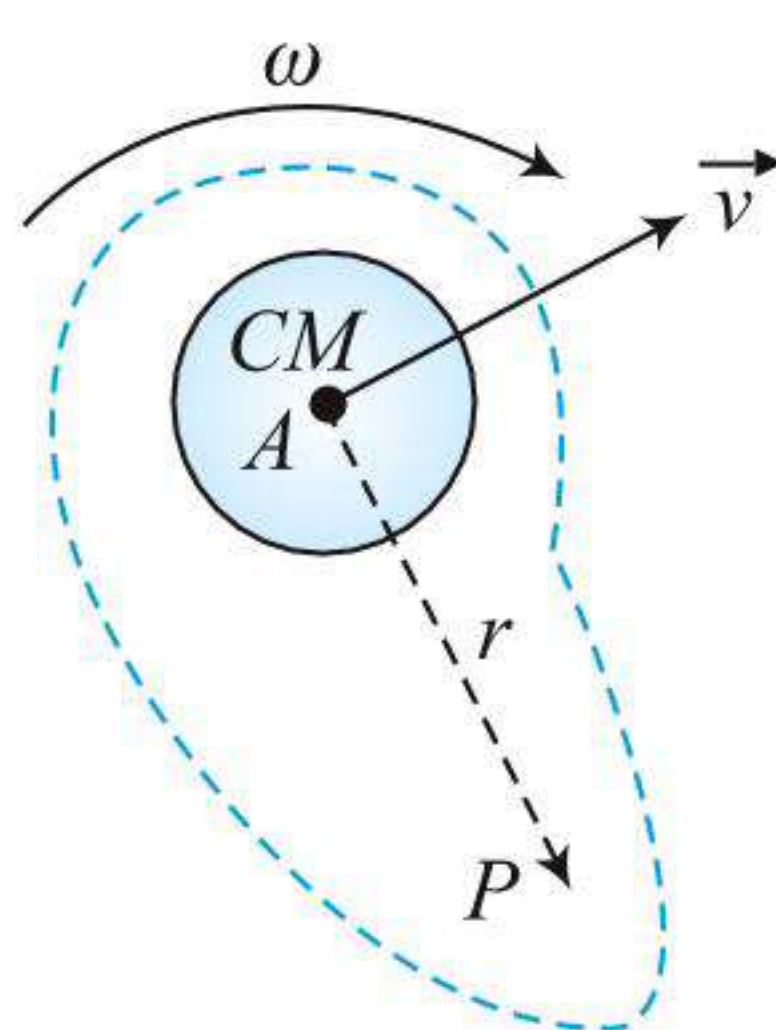
- The instantaneous centre of rotation is stationary with respect to ground.
- It may lie within or outside the body.
- The angular velocity of the body about this point is the same as that about its centre of mass.

FINDING INSTANTANEOUS AXIS OF ROTATION IN DIFFERENT SITUATIONS

When Velocity of a Point on Rigid Body (or Velocity of Centre of Mass) and Its Angular Velocity is Given

The **instantaneous axis of rotation** of the body is the point P of the given (shaded) body or extended body which remains at rest at that instant.

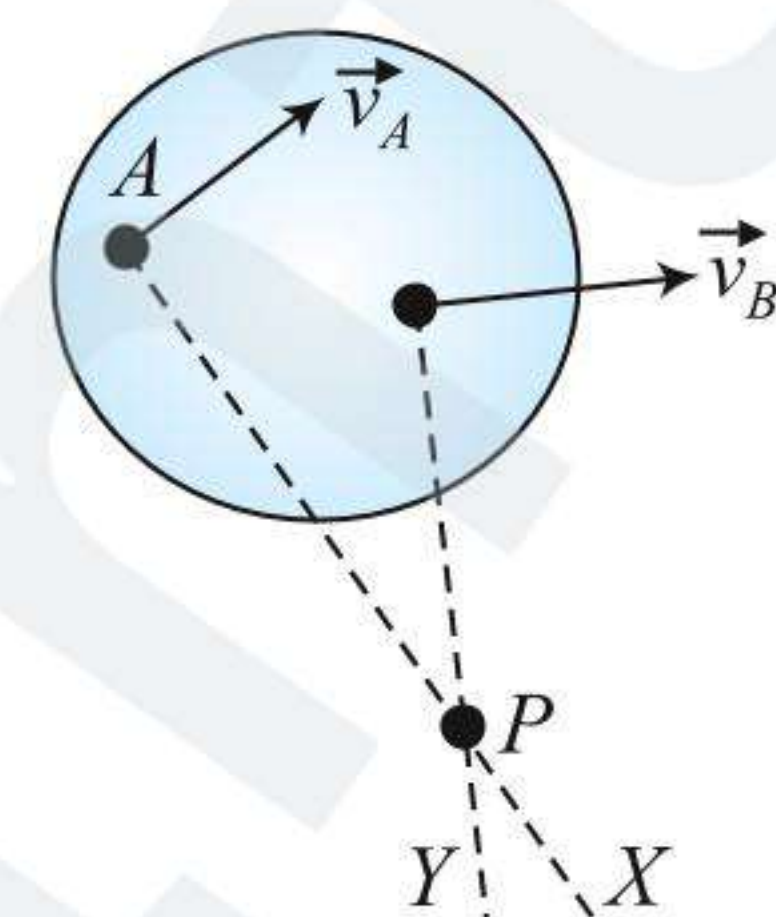
To locate **instantaneous axis of rotation**, drop a perpendicular at a point moving with velocity vector $\vec{v}$. Take point P (ICR) at a distance $r = v/\omega$ in the proper direction along the perpendicular.



Velocities of Any Two Points in the Body are Given

Consider a rigid body moving in a plane. Let A and B are two points having velocities $\vec{v}_A$ and $\vec{v}_B$ respectively at the instant considered.

AX and BY are the perpendicular to the velocities $\vec{v}_A$ and $\vec{v}_B$ respectively. The point of intersection of the two normal AX and BY , P will be the center of zero velocity or instantaneous centre of rotation of the rigid body.



The instantaneous axis of rotation passes through instantaneous centre of rotation and perpendicular to the plane of rotation.

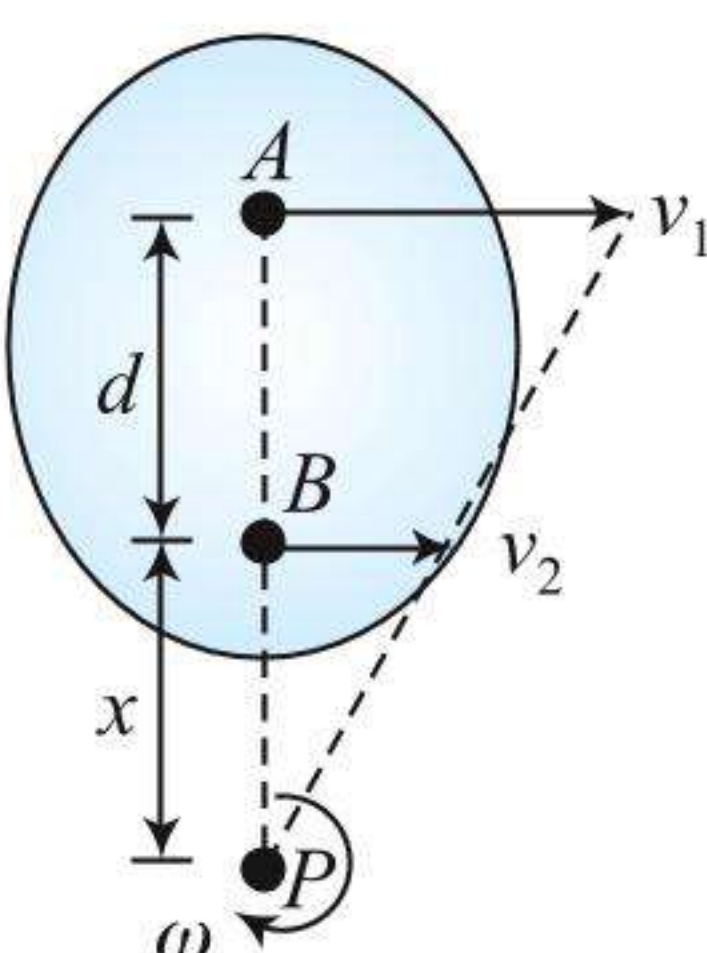
Now we can calculate the angular velocity of the rigid body, we have $v_A = (AP)\omega$. Then, the angular velocity of the rigid body is given as: $\omega = \frac{v_A}{AP} = \frac{v_B}{BP}$

Two Points in a Body Move Along Parallel Lines

Case I: Let two points A and B of a rigid body which are separated by a distance d are moving in same direction with velocities v_1 and v_2 . For finding the position of ICR, we draw the velocity vectors at positions A and B and join their tails and heads and produce the straight lines to meet at the ICR ' P '.

This is the point of zero velocity as shown in figure. The location of instantaneous axis of rotation is given as $\frac{v_1}{d+x} = \frac{v_2}{x} = \omega$

where ω is the angular velocity of the body.



Case II: Join the tails and heads of the velocity vectors $\vec{v}_A$ and $\vec{v}_B$. The meeting point P is the ICR. If antiparallel velocities are given, location of instantaneous axis of rotation is given, as $\frac{v_1}{d-x} = \frac{v_2}{x} = \omega$

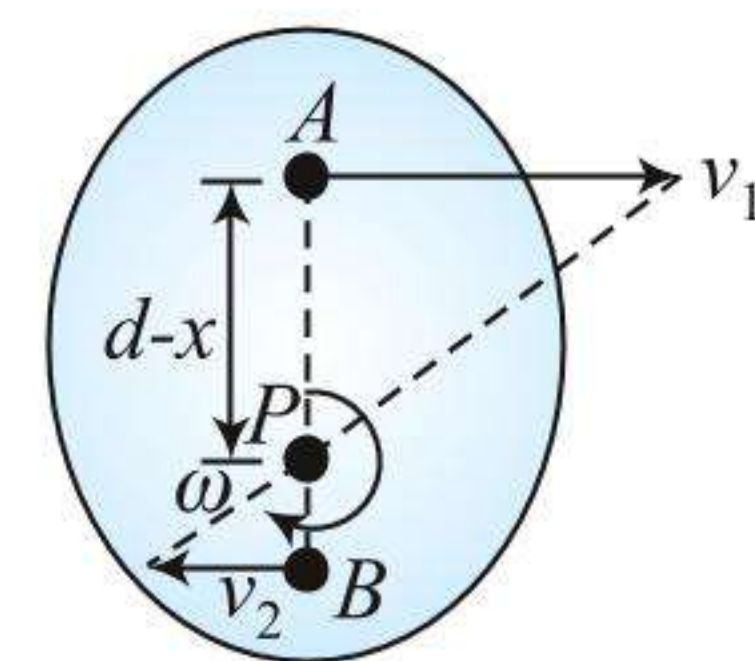
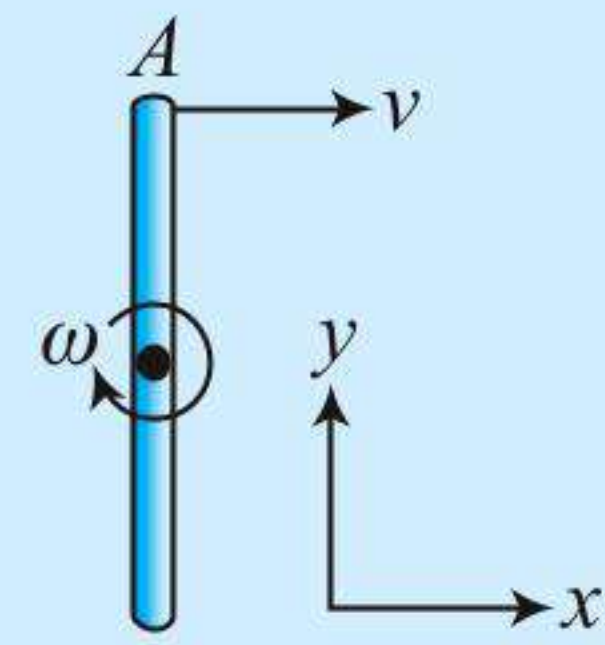


ILLUSTRATION 3.5

Find the instantaneous centre of rotation (ICR) of a rod of length l when its end A moves with a velocity $\vec{v}_A = v\hat{i}$ and the rod rotates with an angular velocity $\vec{\omega} = -\frac{v}{2l}\hat{k}$



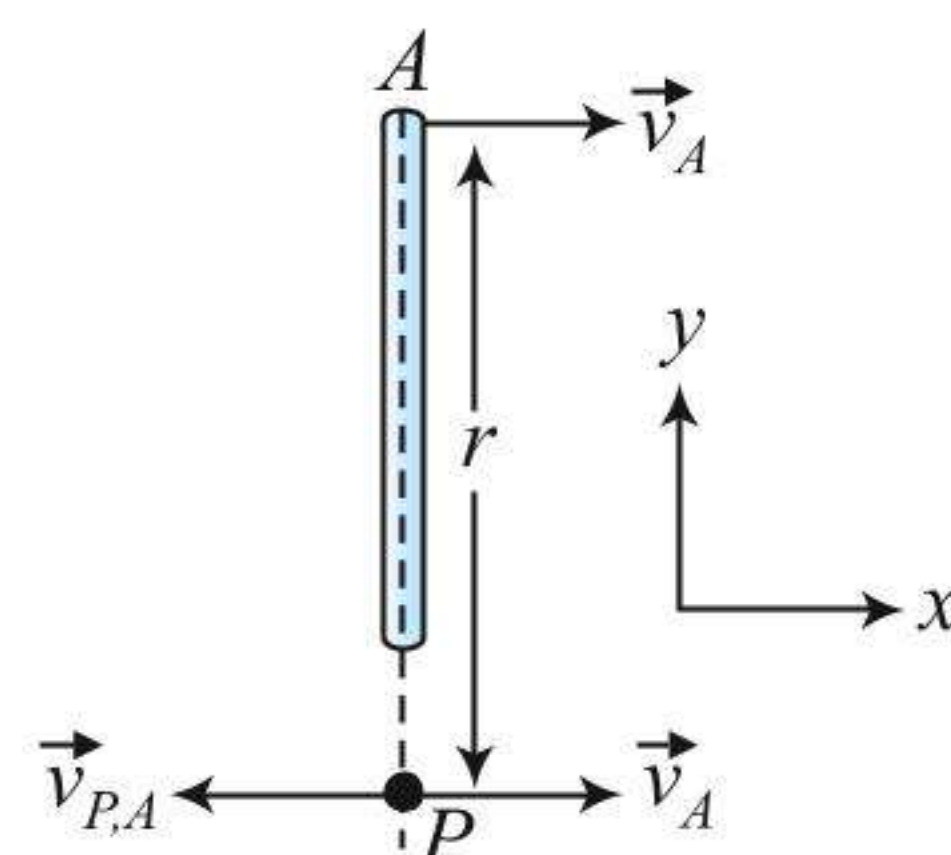
Sol. Method 1: For locating the position of instantaneous centre of rotation (ICR), let us select a point P on the extended rod such that it has zero velocity as ICR is a point of zero velocity, so we can write, $\vec{v}_P = \vec{v}_{P,A} + \vec{v}_A$

We have $\vec{v}_P = 0$; hence $\vec{v}_{P,A} + \vec{v}_A = 0$

Here $\vec{v}_A = v\hat{i}$ and $\vec{v}_{P,A} = -\omega r\hat{i}$

Hence $-\omega r\hat{i} + v\hat{i} = 0 \Rightarrow v = \omega r$

or $r = \frac{v}{\omega} = \frac{v}{(v/2l)} = 2l$



Hence ICR will be located at a distance $2l$ from A .

Method 2: The instantaneous centre of rotation will lie on the perpendicular to $\vec{v}_A$ at point A .

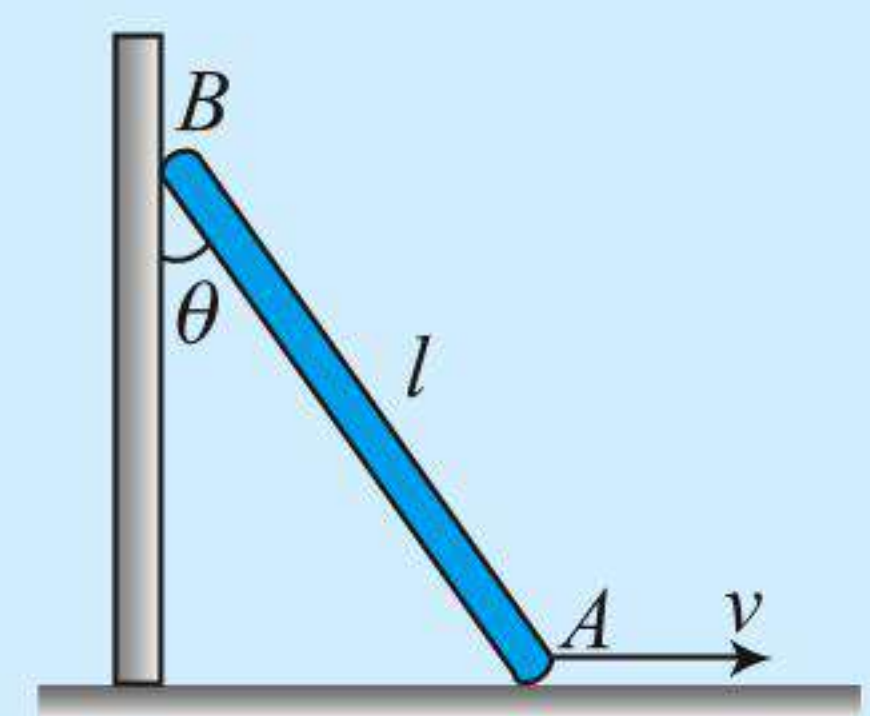
The ICR will lie at distance $AP = \frac{|\vec{v}_A|}{\omega} = \frac{v}{(v/2l)} = 2l$

or at a distance $2l$ from point A .

Note: When we get the IAR outside the given (real) body, we think that the point lying on IAR is fixed with the hypothetically extended body. It may mean that, the rod had been sufficiently long, the IAR would have been situated on the real rod at the distance of $2l$ from the end A of the rod.

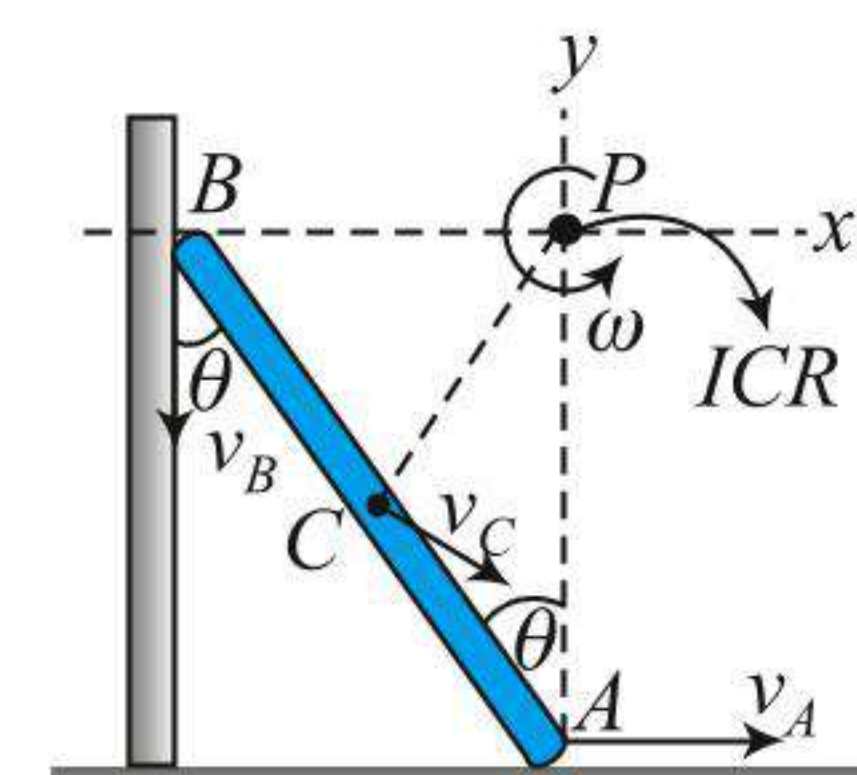
ILLUSTRATION 3.6

A rod ' AB ' of length l leaning against a smooth vertical wall and smooth horizontal surface. Now lower end ' A ' of the rod is pulled with a constant velocity v (say). In consequence, the upper end start sliding against the wall and the rod rotates in the vertical plane. Locate the position of instantaneous center of rotation (ICR) of the rod when it makes an angle θ with vertical and find the angular velocity of the rod.



Sol. Let end B slide on the vertical wall with velocity v_B . AY and BX are the perpendicular to the velocities $\vec{v}_A$ and $\vec{v}_B$ respectively. The point of intersection of the two normal AY and BX , P will be the center of zero velocity or instantaneous centre of rotation of the rod. Each and every point of the rod rotate about P with same angular velocity (say ω) in counter clockwise sense.

We have $\omega_{BP} = \omega_{AP} = (\omega)$,



where $\omega_{AP} = \frac{v_A}{PA}$

Substituting $PA = l \cos \theta$ and $v_A = v$, we have

$$\omega = \frac{v}{l \cos \theta}$$

ROLLING MOTION

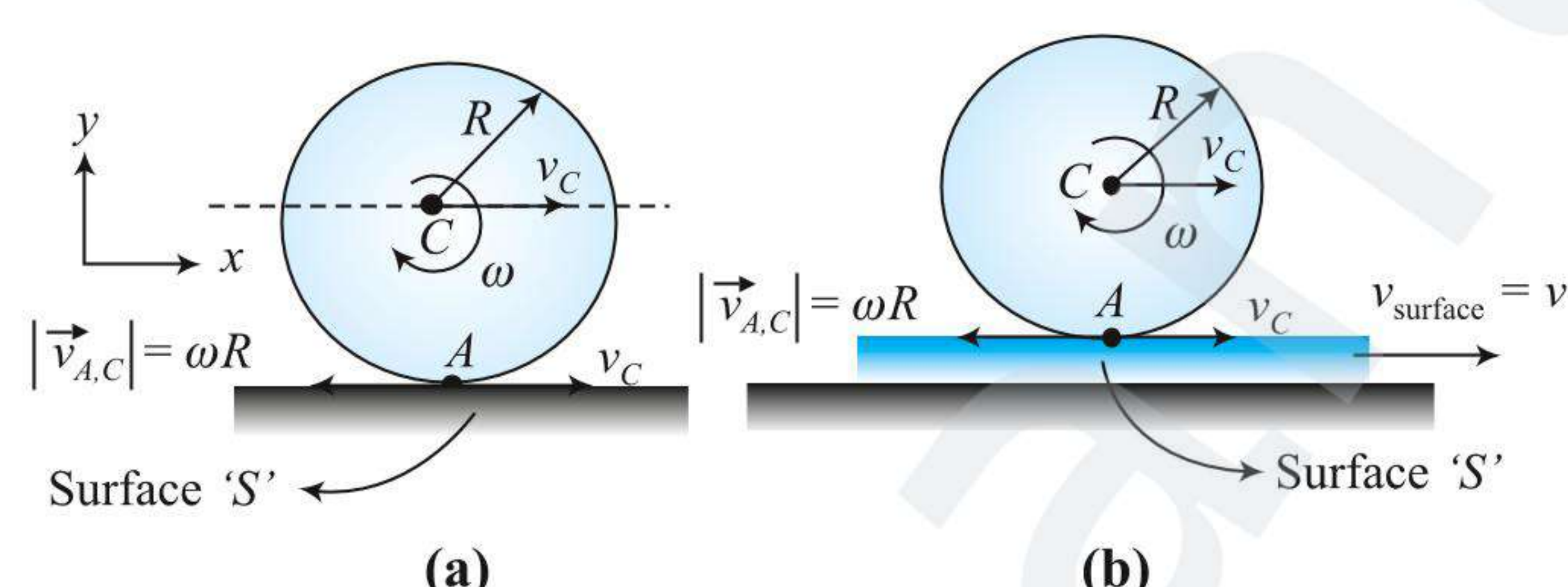
The term rolling means that there is no slipping at the point of contact with the ground—no skidding takes place. The wheel of a car that is moving forward while its tyres are spinning at high speed, leaving behind black strips on the road, is an example of rolling motion with slipping.

A body in combined rotation and translation motion said to be rolling over a surface if the surfaces in contact do not slide relative to each other, it means that the relative velocity between the points of contact of the body with the surface is zero.

Let us take a uniform disc of radius R (say). Place it vertically on a smooth horizontal surface. Spin the disc and simultaneously push it. In consequence, the disc will rotate with an angular velocity ω and its centre of mass C moves with a velocity v_C . The velocity of lowest point A of the disc can be given as: $\vec{v}_A = \vec{v}_{AC} + \vec{v}_C$, where $\vec{v}_{AC} = -R\omega\hat{i}$ and $\vec{v}_C = v_C\hat{i}$. Then velocity of point A , $\vec{v}_A = (v_C - R\omega)\hat{i}$.

The above expression tells us that for different values of v_C and ω , the point A will move with different speeds. Let us assume that A moves with a velocity which is equal to the velocity of the horizontal surface S ; then $\vec{v}_A = \vec{v}_S$. It means, relative velocity between A and S is zero.

Then you can say that P does not slide on the surface S . In other words, the body is said to be rolling without sliding on the horizontal surface.



In the given Fig. (a), the surface on which the disc is rolling, is fixed, then the velocity of point of contact should be zero ($v_A = 0$). Thus, the rolling condition is equivalent to the following relation between the translational speed of the centre of mass and the angular speed.

$$v_C = R\omega \quad \dots(i)$$

Differentiation of Eq. (i) relates the magnitude of the linear acceleration of the centre of the wheel, a_C , to the magnitude of the angular acceleration α about the axis through the centre of the wheel. Similarly,

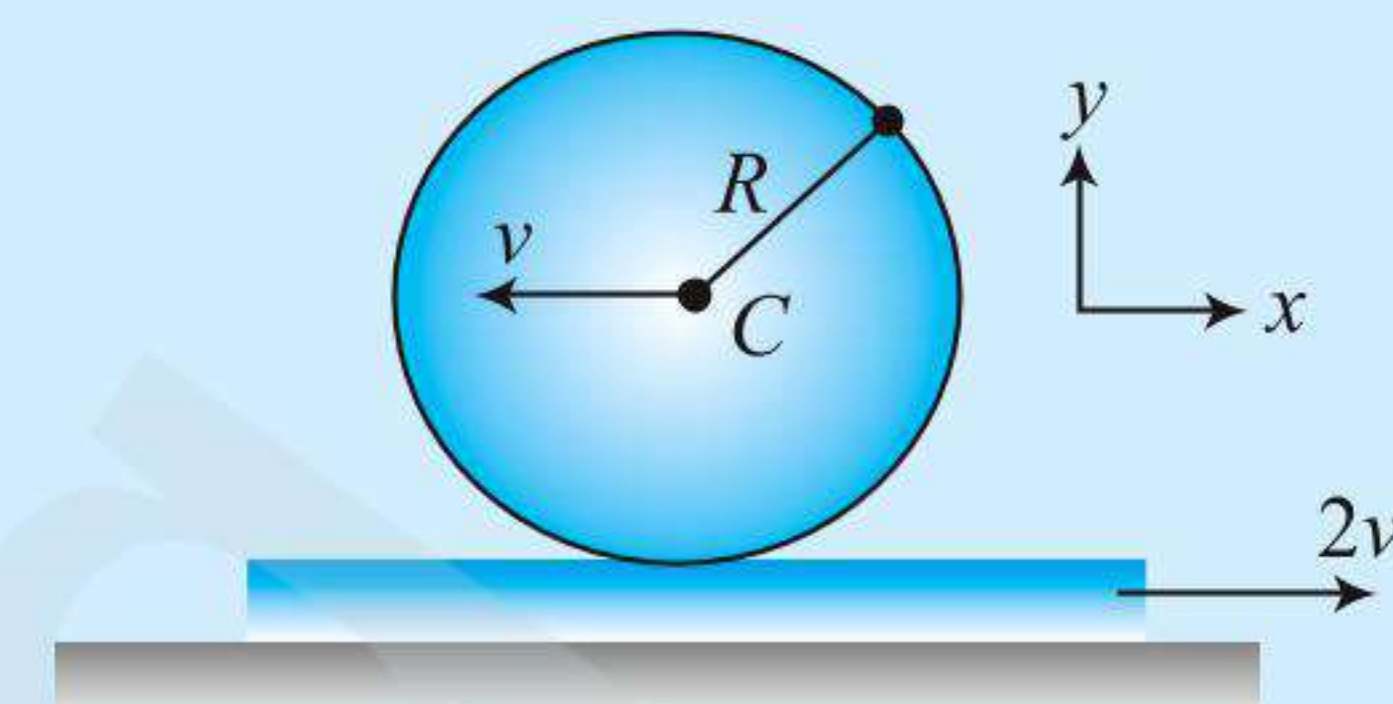
$$a_C = R\alpha \quad \dots(ii)$$

In the given Fig. (b), the surface on which the disc is rolling, is moving with velocity v , then for no sliding case the velocity of point of contact should be same as velocity of the surface

$$v = v_C - R\omega$$

ILLUSTRATION 3.7

A uniform disc is spun with an angular velocity $\vec{\omega}$ and simultaneously projected with a linear velocity v towards left on a plank, while the plank moves towards right with a constant velocity $2v$. If the disc rolls without sliding on the plank just after its spinning, find the magnitude of $\vec{\omega}$.



Sol. For pure rolling of the disc on the plank, $\vec{v}_P = \vec{v}_S$.

The velocity of plank is in rightward direction and disc is moving in leftward direction. As disc is rolling, there should not be relative sliding between plank and disc at point of contact. For this, velocity of point P should be rightwards. It means the rotation of disc should be in counter clockwise sense.

$$\vec{v}_{P,C} + \vec{v}_C = \vec{v}_P$$

or $\omega R - v = 2v$

$$\Rightarrow \omega = \frac{3v}{R} \text{ in counter clockwise sense.}$$

In vector form, $\vec{\omega} = \frac{3v}{R}(\hat{k})$

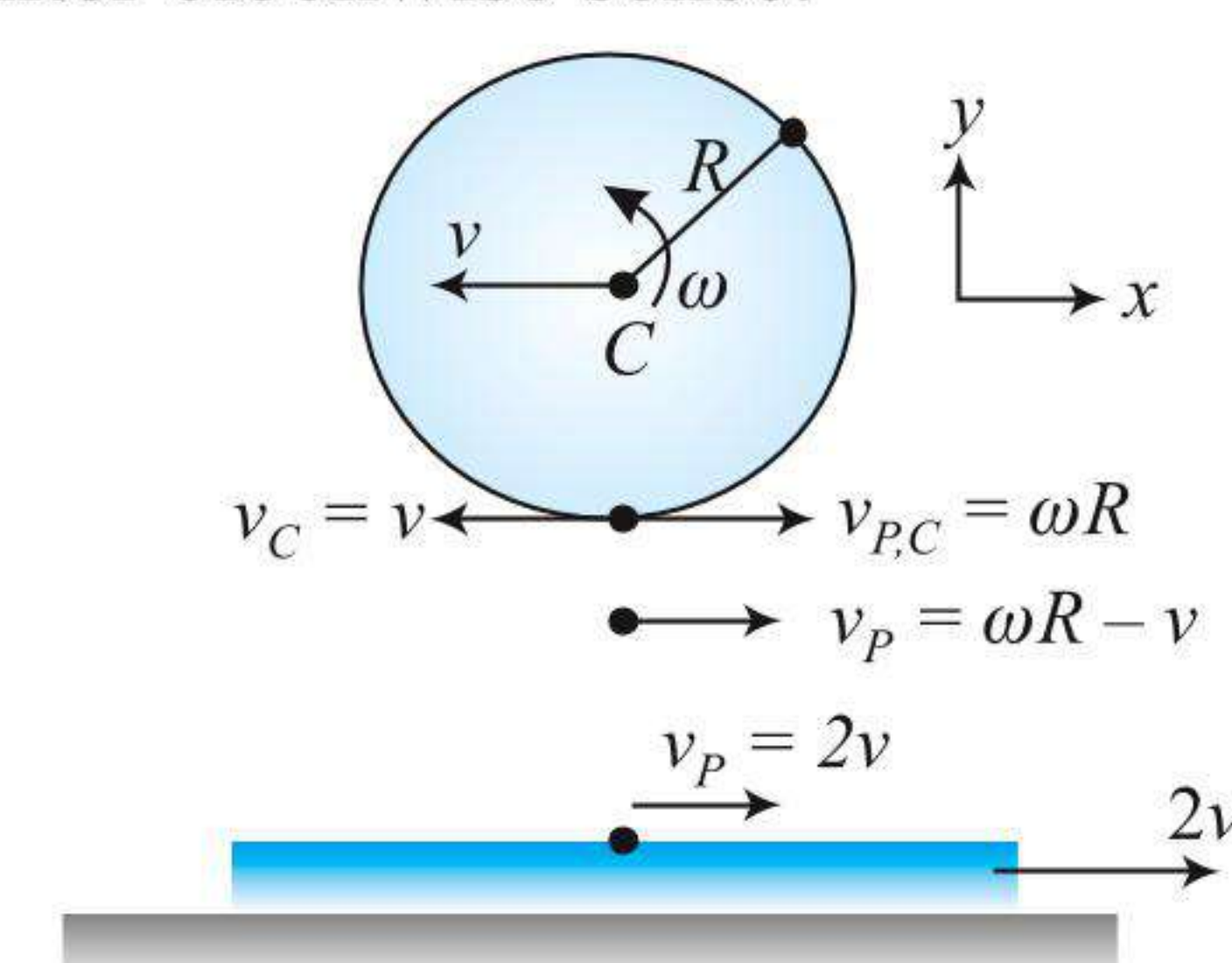
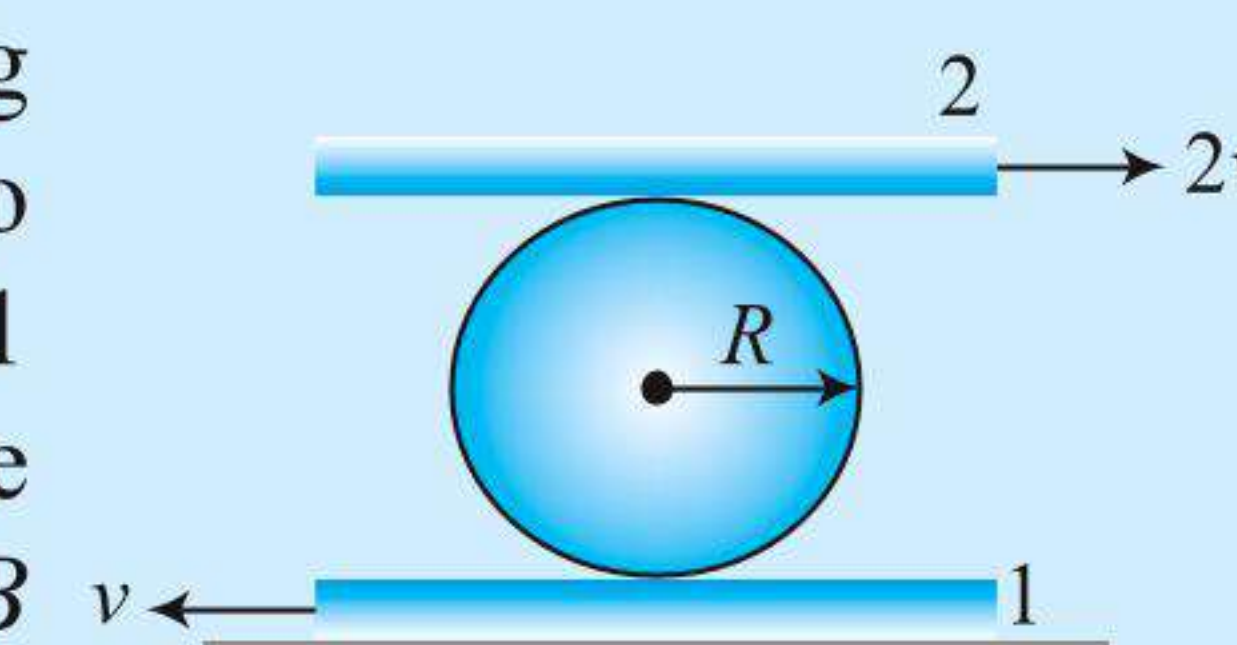


ILLUSTRATION 3.8

A cylinder of radius R is rolling without sliding between two horizontal planks (surfaces) 1 and 2 as shown in figure. If the velocities of the planks A and B are v and $2v$ respectively, find the:

- position of ICR.
- angular velocity of the cylinder.



Sol.

- Since, there is no relative sliding between the cylinder and the planks 1 and 2, the points A and B of the disc will move with velocities equal to the velocities of the respective plank. For finding the position of ICR, we draw the velocity vectors at positions A and B . Joining A and B and A' and B' , we find the point ' P ' as ICR

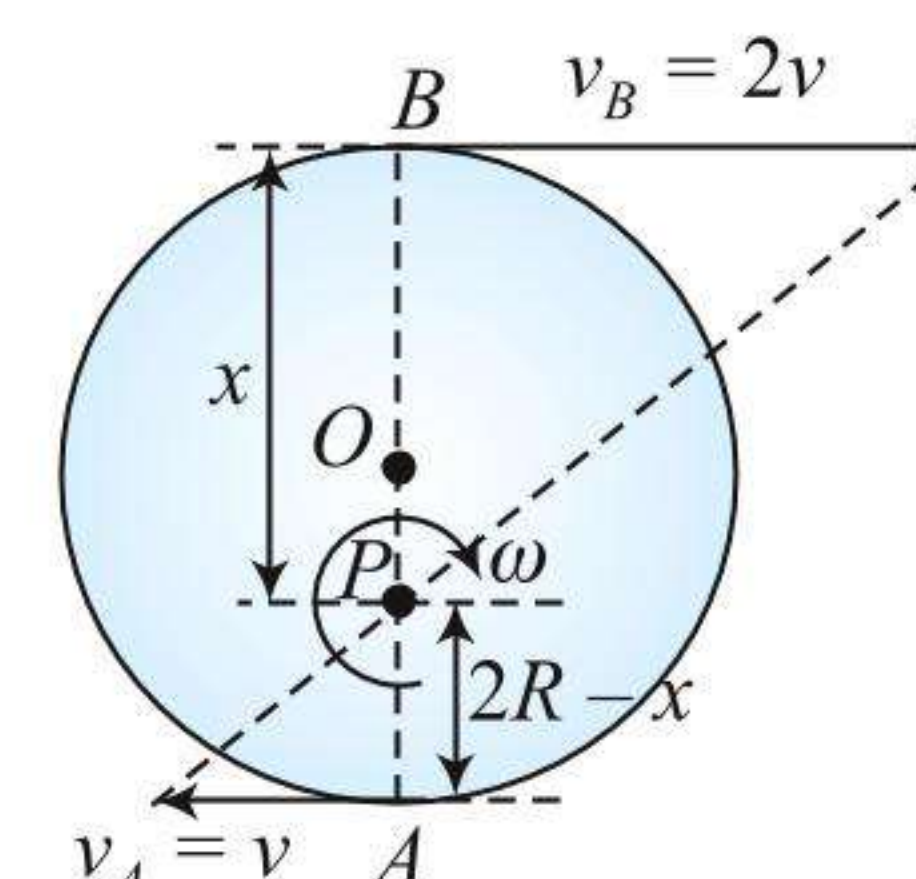
$$\omega = \frac{2v}{x} = \frac{v}{2R - x}$$

This gives, $x = \frac{4R}{3}$

Hence, ICR is located $4R/3$ below the top of the disc.

- Angular velocity of the cylinder $\omega = \frac{2v}{x} = \frac{2v}{4R/3}$

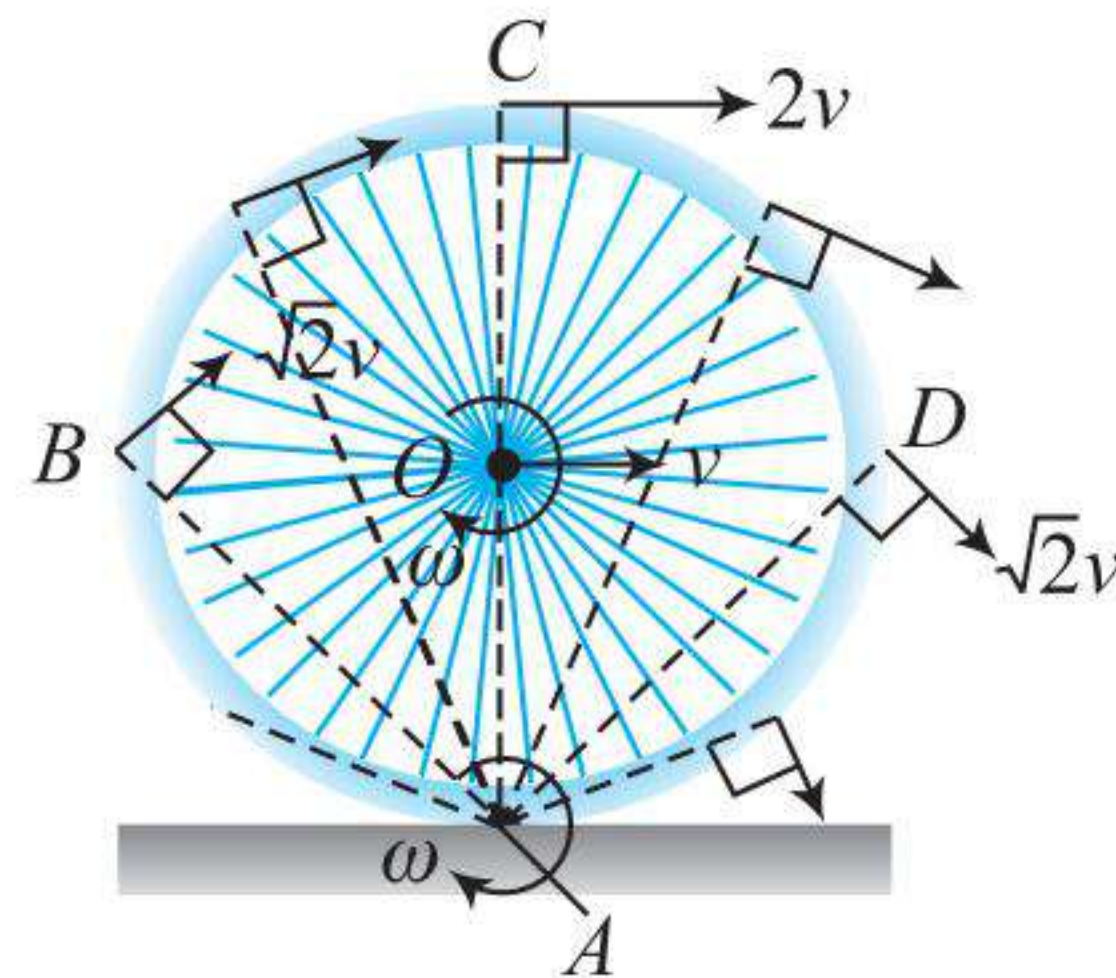
This gives $\omega = \frac{3v}{2R}$ (clockwise)



ROLLING AS PURE ROTATION ABOUT POINT OF CONTACT

Consider a ring rolling on a smooth fixed horizontal surface. A rolling body on a fixed horizontal surface, does not slip, therefore the point of contact A with the surface is instantaneously at rest and the ring momentarily rotates about this point. We can say that at any instant the “lowest point” (point of contact) of the rolling body with the horizontal surface is the ICR. It means, the body rotates perfectly about the point A at a given instant.

It means the rolling can be viewed as pure rotation, with angular speed ω , about an axis that always extends through the point where the wheel contacts the ground as the wheel moves. The vectors show the instantaneous linear velocities of selected points on the rolling wheel.



Velocity of point A , $v_A = 0$

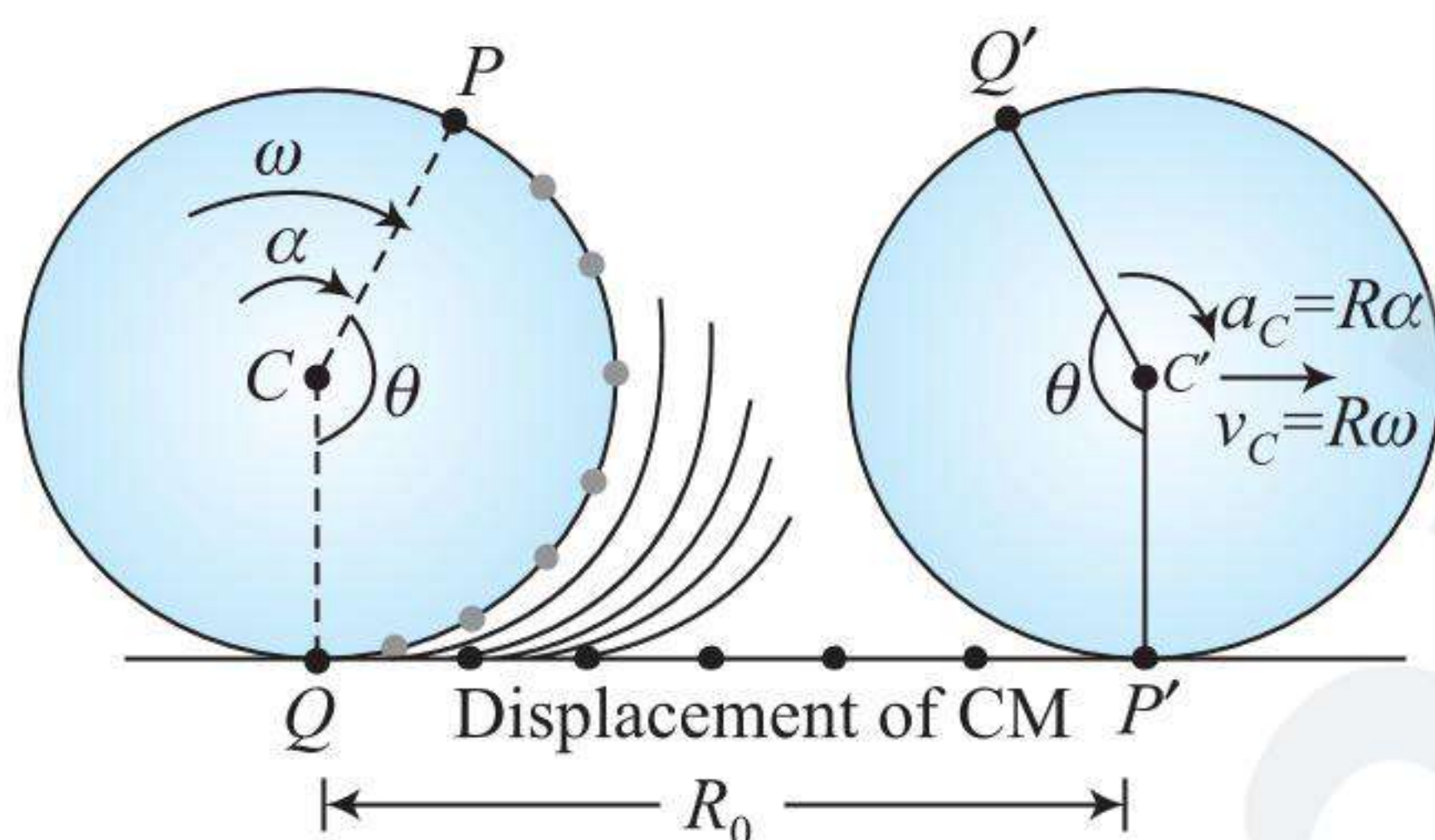
Velocity of centre of mass O , $v_O = \omega R$

Velocity of topmost point C ,

$$v_C = \omega(2R) = 2\omega R = 2v_O$$

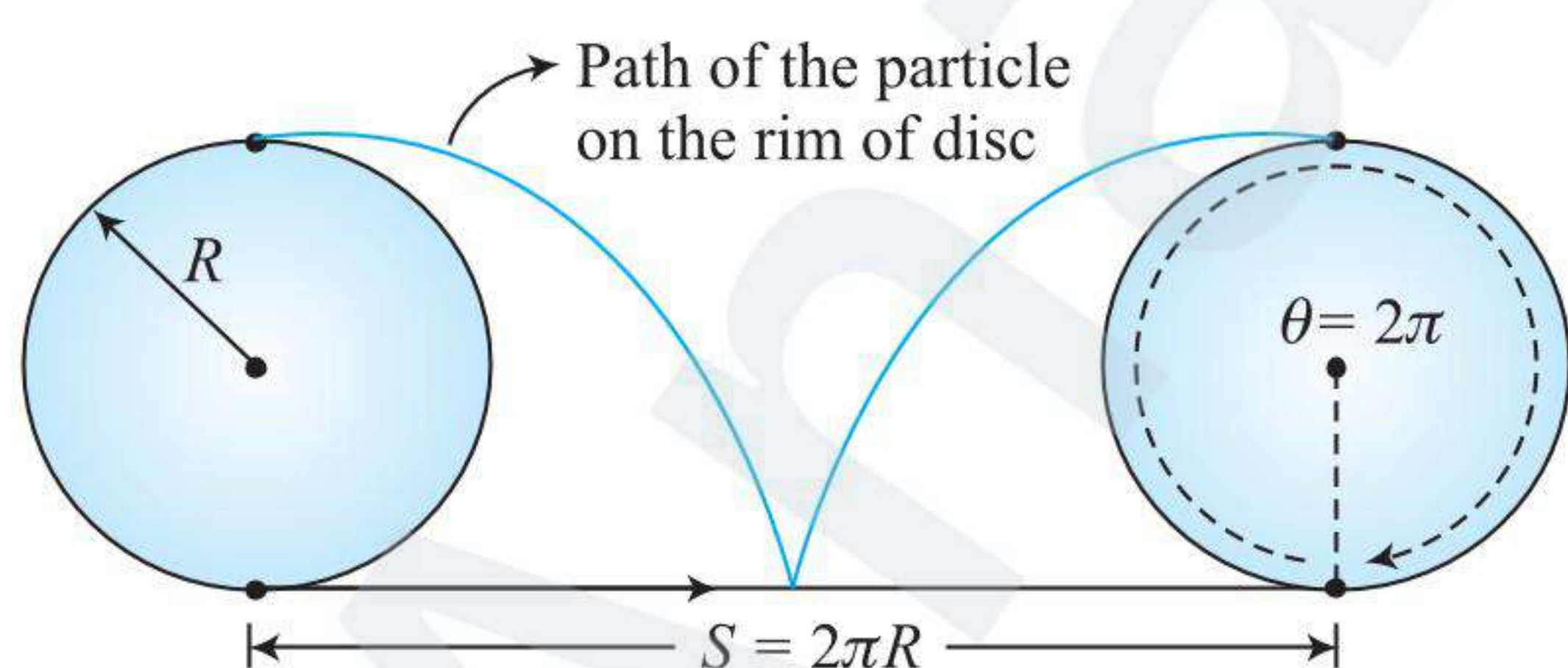
Velocity of points B or D , $v_B = v_D = \omega(\sqrt{2}R) = \sqrt{2}\omega R = \sqrt{2}v_O$

The disc in figure rolls to the right and point C on the axis moves to C' as P moves to P' and Q moves to Q' . The arc length from P to Q equals x just as length $QP' = CC' = x$. Thus, the linear distance travelled by the centre of mass is $x_{CM} = R\theta$

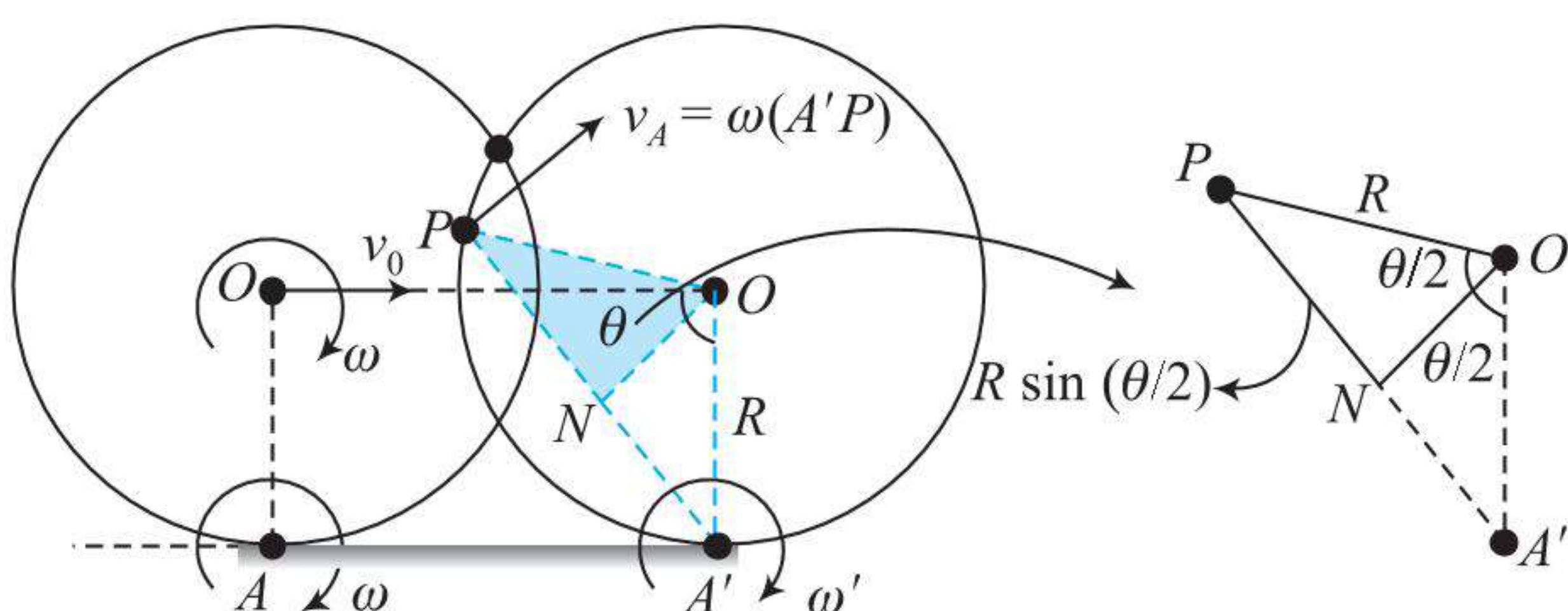


If the disc rotates through an angle θ (in radians), as in figure, then the centre moves through a distance

$$x_{CM} = R\theta$$



One complete revolution of a rolling disc translates the centre of mass by a distance equal to the circumference of the circular cross-section. But what about the length of the path covered by a point on the rim of the disc, let us calculate it.



Magnitude of the velocity of any point on the disc

$$|\vec{v}_P| = \omega \left(2R \sin \frac{\theta}{2} \right) = 2\omega R \sin \frac{\omega t}{2} \quad (\text{As } \theta = \omega t)$$

The length of the arc covered by any point on the rim of the disc is one complete rotation, $s = \int_0^T |\vec{v}_P| dt$,

where T = time for one complete rotation $= \frac{2\pi}{\omega}$

$$\text{Hence } s = \int_0^{\frac{2\pi}{\omega}} 2\omega R \sin \frac{\omega t}{2} \cdot dt = 2\omega R \left[\frac{-\cos \frac{\omega t}{2}}{\frac{\omega}{2}} \right]_0^{\frac{2\pi}{\omega}}$$

$$\Rightarrow s = -4R \left[\cos \frac{\omega}{2} \cdot \frac{2\pi}{\omega} - \cos 0 \right] = 8R$$

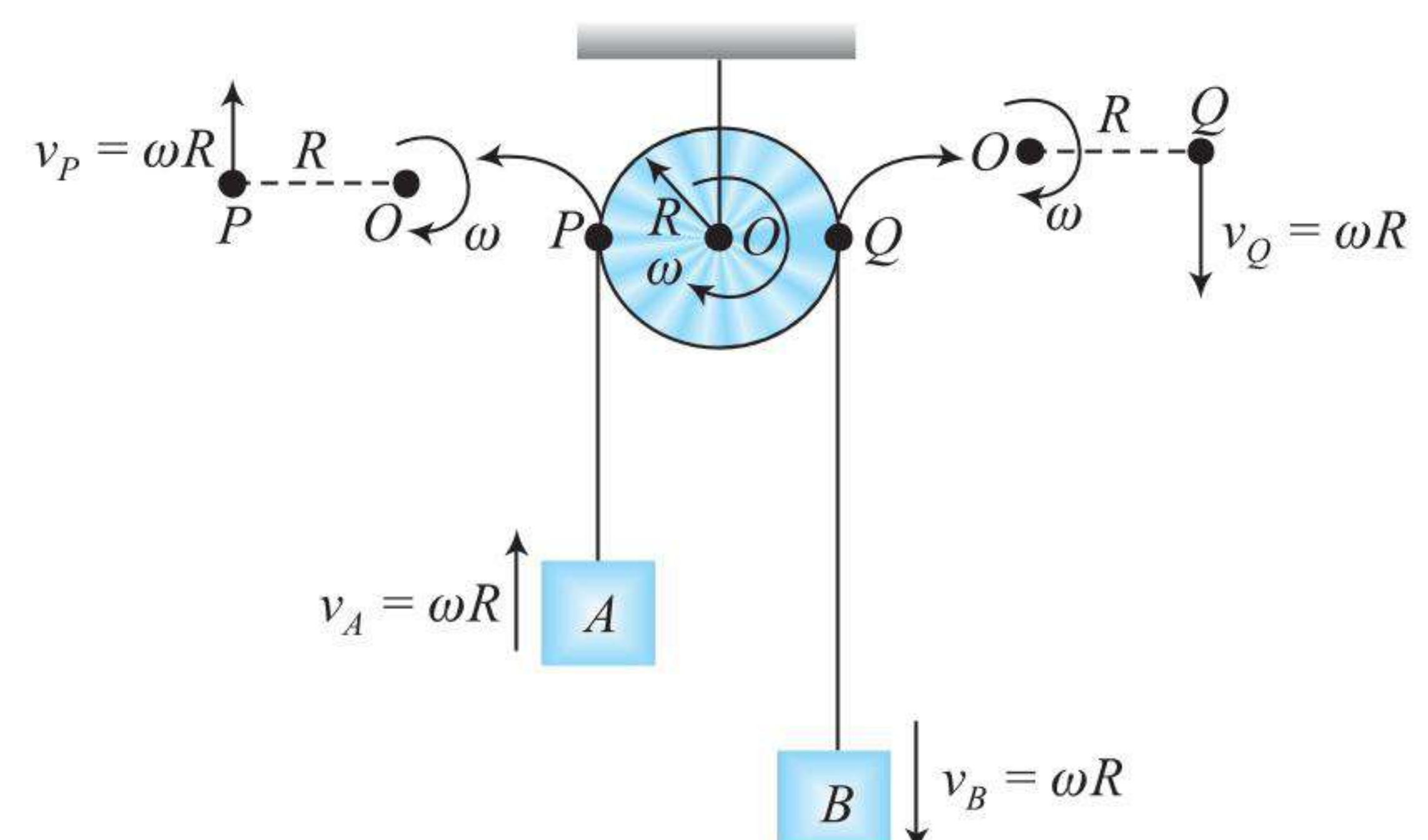
CONSTRAINT RELATIONS IN CASE OF ROTATION AND TRANSLATION COMBINED

In rotational dynamics all the particles of a body move with different velocities and acceleration. For analyzing the motion of a body under combined rotation and translation, we need to relate the velocity and acceleration of any particle in the body with the velocity and acceleration of some other particle of the same body.

In that case we need to write a constraint equation in the same way as we already done in the chapter ‘Newton’s Laws of Motion-I’ in book ‘Mechanics I’. Here we are going to learn writing constraint relations in combined rotation and translation through some selected illustrations.

CONSTRAINTS DUE TO ROPES AND PULLEYS

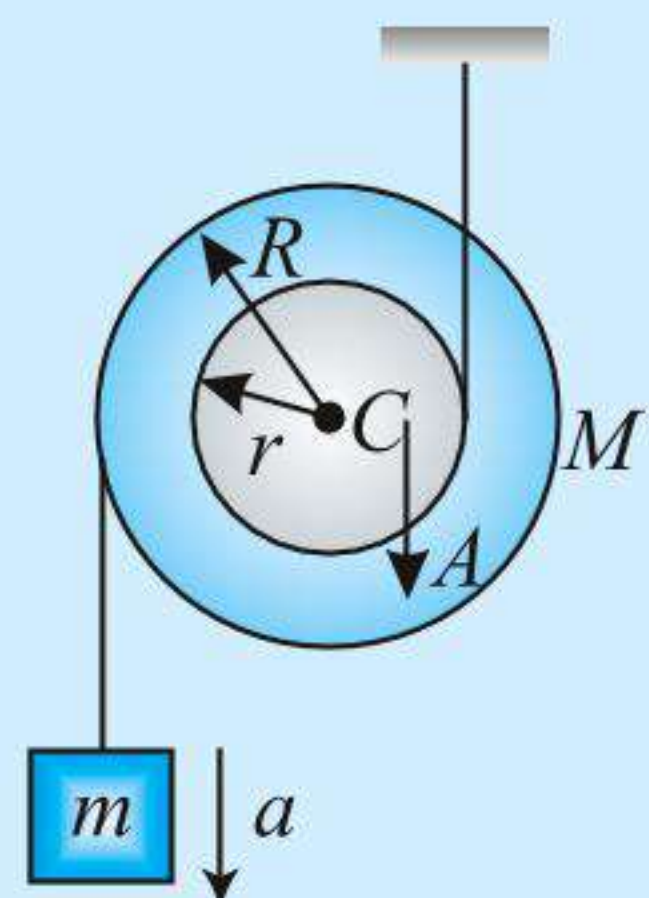
While solving the problems of rotational dynamics we encounter situations where pulleys are connected via strings or belts to other objects. We assume that the string turns on the pulley without slipping. Due to this constraint the string’s speed v_{string} must be equal to the speed of the rim of the pulley which has magnitude $|\vec{v}_{\text{rim}}| = |\vec{\omega}| R$. If the pulley has angular acceleration α , the string acceleration a_{string} must be equal to the tangential acceleration of the rim of the pulley $|\vec{a}_t| = |\vec{\alpha}| R$



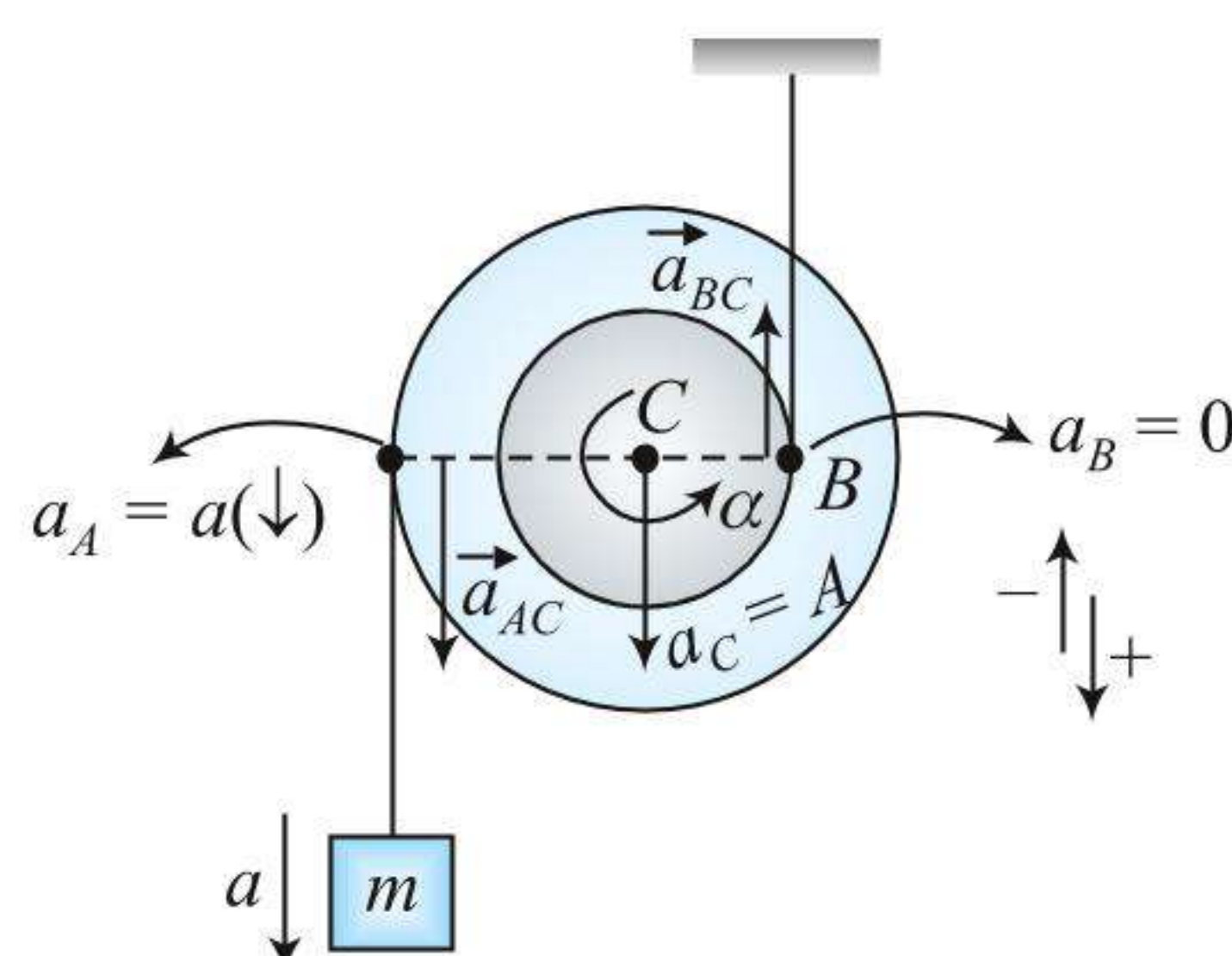
In nutshell we can write $\vec{v}_{\text{block}} = |\vec{\omega}| R$; $\vec{a}_{\text{block}} = |\vec{\alpha}| R$.

ILLUSTRATION 3.9

A bobbin with inner radius r and outer radius R is arranged with light strings and a block as shown in the figure. The string is not sliding over the cylinder. The system is released from rest. The block and bobbin move down with accelerations a and A respectively. Find the ratio a/A .



Sol. Let the angular acceleration of the bobbin is α as shown. Point B is directly connected with roof through string. It means net acceleration of point B should be zero.



$$\vec{a}_B = \vec{a}_{B,C} + \vec{a}_C$$

$$0 = -\alpha r + A \quad \text{or} \quad \alpha = \frac{A}{r} \quad \dots(i)$$

Point A on the bobbin is directly connected with block, it means the acceleration of point A should be same as the acceleration of the block.

$$\vec{a}_A = \vec{a}_{A,C} + \vec{a}_C$$

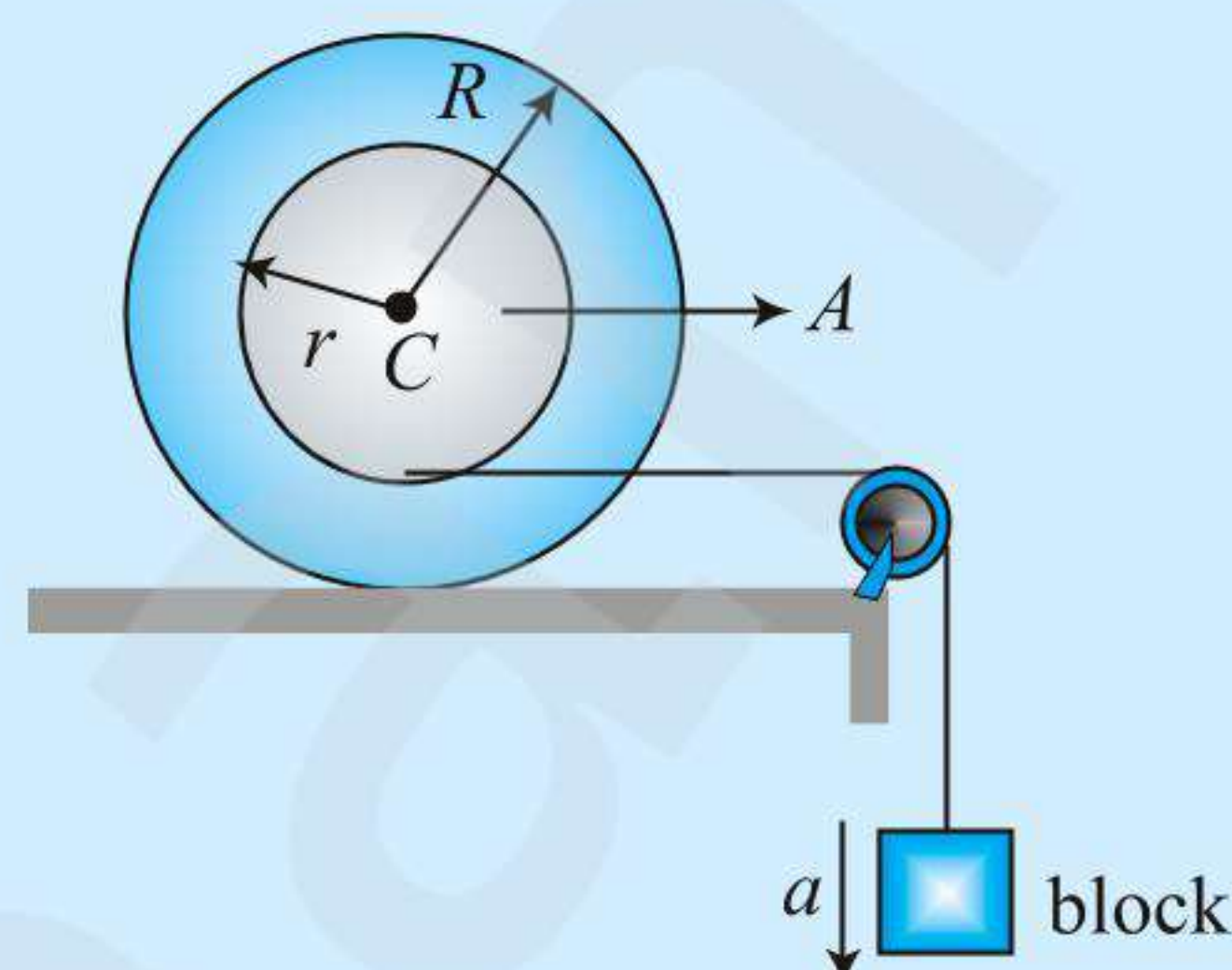
$$a = \alpha R + A \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

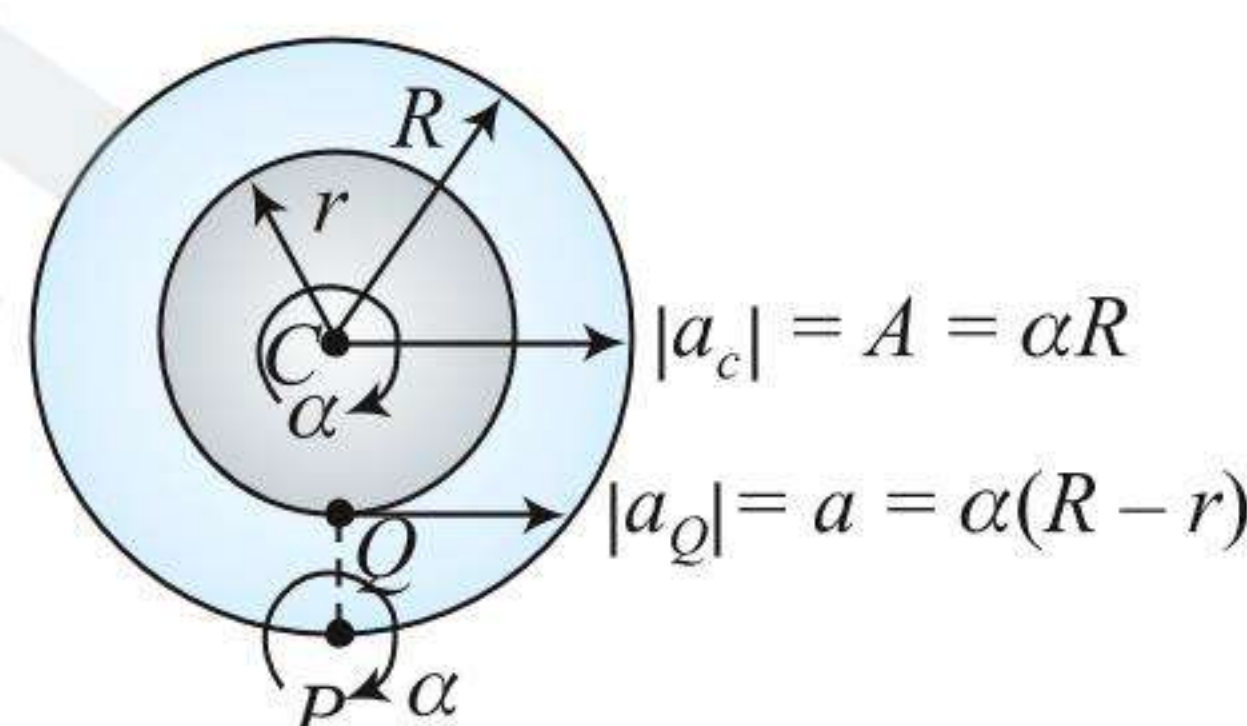
$$a = \left(\frac{A}{r}\right)R + A \quad \text{or} \quad \frac{a}{A} = \frac{R+r}{r}$$

ILLUSTRATION 3.10

A bobbin has inner radius r and outer radius R is placed on a rough horizontal surface. A light string is wrapped over inner core, connects a block with bobbin as shown in the figure. Now system is released from rest and bobbin moves on the horizontal surface without sliding. If the string does not slide from bobbin, find the ratio of the acceleration of the block and bobbin.



Sol. As the motion of the bobbin is pure rolling, point P should be instantaneous centre of rotation (ICR) of the bobbin. Hence, we can consider pure rotation of the bobbin about P . The acceleration of point C ; $a_C = \alpha R$.



But the bobbin is moving with acceleration A ,

$$\text{Hence } A = \alpha R \quad \dots(i)$$

The acceleration of point Q ; $a_Q = \alpha(R-r)$

But point Q is directly connected with block, it means the magnitude of acceleration of the point Q should be equal to the acceleration of the block

$$a_Q = a = \alpha(R-r) \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

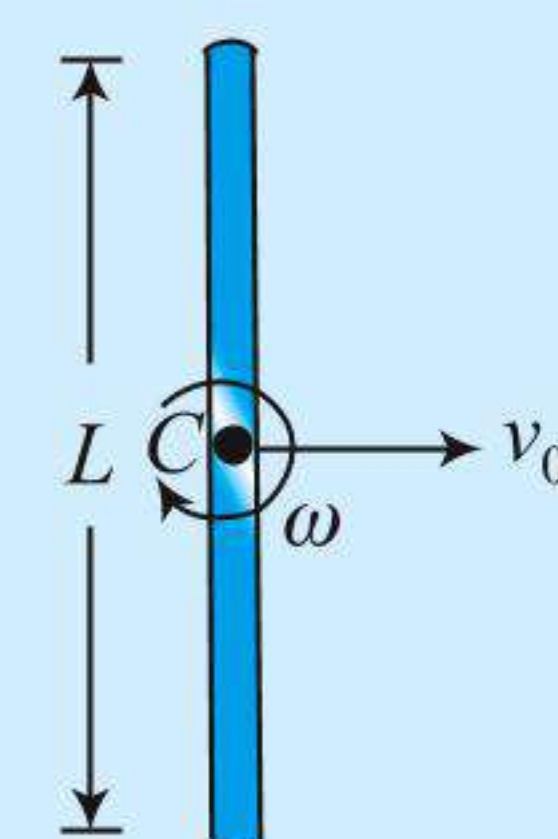
$$\frac{a}{A} = \frac{R-r}{R} = \left(1 - \frac{r}{R}\right)$$

CONSTRAINTS ON A ROD HAVING COMBINED TRANSLATIONAL AND ROTATIONAL MOTION

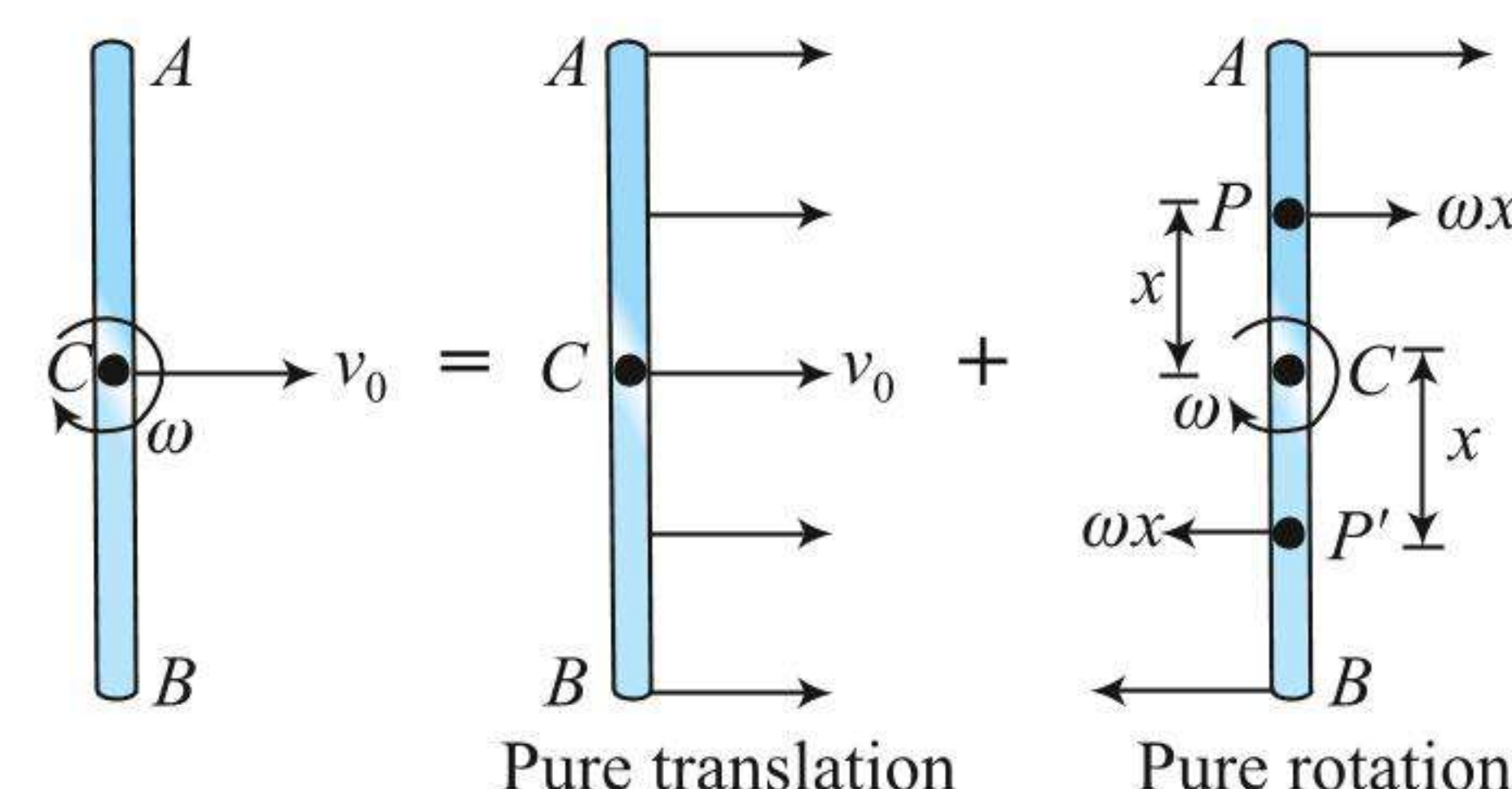
In many situations in rotational dynamics we need to analyze the situations where a rod moves in combined translational and rotational motion. We need to relate the velocity and acceleration of a particle on the rod with the velocity and acceleration of centre of mass of the rod. Let us learn this through following illustrations.

ILLUSTRATION 3.11

A rod of length L is placed on a smooth horizontal surface. An impulse is given to rod, as a result the rod translates and rotates about its centre of mass as shown in figure. Find the point on the rod which has zero velocity.



Sol. The motion of the rod can be considered as superposition of pure translation and pure rotation.



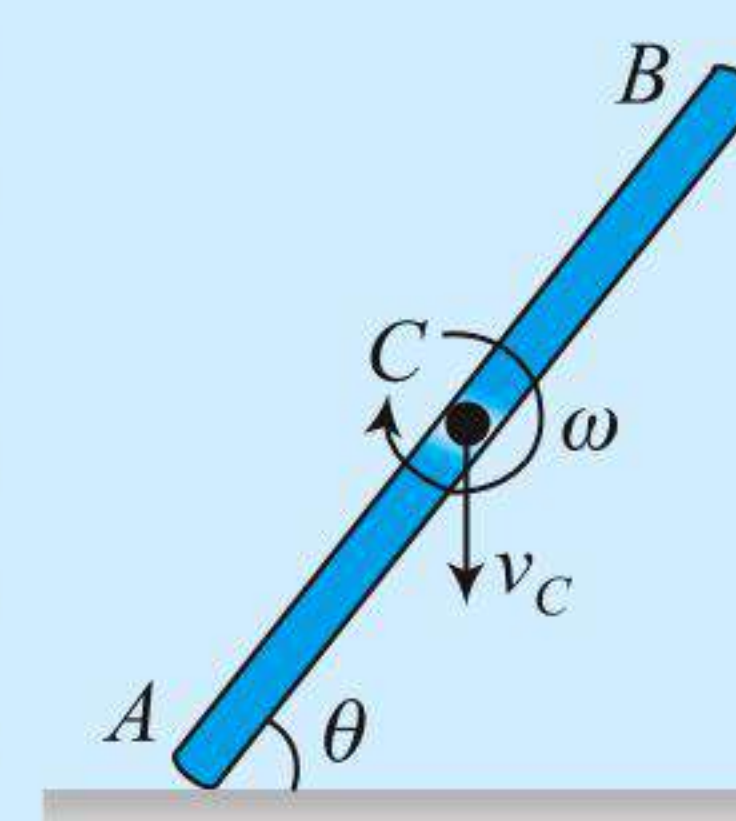
From figure, the point of zero velocity on the rod should be between C and B . Let point P' is the point of zero velocity.

$$\vec{v}_{P'} = \vec{v}_{P',C} + \vec{v}_C$$

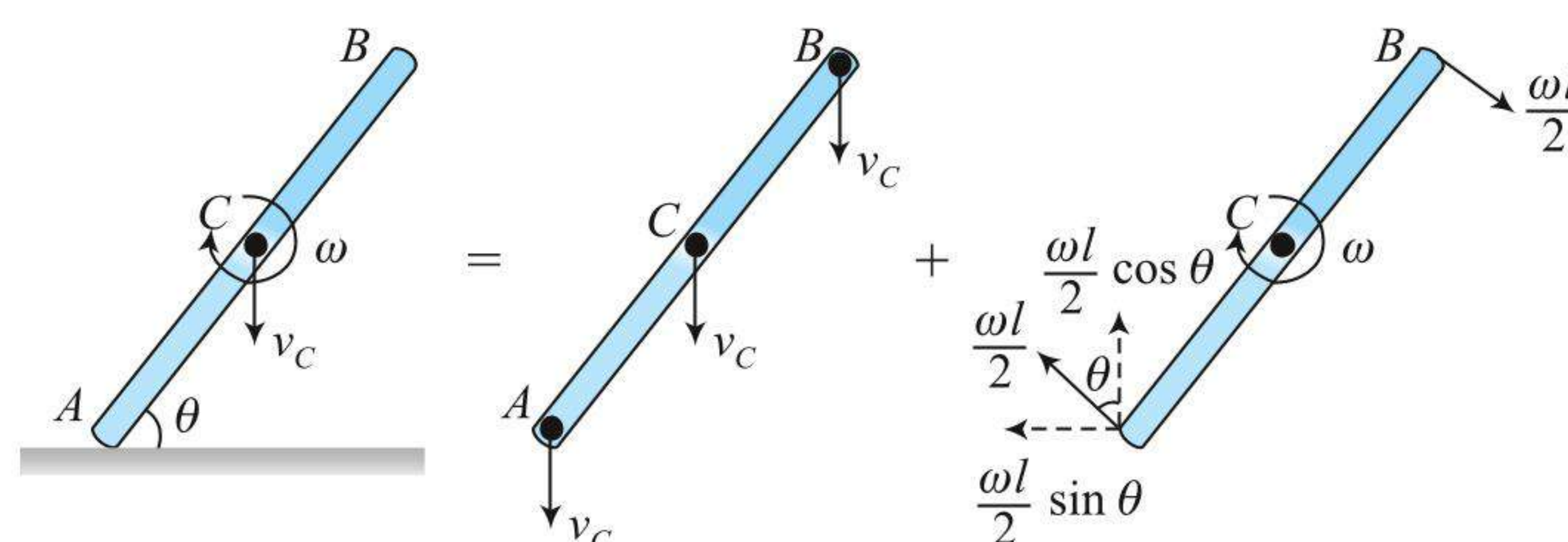
$$0 = -\omega x + v_0 \Rightarrow x = \frac{v_0}{\omega}$$

ILLUSTRATION 3.12

A rod AB of length L is supported against smooth horizontal surface. The rod is inclined at angle θ with horizontal. Now it is released so that end ' A ' starts sliding on horizontal surface. Make a relation between velocity of centre of mass of the rod and its angular velocity.



Sol. The motion of the rod can be considered as superposition of pure translation and pure rotation.

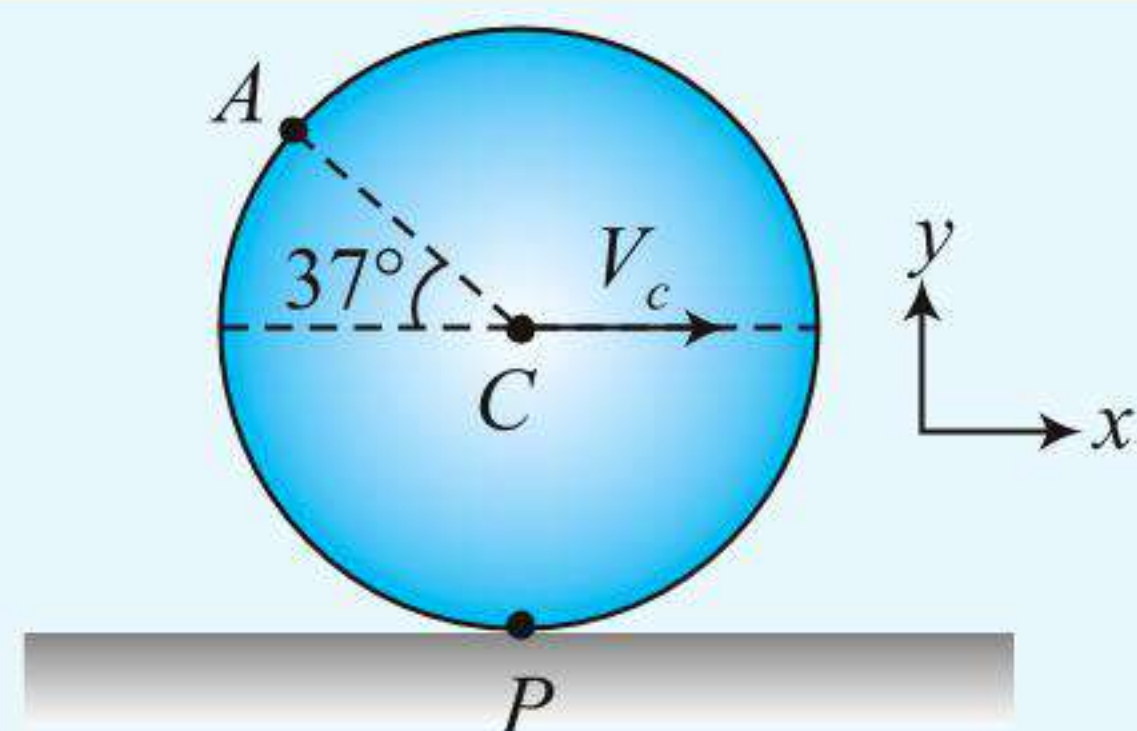


As end A , the rod slide along horizontal surface, the component of its velocity normal to ground should be zero.

$$\text{i.e., } -v_c + \frac{\omega l}{2} \cos \theta = 0 \text{ or } v_c = \frac{\omega l}{2} \cos \theta$$

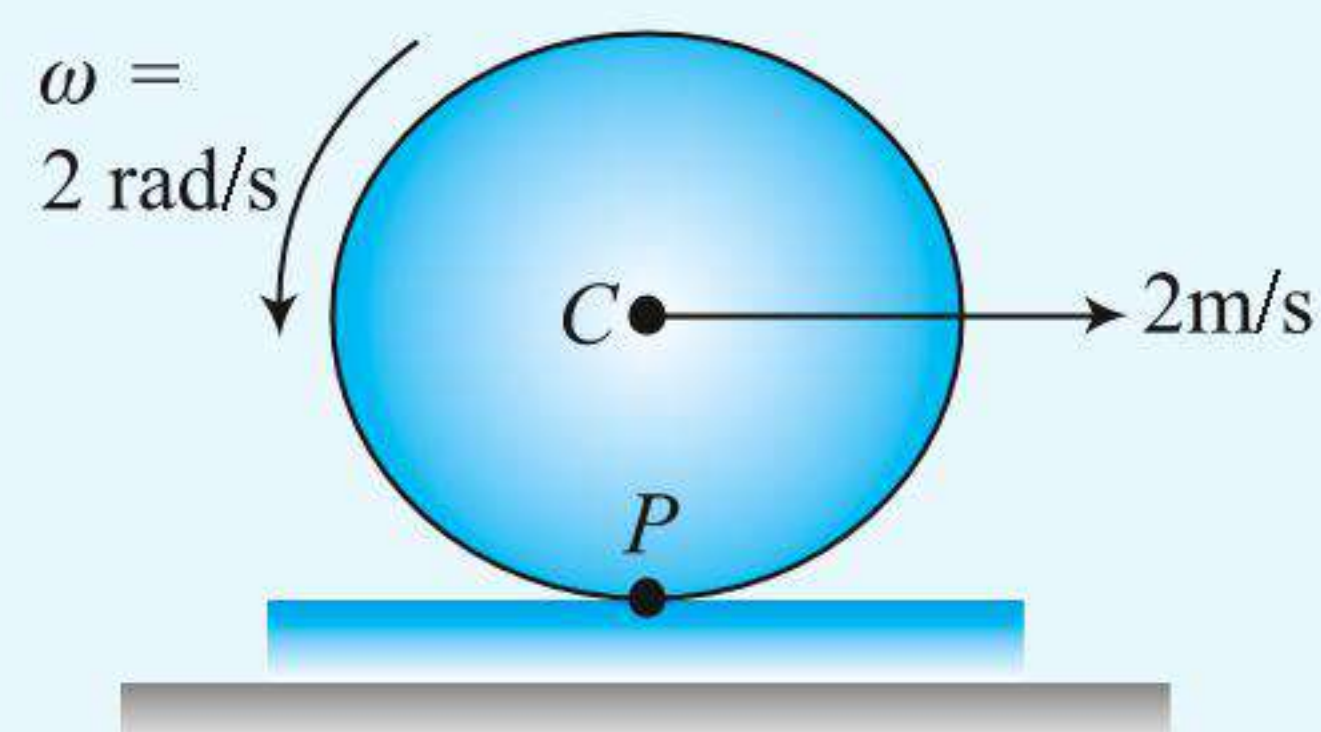
CONCEPT APPLICATION EXERCISE 3.2

1. A cylinder of radius 5 m rolls on a horizontal surface. If the velocity of its center is 25 m/s. Find its angular velocity and velocity of the point A .

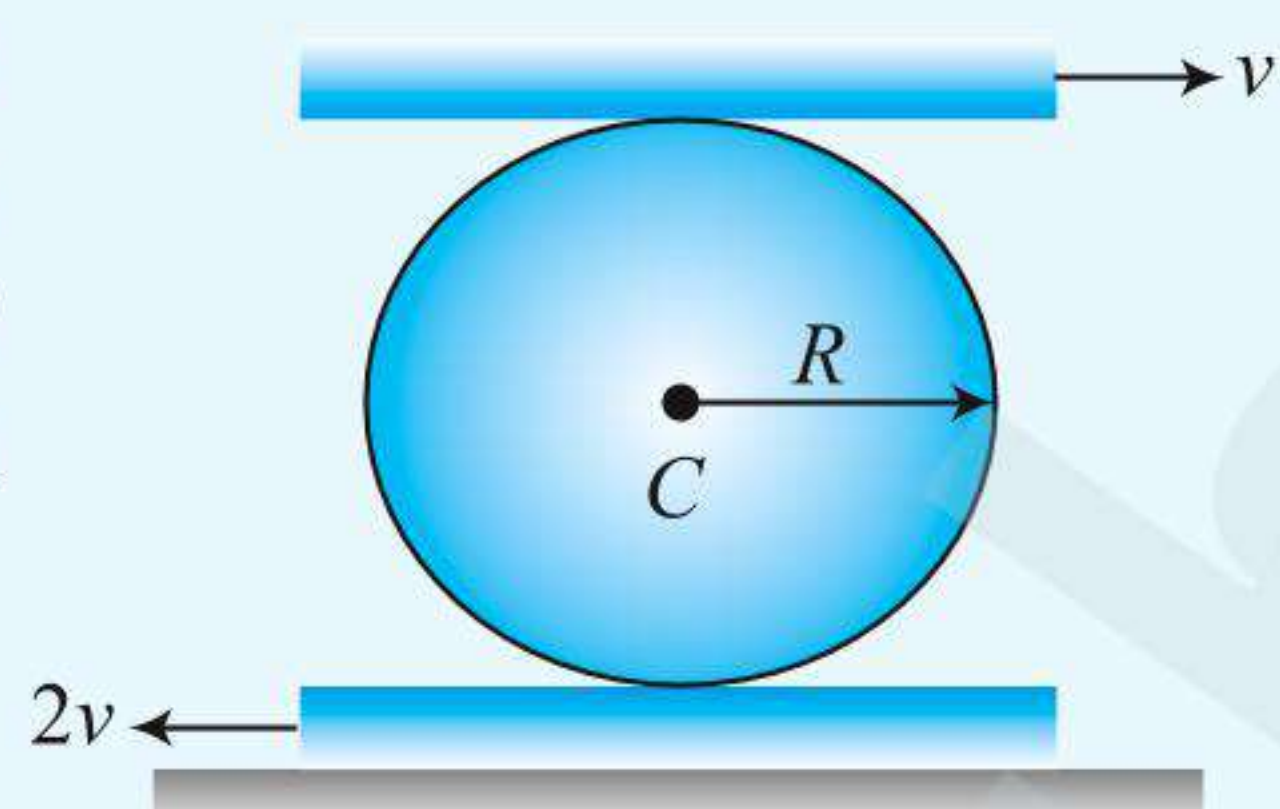


2. A disc of radius $R = \frac{1}{2}$ m

is rolling on a plank. If the disc is moving on the plank with angular velocity $\omega = 2$ rad/s and speed $v_c = 2$ m/s. Find the speed of the plank for pure rolling.



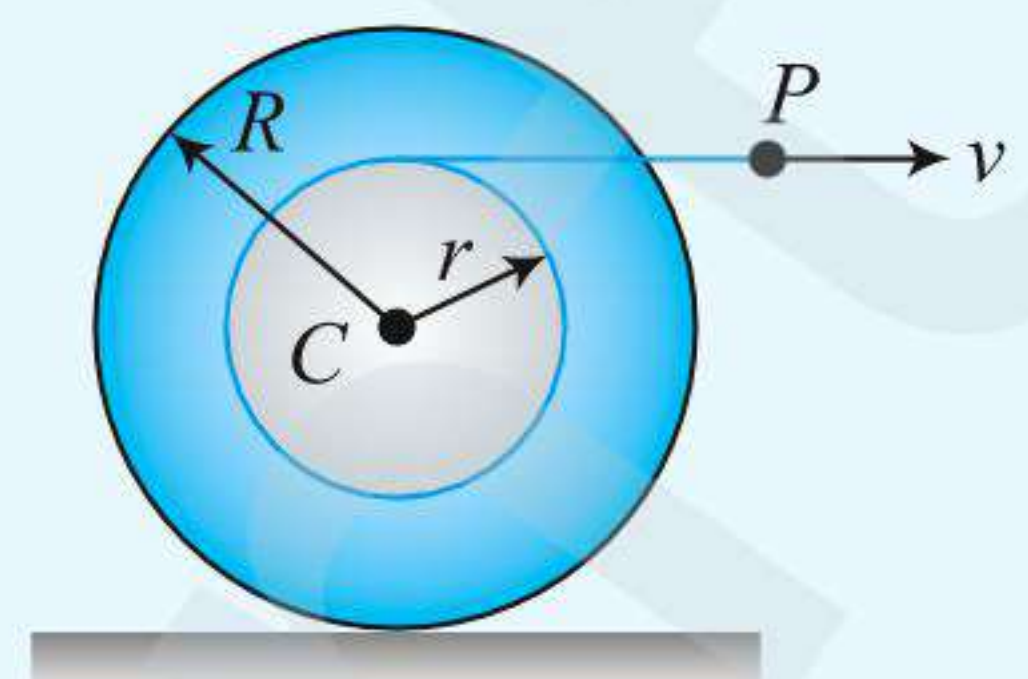
3. A uniform disc of radius R rolls perfectly over two horizontal plank A and B moving with velocities v and $2v$, respectively. Find the



- (a) velocity of CM of the disc.

- (b) angular velocity of the disc.

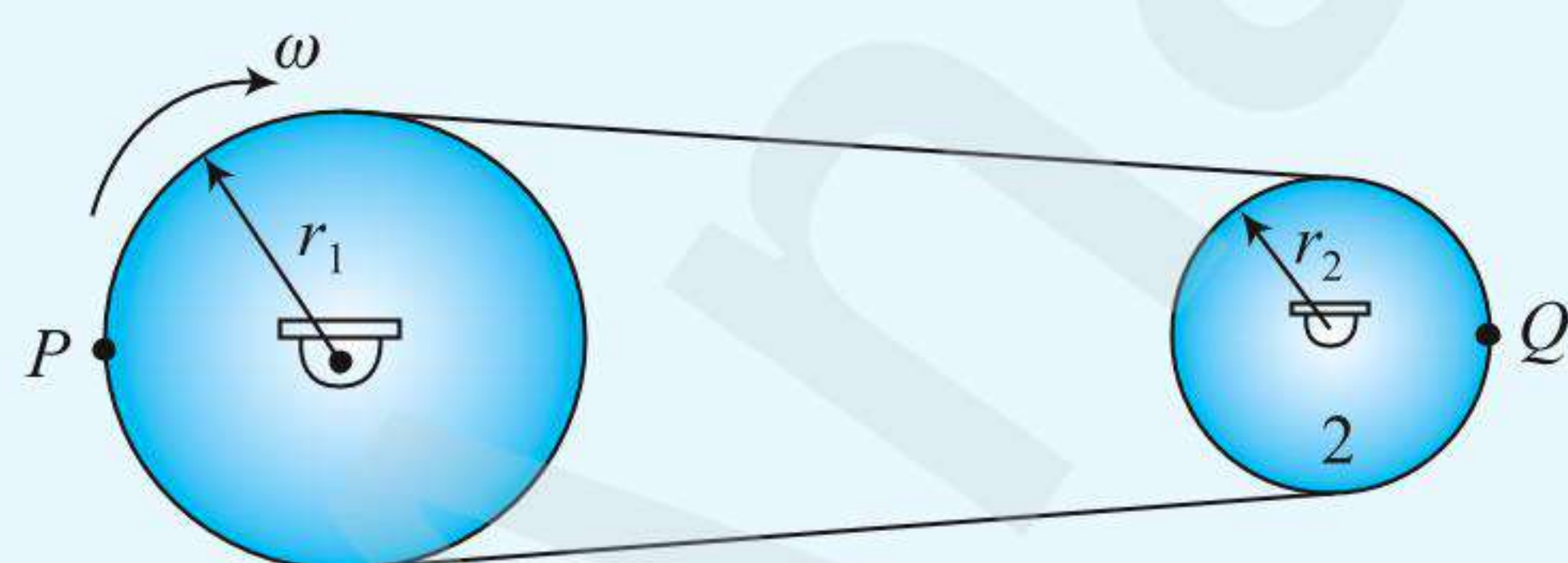
4. A cotton reel rolls without sliding such that the point P of the string has velocity v . Find the:



- (i) velocity of its centre C

- (ii) angular velocity of the cotton reel.

5. Two wheels 1 and 2 of radii r_1 and r_2 are connected by a belt. If the belt does not slide over the wheels, find the:

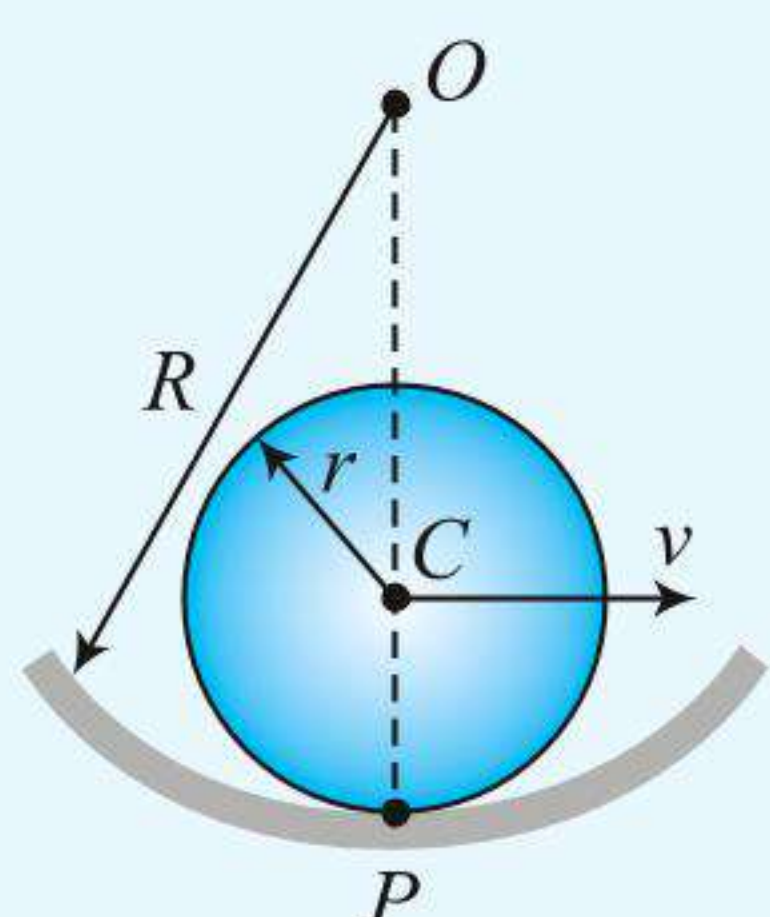


- (i) speed of a point on the belt.

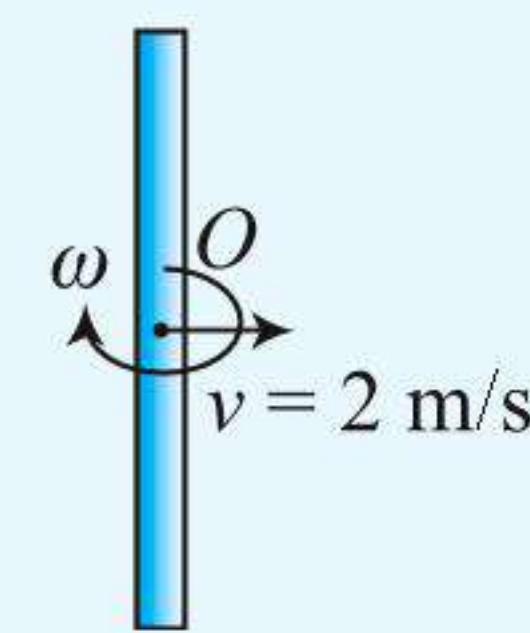
- (ii) angular velocity of the wheel 2.

- (iii) acceleration of P relative to Q .

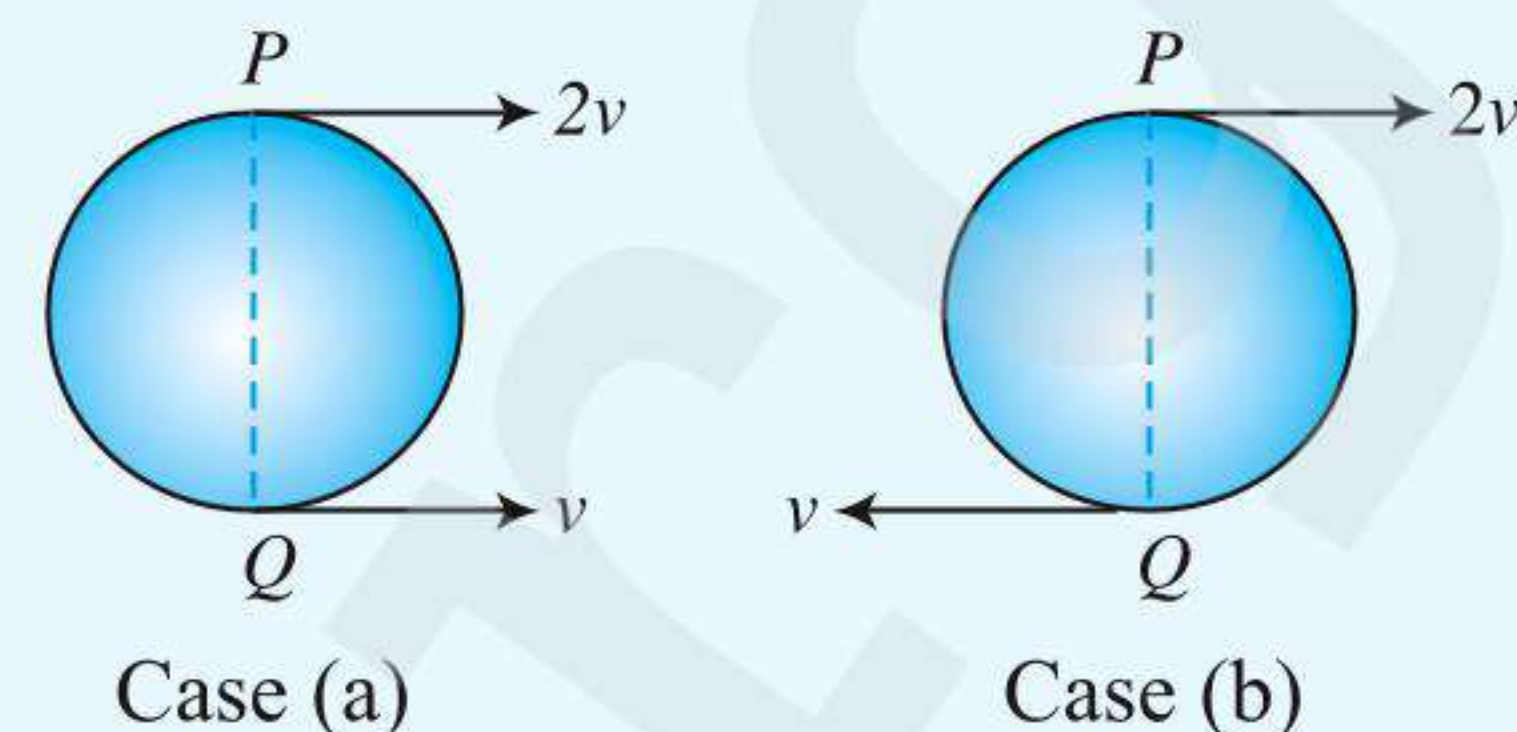
6. A cylinder of radius r rolls perfectly with a speed v on a concave surface of radius R . Find the acceleration of the bottom of the rolling body.



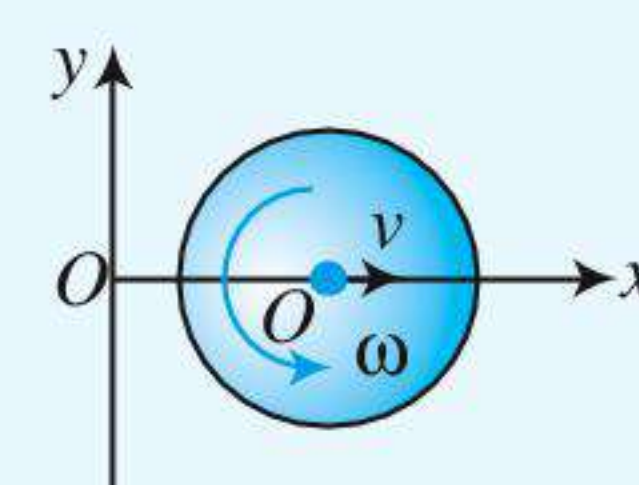
7. Shown in the figure is a rod which moves with $v = 2$ m/s⁻¹ and rotates with $\omega = 2\pi$ rad/s⁻¹. Find the instantaneous axis of rotation.



8. Find the position of instantaneous centre of rotation and angular velocity of the disc in the following cases as shown. Radius of disc is R in each case.



9. A rotating disc moves in the positive direction of x -axis as shown. Find the equation $y(x)$ describing the position of the instantaneous axis of rotation if at the initial moment the centre C of the disc was located at origin after which



- (a) it moved with constant acceleration ' a ' (initial velocity zero) while the disc rotating anticlockwise with constant angular velocity ω .
(b) it moved with constant velocity v while the disc started rotating anticlockwise with a constant angular acceleration α (with initial angular velocity zero).

ANSWERS

1. 5 rad/s, $(40\hat{i} + 20\hat{j})$ m/s 2. 3 m/s 3. (a) $v/2$ (b) $3v/2R$

4. $vR/(R+r)$ 5. (i) ωr_1 (ii) $\frac{\omega r_1}{r_2}$ (iii) $\omega r_1 \left(1 + \frac{r_1}{r_2}\right)$

6. $v^2 \left[\frac{1}{r} + \frac{1}{R-r} \right] \hat{j}$ 9. (a) $y^2 = \left(\frac{2a}{\omega^2} \right) x$ (b) $xy = \text{constant}$

MOMENT OF INERTIA

According to Newton's first law of motion, no body, by itself, can change its state of rest or of uniform motion in a straight line. This inertness or inability of the body is called inertia. Obviously, an external force is required to change the state of rest or of uniform motion in a straight line of the body. Larger the force required to change the state of the body, greater is its inertia. Therefore, to produce or to destroy a given acceleration, force required is directly proportional to the mass of the body. Clearly, heavier the body, greater is the force required to change its state and hence greater is its inertia. The reverse is also true. Mass of the body is, therefore, a measure of its inertia in translatory motion.

Similarly, a body rotating about an axis is unable to produce a change in its rotational motion by itself and this inertness in case of rotational motion is known as moment of inertia. It is the resistance to change in rotational motion.

The moment of inertia of a body about a given axis plays the same role in rotational motion about that axis as the mass of a body does in translational motion. However, the two are not the same whereas the inertia of a body depends only upon its mass, the moment of inertia of a body about a given axis depends upon:

- (i) its mass
- (ii) position and direction of the axis of rotation.
It accounts the fact that two bodies of the same mass and shape but rotating about different axes of rotation, will have different moments of inertia.
- (iii) shape of the body (i.e., distribution of the mass of the body about the said axis).

It is due to the reason that two bodies having same mass, but different shapes will have different moments of inertia about a given axis of rotation.

Quantitatively, moment of inertia of a body about a given axis is equal to the sum of the products of the masses of the constituent particles and squares of their respective perpendicular distances from the axis of rotation.

If $m_1, m_2, m_3, \dots$ are the masses of the particles distant $r_1, r_2, r_3, \dots$ respectively from the axis of rotation, then moment of inertia of the body,

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots$$

$$\text{i.e., } I = \Sigma m r^2$$

where Σ denotes the summation of the said products for all the particles constituting the body.

The unit of moment of inertia is g cm^2 in cgs system and kg m^2 in SI.

As moment of inertia = mass $\times$ (distance)², the dimensional formula for moment of inertia is $[ML^2]$. The moment of inertia is scalar quantity.

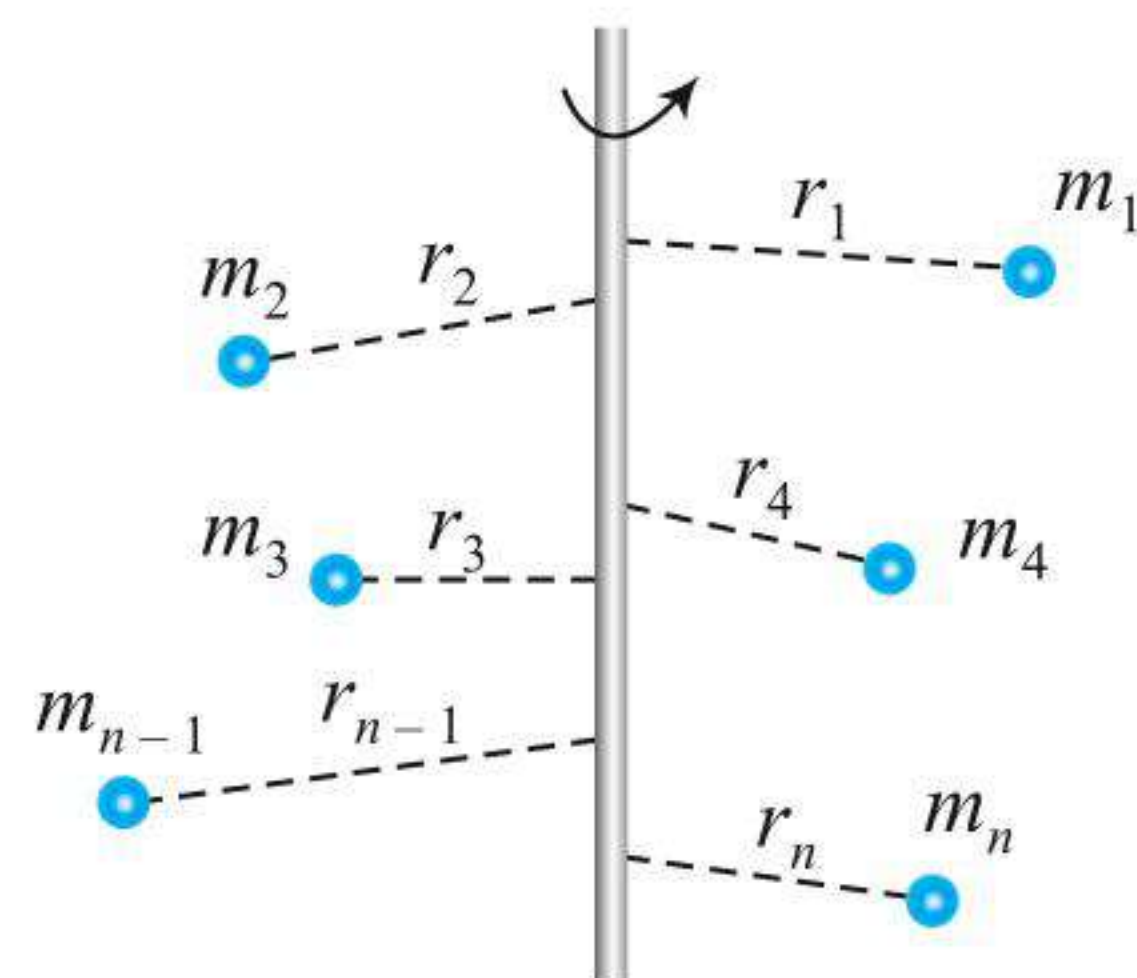
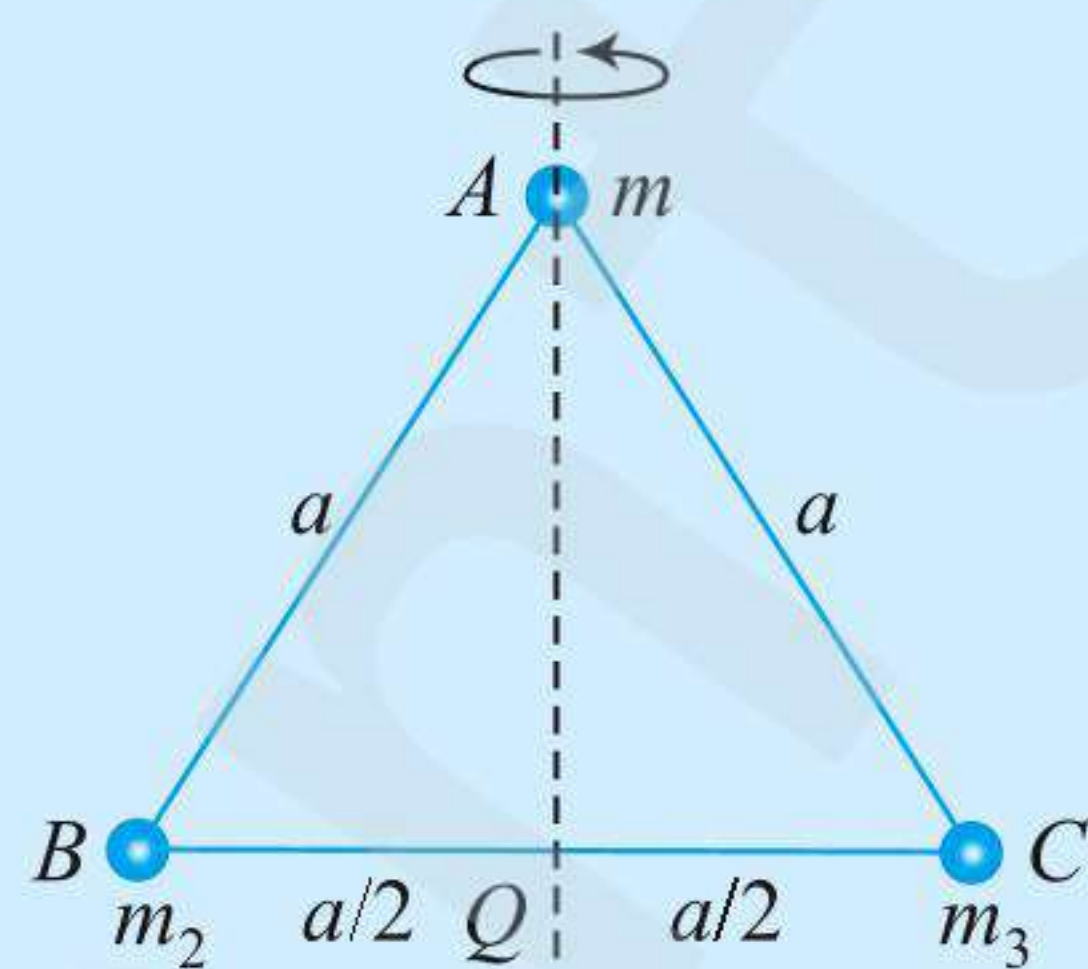


ILLUSTRATION 3.13

Three mass points m_1, m_2, m_3 are located at the vertices of an equilateral triangle of length a as shown in figure. What is the moment of inertia of the system about an axis along the altitude of the triangle passing through m_1 ?



Sol. Let the mass points m_1, m_2 and m_3 be situated at vertices A, B and C of the equilateral triangle of each side a . AQ is altitude of the triangle passing through mass m_1 . Then, moment of inertia of the system about the altitude AQ ,

$$I = m_1 \times (\text{distance of } m_1 \text{ from } AQ)^2 + m_2 \times (\text{distance of } m_2 \text{ from } AQ)^2 + m_3 \times (\text{distance of } m_3 \text{ from } AQ)^2$$

$$I = m_1 (0)^2 + m_2 \left(\frac{a}{2}\right)^2 + m_3 \left(\frac{a}{2}\right)^2 = \frac{1}{4}(m_2 + m_3)a^2$$

MOMENT OF INERTIA OF A RIGID BODY

The moment of inertia (MI) of a symmetrical body rotating about its axis of symmetry can be found by finding the moment of inertia of a small element of it about the axis of rotation and then integrating it within the proper limits for whole of the body.

Consider that a small elementary portion of mass dm of the rigid body rotates about the axis of rotation at a distance r from it. Then, moment of inertia of the small elementary portion about the axis of rotation,

$$dI = dm r^2 \quad \dots(i)$$

The moment of inertia of the whole rigid body about the axis of rotation,

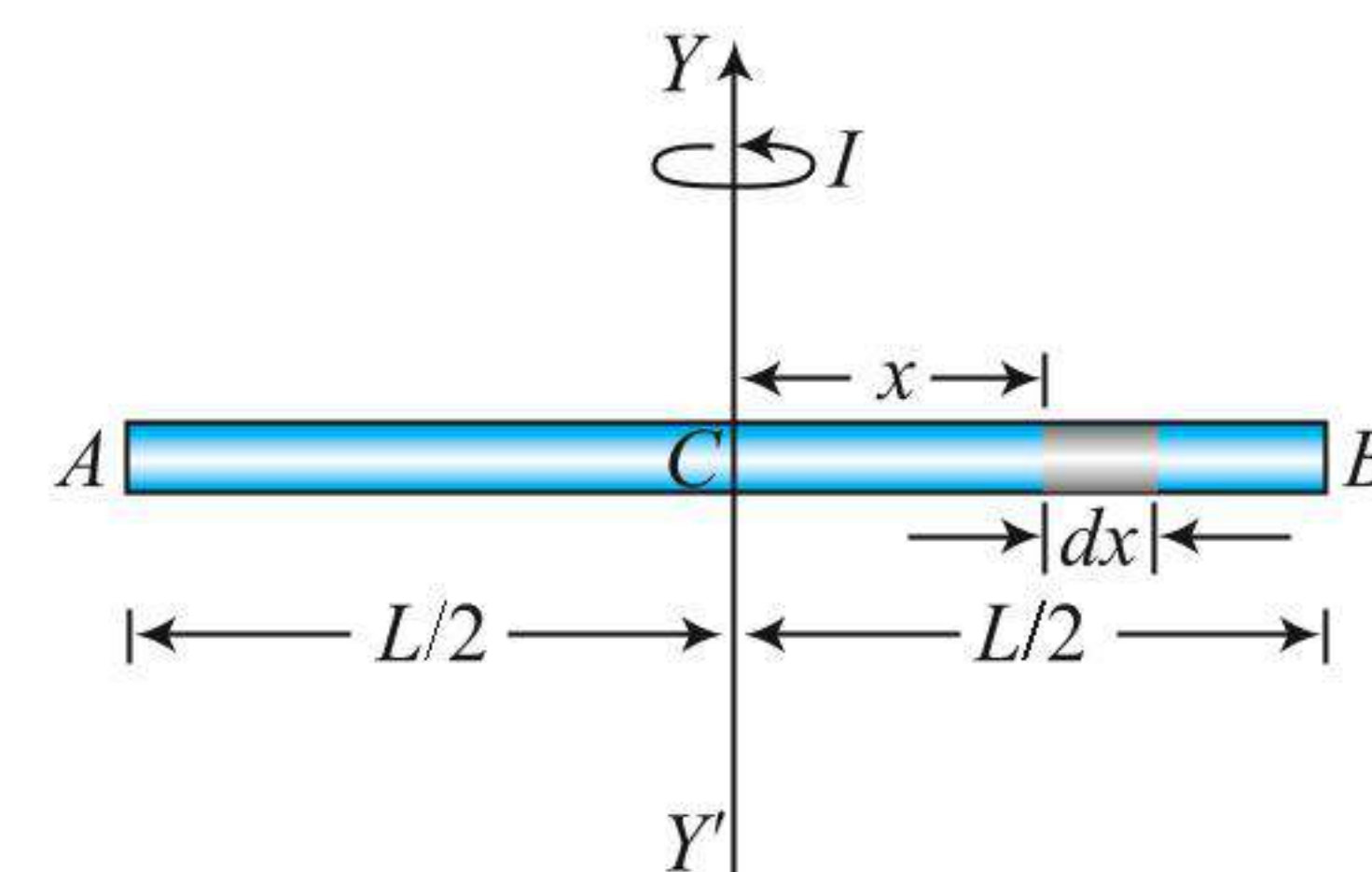
$$I = \int dI = \int dm r^2 \quad \dots(ii)$$

Note:

- MI is not constant for a body. It depends on the axis of rotation.
- MI depends on the mass of the body. The higher the mass, the higher is the MI.
- MI depends on the distribution of the mass about an axis. The farther the mass is distributed from the axis, the higher will be the MI.
- Moment of inertia does not change if the mass
 - (a) is shifted parallel to the axis of the rotation.
 - (b) is rotated with constant radius about axis of rotation clockwise or anticlockwise (i.e., the MI does not depend upon sense of rotation).

MOMENT OF INERTIA OF A STRAIGHT ROD ABOUT AN AXIS THROUGH ITS CENTRE

Consider a uniform straight rod AB of length L , mass M and centre C . Then, mass per unit length of the rod $= \frac{M}{L}$



Let the moment of inertia of the rod about the axis YY' passing through its centre C and perpendicular to its length be I . Consider a small element of length dx of the rod at a distance x from the point C .

The mass of elementary portion of the rod, $dm = \frac{M}{L} dx$

The moment of inertia of the elementary portion of the rod about the axis YY' ,

$$dI = \text{mass of the elementary portion} \times (\text{distance from the axis } YY')^2$$

$$\Rightarrow dI = \left(\frac{M}{L} dx\right) \times x^2 = \frac{M}{L} x^2 dx$$

The moment of inertia (I) of the rod AB about the axis YY' can be found by integrating the above for all such elementary portions between the points A and B i.e., between the limits $x = -L/2$ to $x = +L/2$.

$$\begin{aligned} I &= \int_{-L/2}^{+L/2} dI = \int_{-L/2}^{+L/2} \frac{M}{L} x^2 dx = \frac{M}{L} \int_{-L/2}^{+L/2} x^2 dx \\ &= \frac{M}{L} \left[\frac{x^3}{3} \right]_{-L/2}^{+L/2} = \frac{M}{3L} \left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3 \right] = \frac{M}{3L} \times \frac{2L^3}{8} \end{aligned}$$

$$\text{or } I = \frac{1}{12} ML^2$$

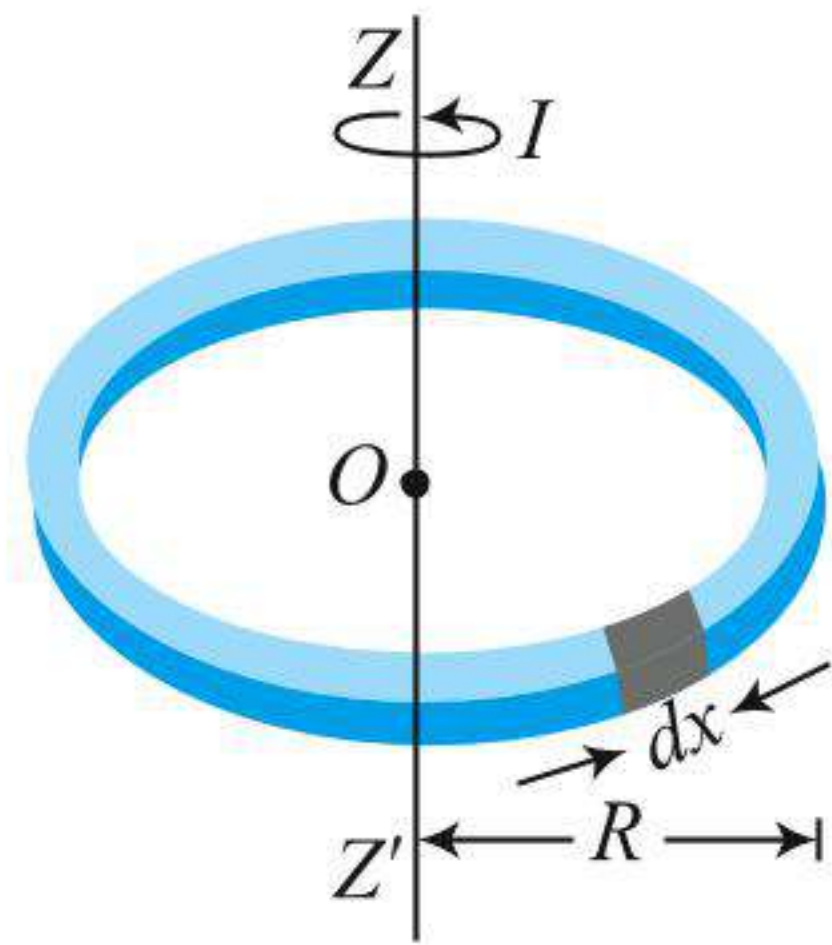
MOMENT OF INERTIA OF A CIRCULAR RING ABOUT AN AXIS PASSING THROUGH ITS CENTRE AND PERPENDICULAR TO ITS PLANE

Consider a circular ring of mass M , radius R and centre O . Let the moment of inertia of the circular ring about the axis ZZ' passing through its centre O and perpendicular to its plane be I (see figure). Then,

$$\text{Mass per unit length of the circular ring} = \frac{M}{2\pi R}$$

Consider an elementary portion of the ring of length dx .

$$\text{Then, mass of the elementary portion} = \frac{M}{2\pi R} dx$$



As the ring rotates about axis ZZ' , the elementary portion always remains at a distance R from it. Therefore, moment of inertia of elementary portion about ZZ' ,

$$dI = \left(\frac{M}{2\pi R} dx \right) R^2 = \frac{M}{2\pi} R dx$$

The moment of inertia I of the ring about axis ZZ' can be found by integrating the above for whole circumference of the ring i.e., between limits $x = 0$ to $x = 2\pi R$.

$$\therefore I = \int_0^{2\pi R} dI = \int_0^{2\pi R} \frac{M}{2\pi} R dx = \frac{M}{2\pi} R \int_0^{2\pi R} dx$$

$$\Rightarrow I = \frac{M}{2\pi} R [x]_0^{2\pi R} = \frac{M}{2\pi} R [2\pi R - 0] \text{ or } I = MR^2$$

Note: Answer will remain the same even if the mass is non-uniformly distributed because $\int dm = M$ always.

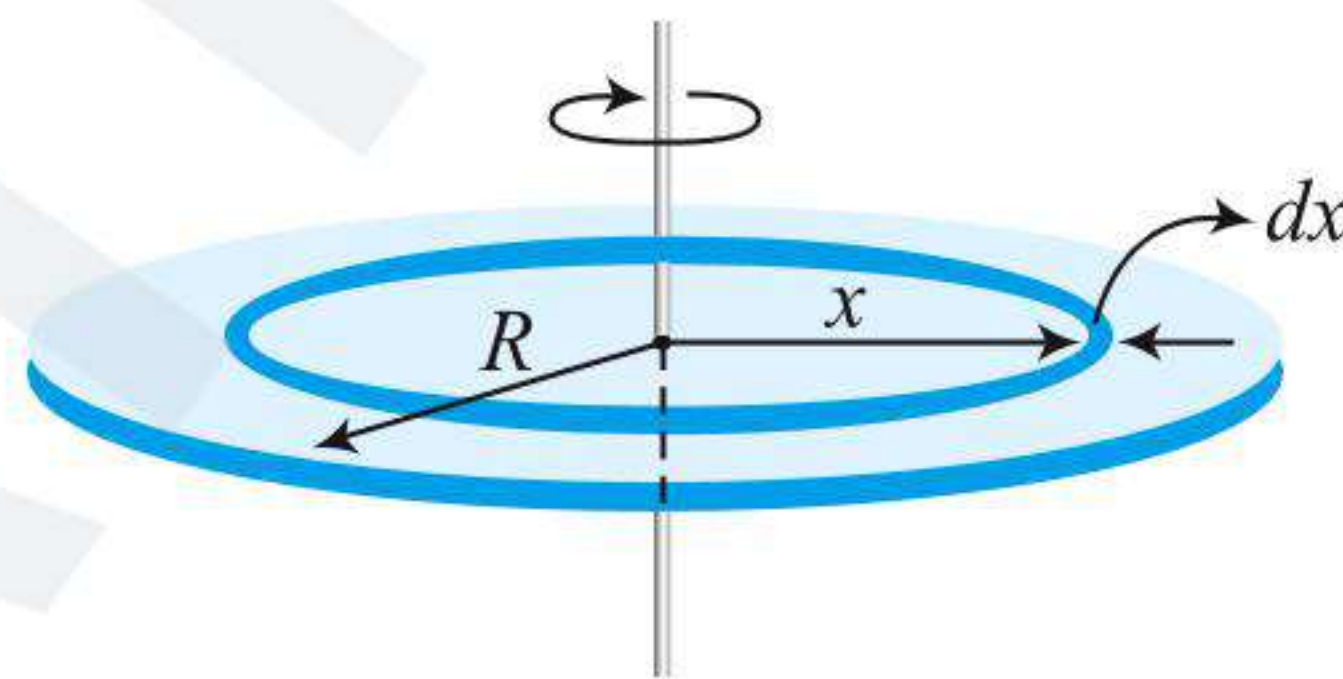
MOMENT OF INERTIA OF A DISC

Consider a disc of mass M and radius R , shown in figure. To find the moment of inertia, we consider an elemental ring of radius x and width dx . Its mass dm is given by

$$dm = \frac{M}{\pi R^2} (2\pi x dx)$$

Now the moment of inertia of this elemental ring is given as

$$dI = dm x^2 = \frac{2M}{R^2} x^3 dx$$



Moment of inertia of whole disc is given by integration of above relation from 0 to R .

$$I = \frac{2M}{R^2} \int_0^R x^3 dx = \frac{2M}{R^2} \left[\frac{x^4}{4} \right]_0^R$$

$$\text{Hence, } I = \frac{1}{2} MR^2$$

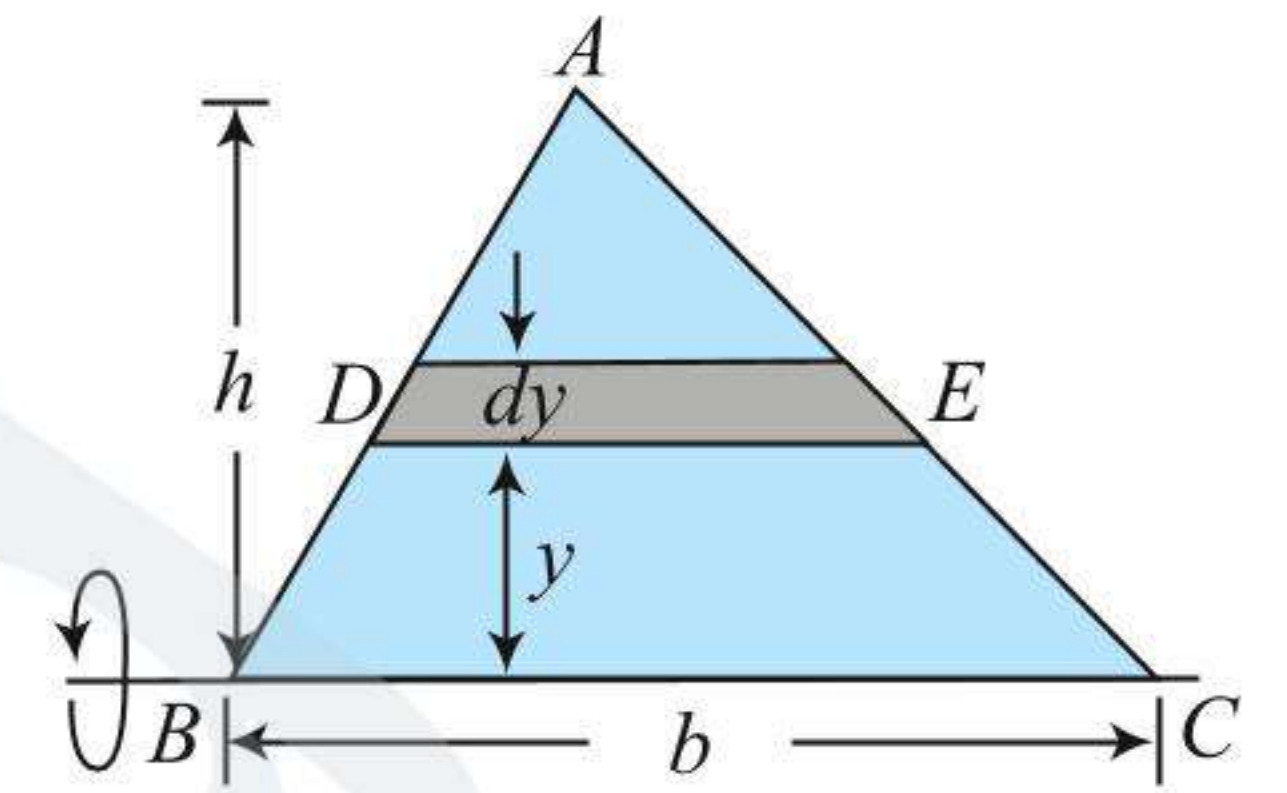
MI OF A TRIANGULAR LAMINA ABOUT ITS BASE

Consider a triangular lamina of base b , altitude h and mass M .

Surface mass density (mass per unit area of the lamina)

$$\sigma = \frac{M}{\left(\frac{1}{2} bh \right)} = \frac{2M}{bh}$$

Consider a rectangular differential strip parallel to the base of width dy , at a distance ' y ' from the axis of rotation (base).



From similar triangles $\triangle ADE$ and $\triangle ABC$, we have

$$\frac{DE}{BC} = \frac{h-y}{h} \Rightarrow DE = (h-y) \frac{b}{h}$$

$$\text{Area of strip } DE, dA = (h-y) \frac{b}{h} dy$$

MI of the strip DE about the axis of rotation BC is given by

$$dI_{BC} = (\text{Mass of strip } DE) \times (\text{distance from axis})^2 \\ = \left(\frac{2M}{bh} (h-y) \frac{b}{h} dy \right) y^2 = \frac{2M}{h^2} y^2 (h-y) dy$$

Integrating the above expression between the limits from $y = 0$ to $y = h$.

$$I_{BC} = \int_0^h \frac{2M}{h^2} (hy^2 - y^3) dy = \frac{2M}{h^2} \left[\left(\frac{hy^3}{3} \right) - \left(\frac{y^4}{4} \right) \right]_0^h$$

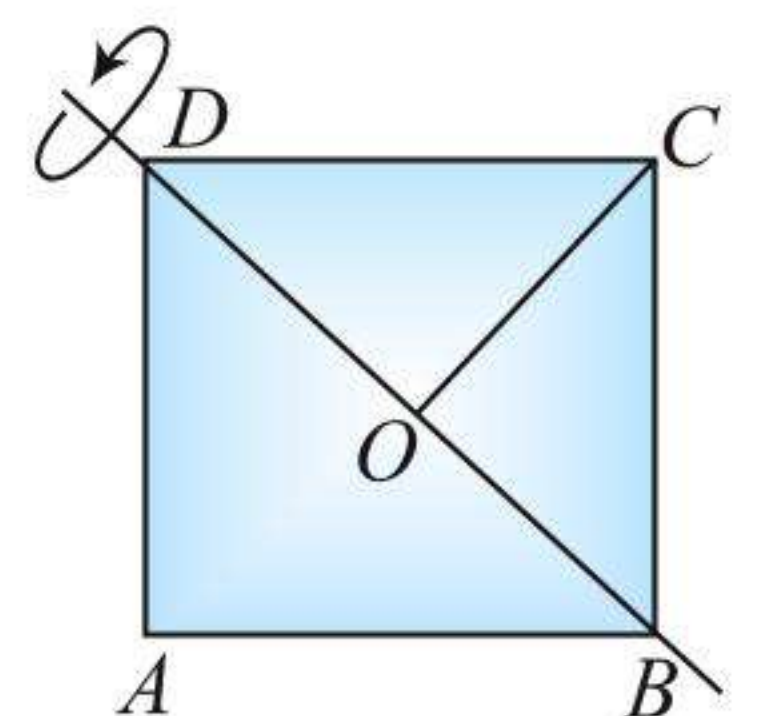
$$\Rightarrow I_{BC} = \frac{Mh^2}{6}$$

MI OF A SQUARE LAMINA ABOUT ITS DIAGONAL

Let the mass of the lamina be M and each of its edge has length l . The diagonal divides the square lamina into two triangular laminas each of equal mass $M/2$.

Consider the triangular lamina BCD , with base BD and altitude OC .

$$OC = \frac{1}{2} (\text{diagonal}) = \frac{1}{2} \left(\frac{l}{\sqrt{2}} \right) = \frac{l}{\sqrt{2}}$$



MI of the lamina BCD about BD is given by

$$\frac{1}{6} \left(\frac{M}{2} \right) \left(\frac{l}{\sqrt{2}} \right)^2 = \frac{Ml^2}{24}$$

MI of the square lamina about the diagonal BD algebraic sum of MI of laminas ABD and BCD each about their respective bases BD . Thus,

$$I_{\text{total}} = 2 \left(\frac{Ml^2}{24} \right) = \frac{Ml^2}{12}$$

ILLUSTRATION 3.14

A uniform rod of mass m and length l_0 is rotating with a constant angular speed ω about a vertical axis passing through its point of suspension. Find the moment of inertia of the rod about the axis of rotation if it makes an angle θ to the vertical (axis of rotation).

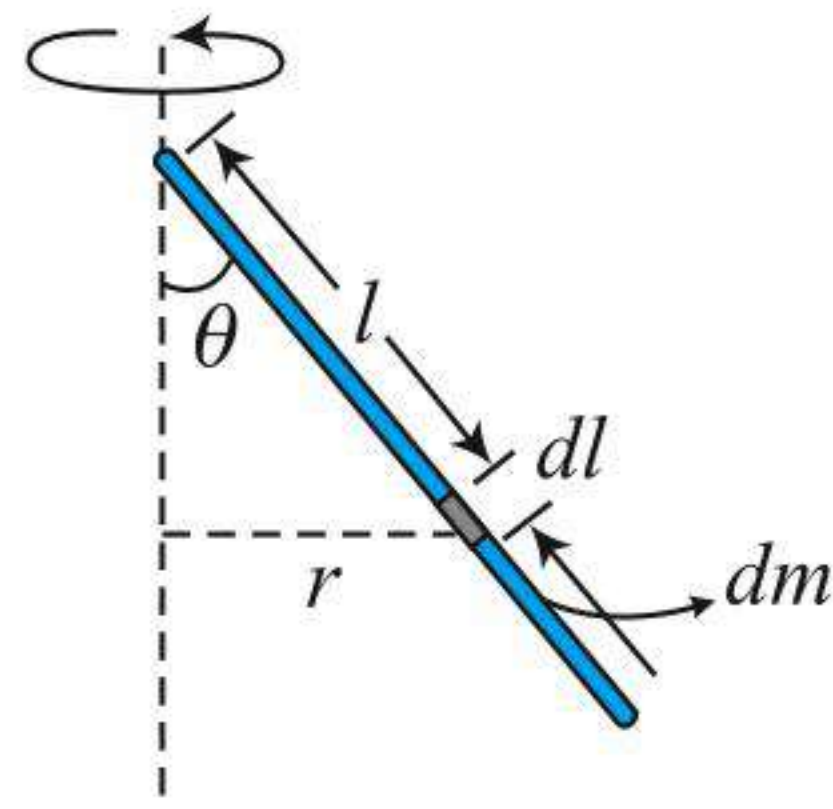
Sol. We can observe that every element of rod is rotating with different radius about the axis of rotation

Take an elementary mass dm of the rod.

$$dm = \frac{m}{l_0} dl$$

The moment of inertia of the elementary mass is given as $dI = (dm)r^2$

The moment of inertia of the rod $= I = \int dI \Rightarrow I = \int r^2 dm$

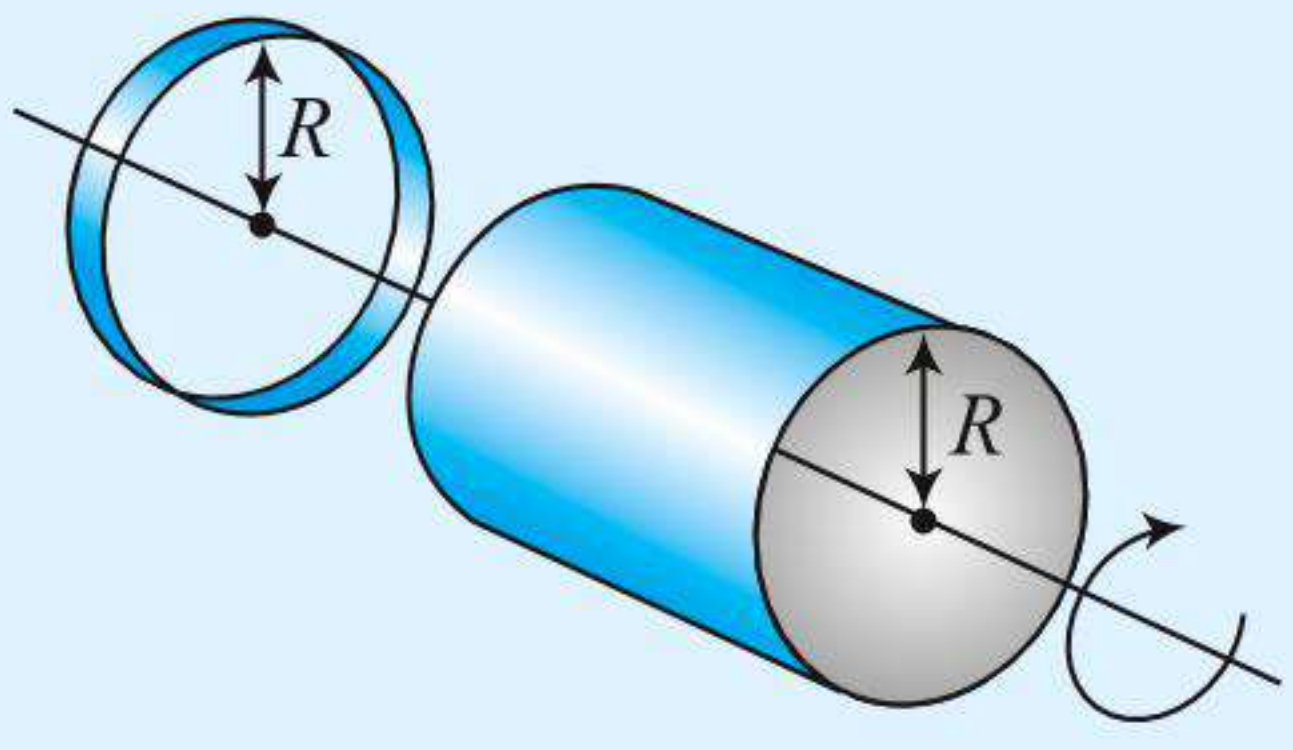
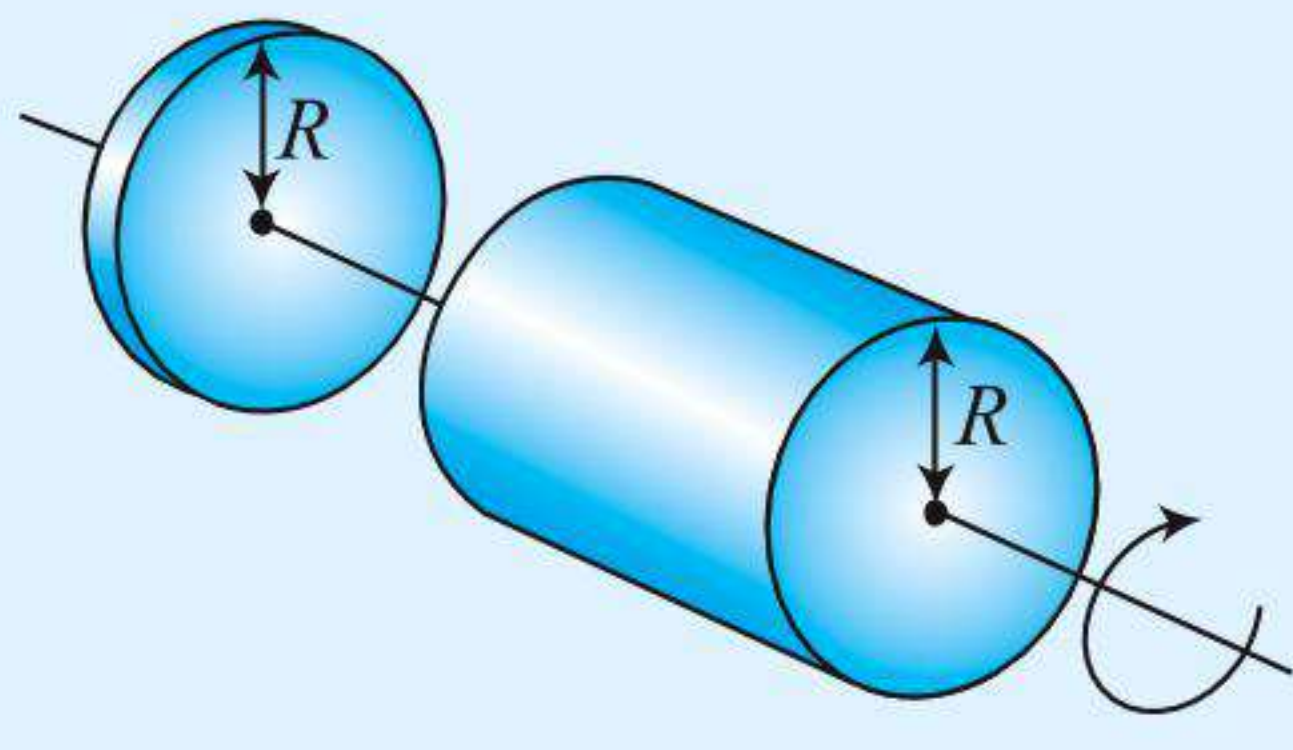
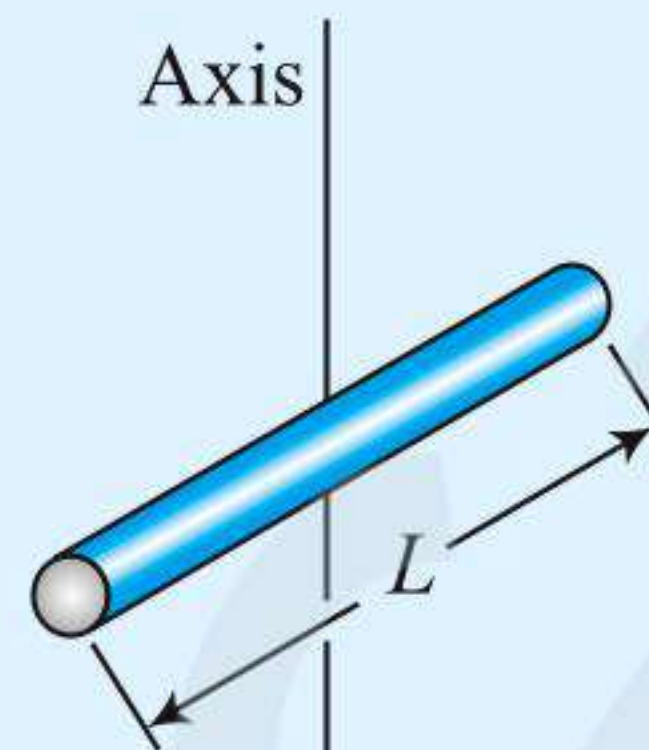
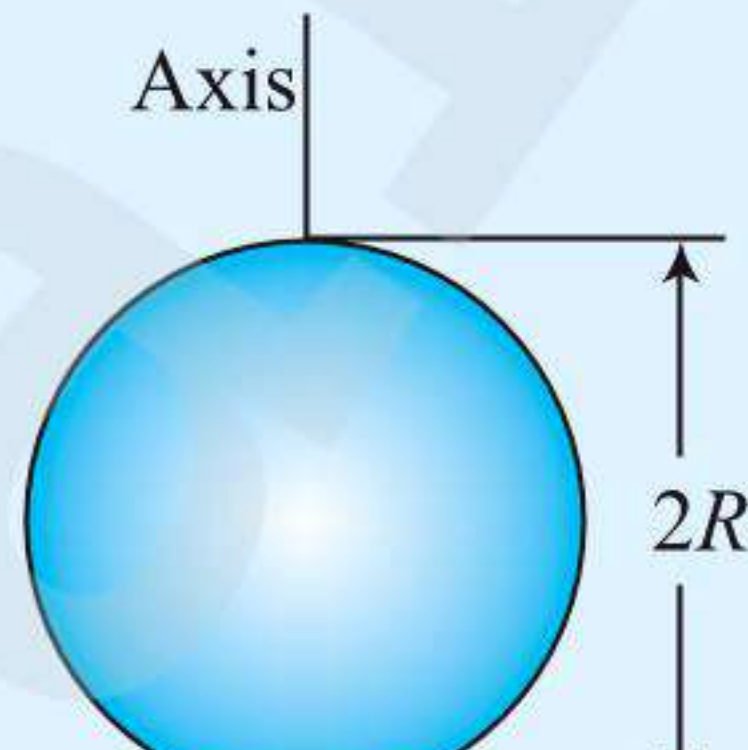
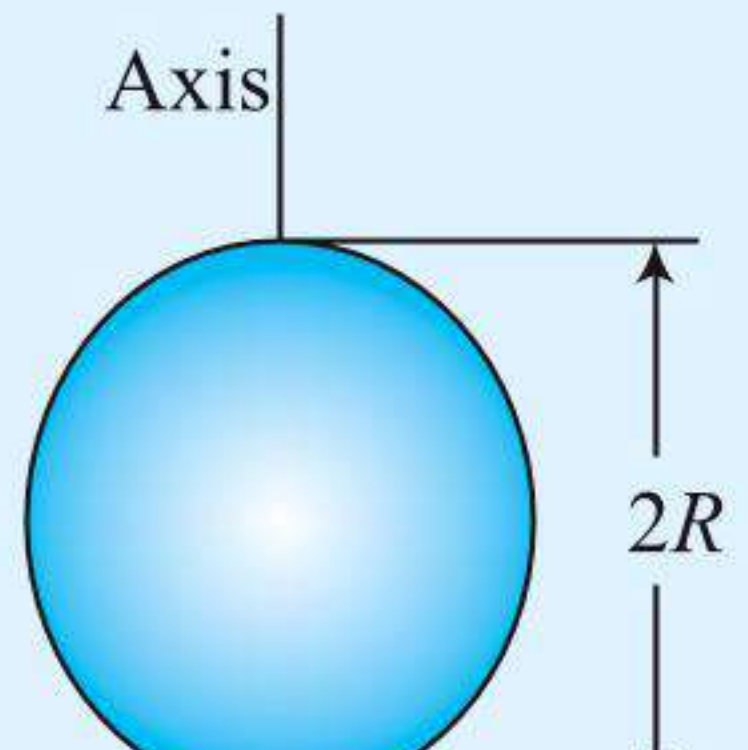


Substituting $r = l \sin \theta$ and $dm = \frac{m}{l_0} dl$ we obtain

$$I = \int (l^2 \sin^2 \theta) \frac{m}{l_0} dl = \frac{m \sin^2 \theta}{l_0} \int_0^{l_0} l^2 dl = \frac{ml_0^3}{3l_0} \sin^2 \theta$$

$$\Rightarrow I = \frac{ml_0^2 \sin^2 \theta}{3}$$

Some important results of moment of inertia

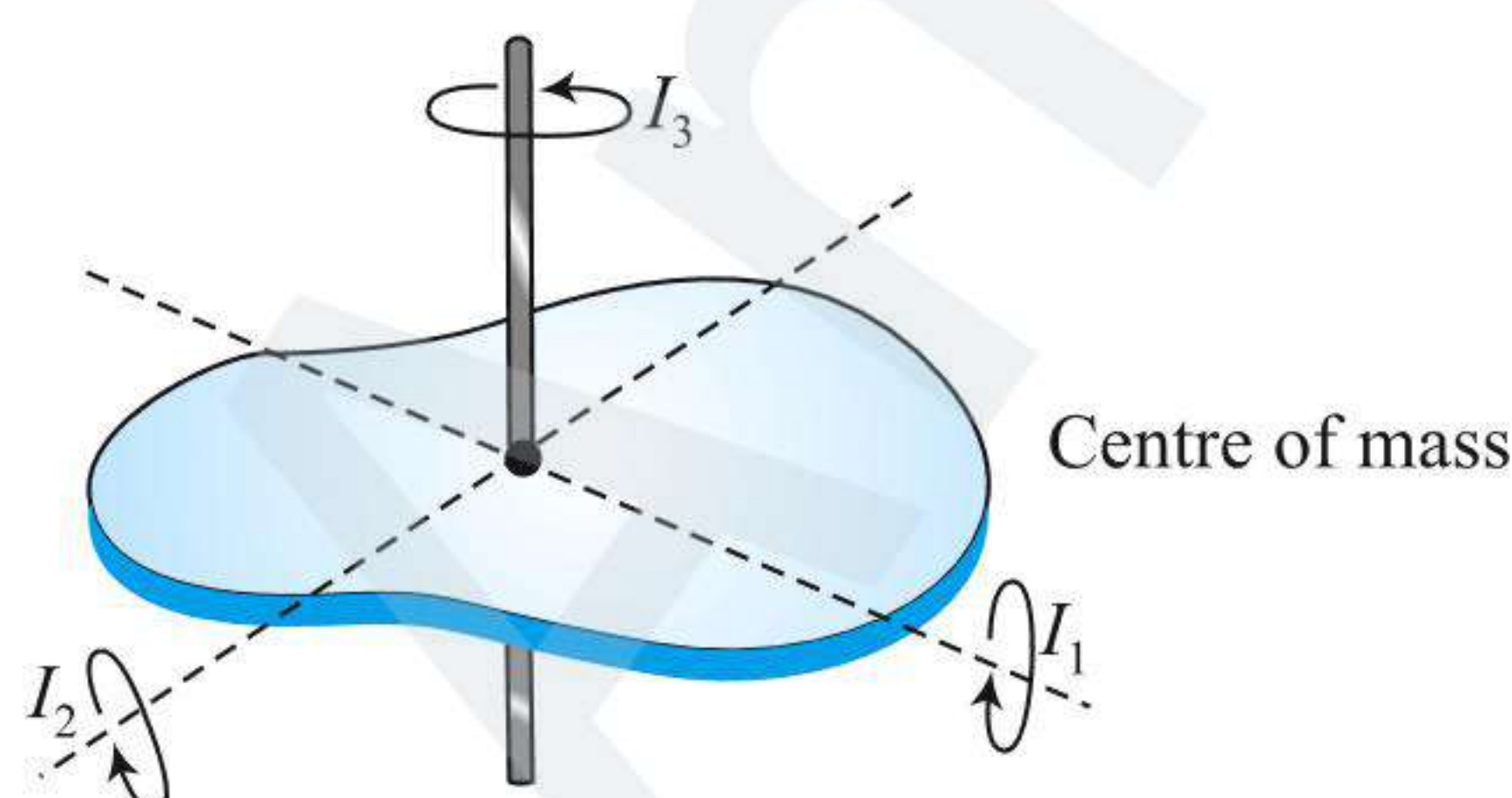
Hoop and annular cylinder	Disc and solid cylinder	Thin rod	Solid sphere	Thin spherical shell
				
$I = MR^2$	$I = \frac{MR^2}{2}$	$I = \frac{1}{12} ML^2$	$I = \frac{2}{5} MR^2$	$I = \frac{2}{3} MR^2$

MOMENT OF INERTIA ABOUT A GENERAL AXIS OF ROTATION

Up to this point we have learnt to calculate the moment of inertia of different objects using the method of integration. In most of the cases we have evaluated the moment of inertia about the axis of symmetry of the objects, generally passing through the center of mass. For evaluation of moment of inertia about any other axis of rotation of body, we use two theorems. Through these two theorems we can easily calculate the moment of inertia about any randomly selected axis of rotation of a given body.

PERPENDICULAR AXIS THEOREM

This theorem is valid only for two dimensional objects or lamina (thin sheet kind of) object, which are rotating about an axis passing through their centre of mass. Let us consider a two-dimensional object as shown in figure.



If the body is rotated about x -axis, lying in the plane on body and passing through its centre of mass, the moment of inertia about this axis be I_1 (say). If the body is rotated about y -axis, which is also in its plane and passing through its centre of mass, moment of inertia about this axis be I_2 (say). Now, if the body is rotated about a third axis, which is perpendicular to both previous axes and perpendicular to the plane of the body, its moment of inertia I_3 is given by the sum of I_1 and I_2 .

$$I_3 = I_1 + I_2 \quad \dots(i)$$

Thus, the sum of moments of inertia of a lamina object about two mutually perpendicular axes lying in the plane of lamina is equal to the moment of inertia about an axis normal to the plane of the lamina and passing through the two perpendicular axes. This equation is known as the **perpendicular axis theorem**.

We shall now prove it.

We consider an elemental mass dm in the body at a distance r from the main axis, which is perpendicular to the plane of body, as shown in figure. Let the distance of dm from x - and y - axes are a and b respectively. The moment of inertia of the whole body about main axis will be according definition of momentum of inertia.

$$I = \int dm r^2 = \int dm (a^2 + b^2)$$

$$I = \int dma^2 + \int dmb^2 = I_x + I_y$$

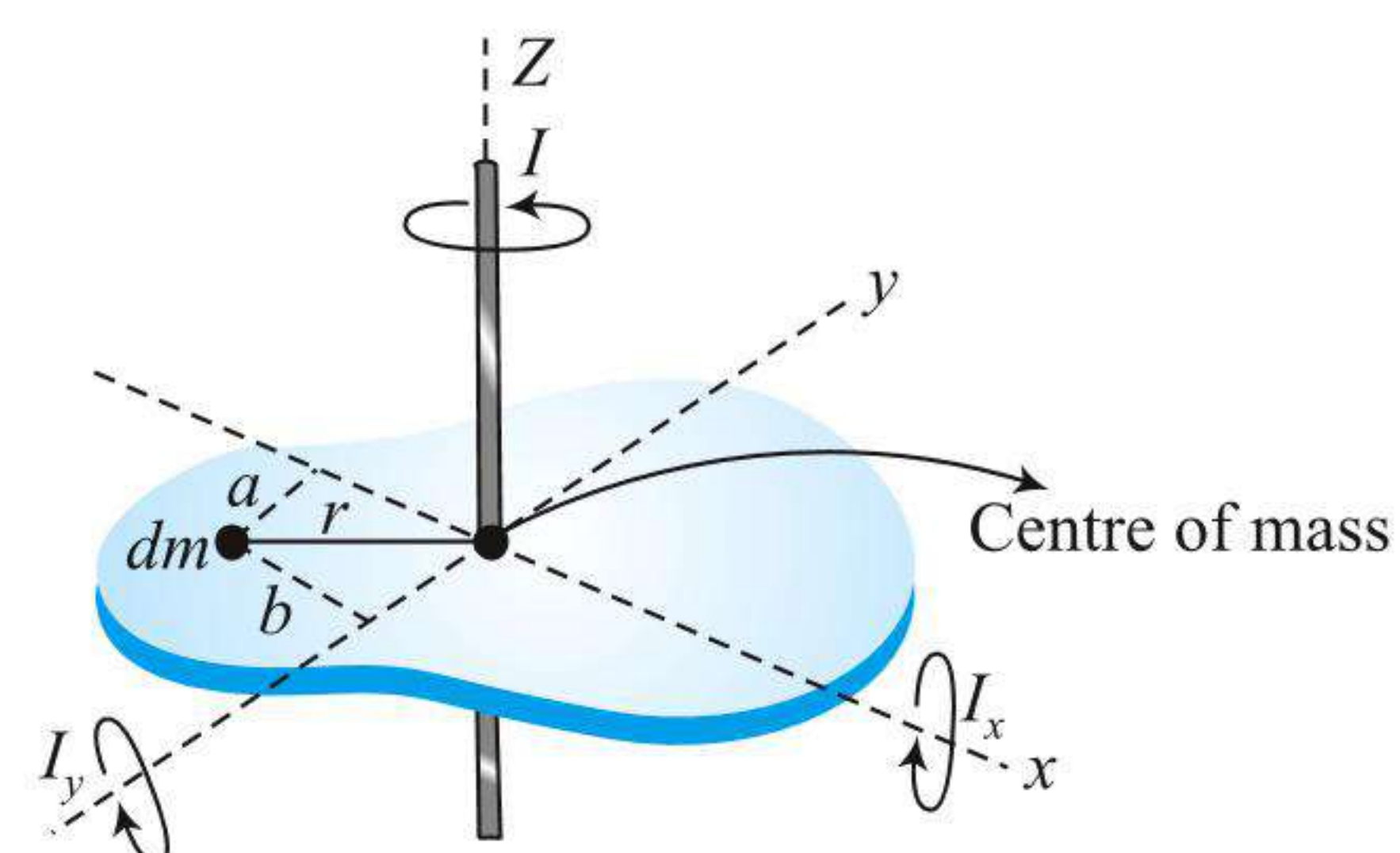


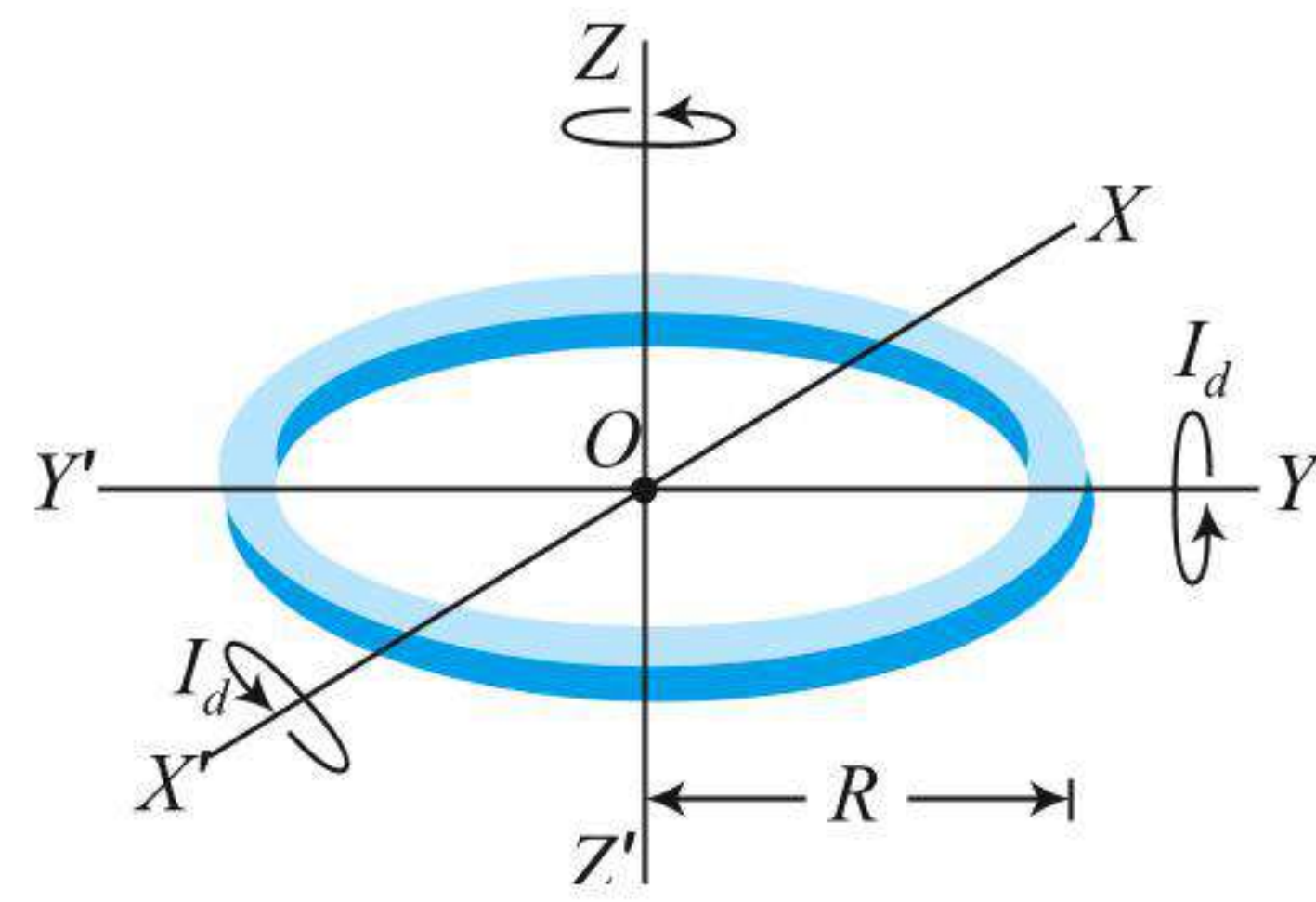
ILLUSTRATION 3.15

Calculate the moment of inertia of

- a ring of mass M and radius R about an axis coinciding with the diameter of the ring.
- a thin disc about an axis coinciding with the diameter.

Sol.

- (i) Let I_d be the moment of inertia of the circular ring about its diameter XX' . Let YY' be another diameter of the ring perpendicular to the diameter XX' .

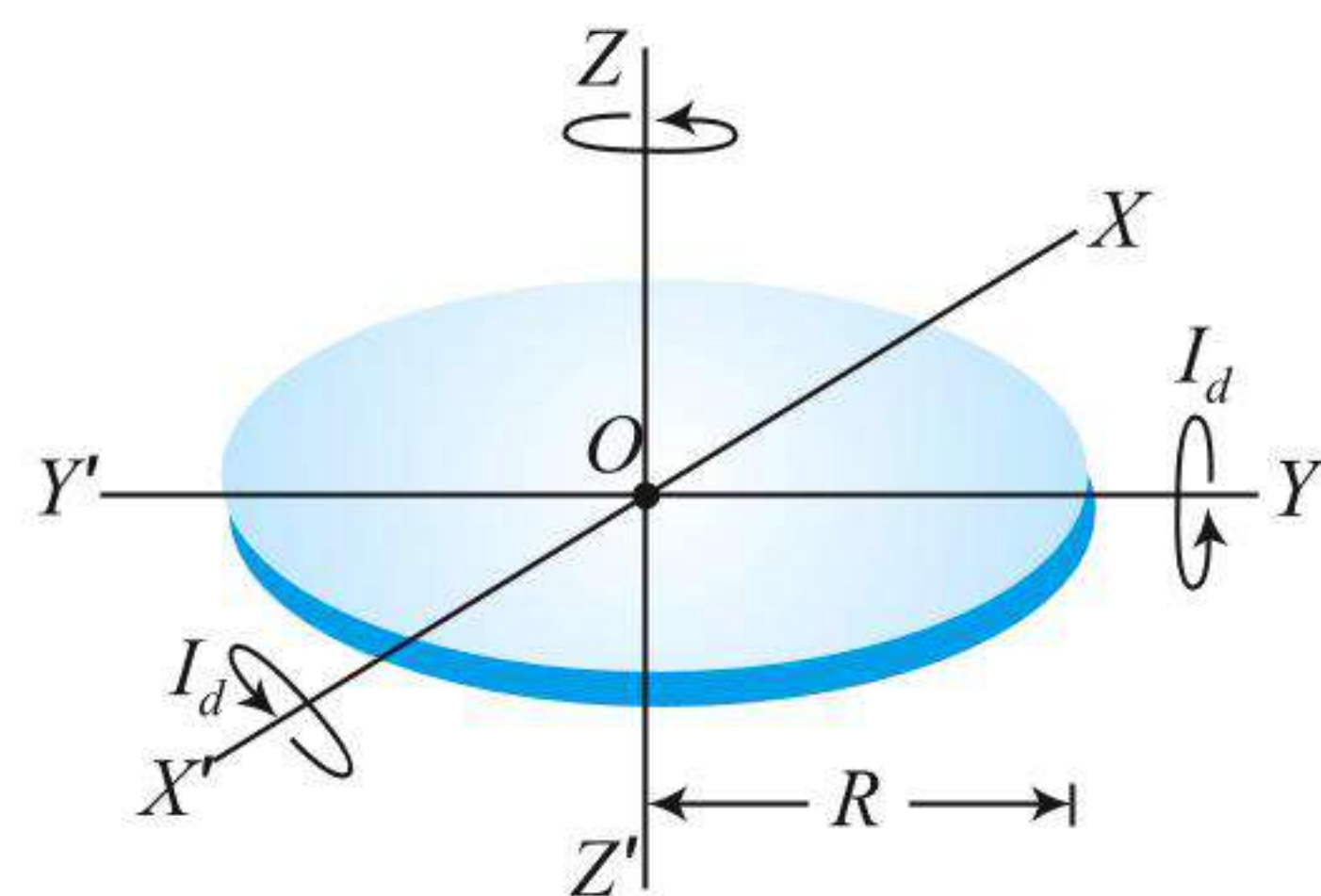


Since the ring is symmetrical about its any diameter, the moment of inertia of the ring about the diameter YY' is also equal to I_d . According to the theorem of perpendicular axes,

MI of the ring about $ZZ' = \text{MI about } XX' + \text{MI about } YY'$
i.e., $I = I_d + I_d$

$$\text{or } I_d = \frac{1}{2}I = \frac{1}{2}MR^2$$

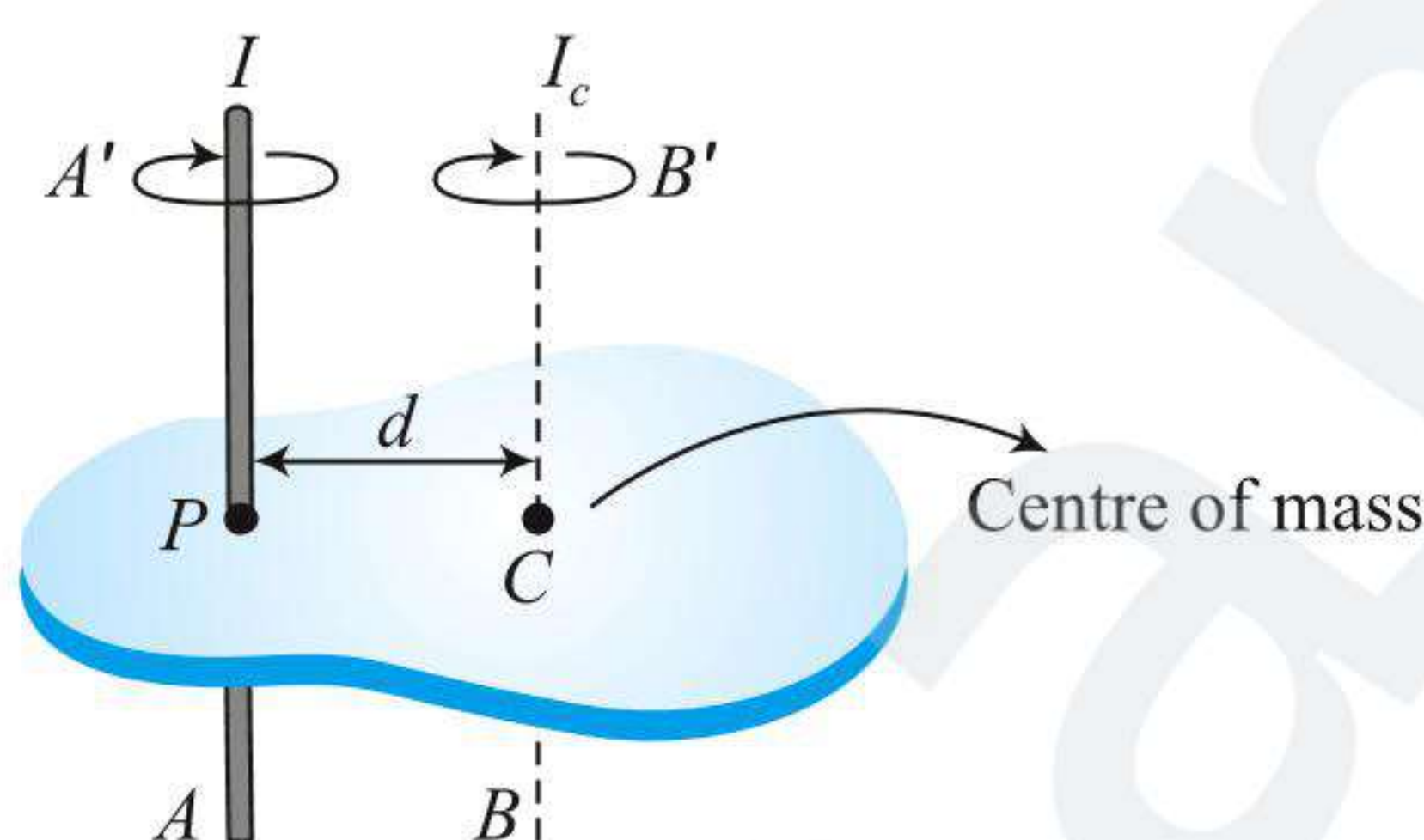
- (ii) From previous part, it can be obtained that the moment of inertia of the circular disc about its diameter,



$$I_d = \frac{1}{2}I = \frac{1}{2} \times \left(\frac{1}{2}MR^2 \right) = \frac{1}{4}MR^2$$

PARALLEL AXIS THEOREM

If we want to find the moment of inertia I of a body of mass M about a given axis. In principle, we can always find it by method of integration. However, there is a neat shortcut if we already know the moment of inertia I_c of the body about a parallel axis that extends through the body's centre of mass.

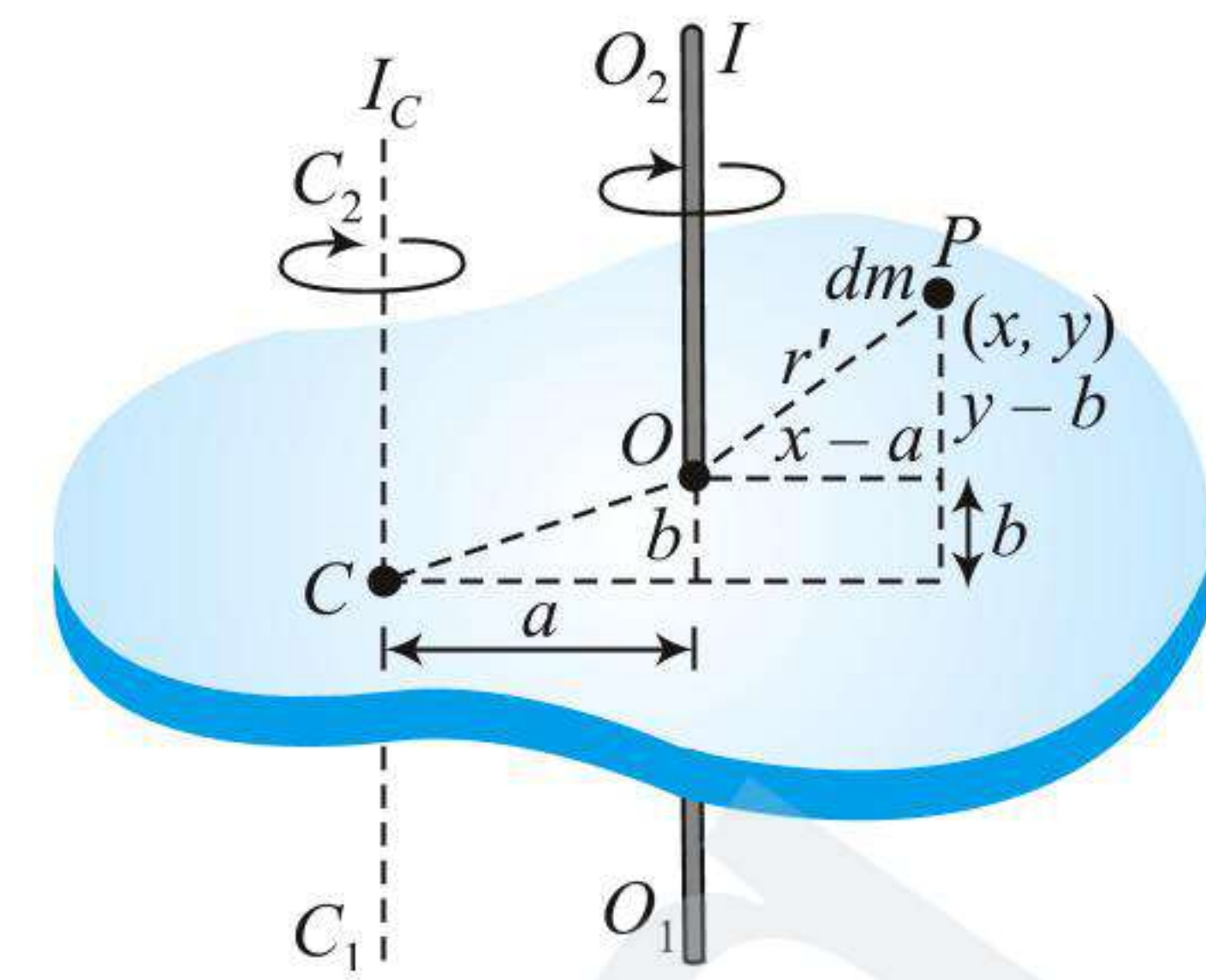


In the figure given, consider a body rotating about an axis AA' passing through it. To find moment of inertia about this axis of rotation, first we find an imaginary axis BB' which is parallel to AA' and passing through centre of mass of the body. Now we evaluate the moment of inertia of the body if it were rotated about BB' , let it be I_c . The moment of inertia of the body about the axis AA' is given as

$$I = I_c + Md^2 \quad \dots(i)$$

where d is perpendicular distance between the two axes AA' and BB' . This equation is known as the **parallel axis theorem**. We shall now prove it.

Let us take a body as shown in figure. C is the centre of mass of the body and O is the point in x - y plane on the axis about which we are required to evaluate the moment of inertia.



Consider an elemental mass dm at coordinate (x, y) with respect to C and let (a, b) be the coordinate of the point O . The rotational inertia of the body about the axis through O (about the axis O_1O_2) is given as, $I = \int dm r_1^2$

$$\text{We have } r_1^2 = (x-a)^2 + (y-b)^2$$

$$\Rightarrow I = \int r_1^2 \cdot dm = \int [(x-a)^2 + (y-b)^2] dm,$$

which we can rearrange as

$$I = \int (x^2 + y^2) dm - 2a \int x dm - 2b \int y dm + \int (a^2 + b^2) dm \quad \dots(ii)$$

From the definition of the centre of mass, the middle two integrals of Eq. (ii) give the coordinates of the center of mass (multiplied by a constant) and thus must each be zero. Because $x^2 + y^2$ is equal to r^2 , where r is the distance from O to dm , the first integral is simply I_c , the rotational inertia of the body about an axis through its center of mass. Figure shows that the last term in Eq. (ii) is Md^2 , where M is the body's total mass.

So, we have

$$I = I_c + Md^2$$

Thus, Eq. (ii) reduces to Eq. (i), which is the required relation.

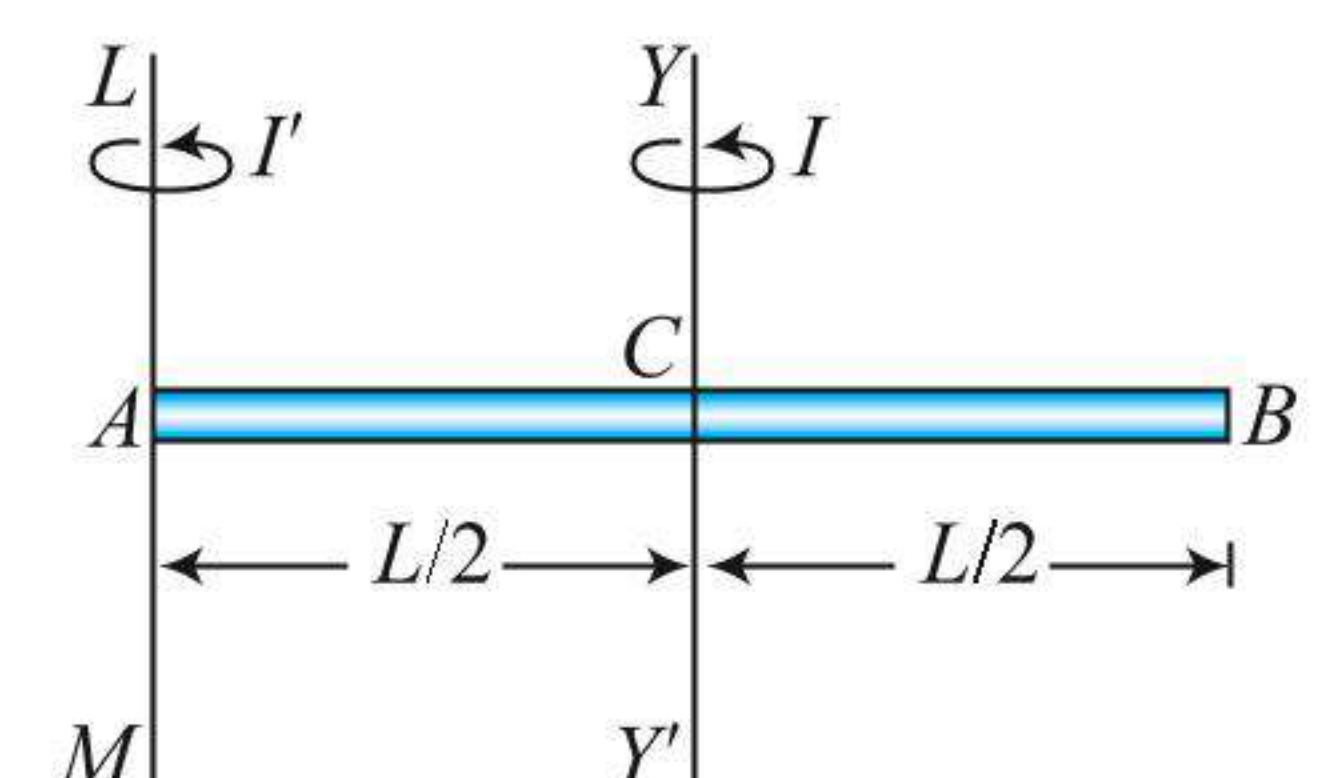
Important Points:

- Theorem of parallel axes is applicable for any type of rigid body whether it is two dimensional or three dimensional, while the theorem of perpendicular axes is applicable for lamina type or two-dimensional bodies only.
- In theorem of perpendicular axes, the point of intersection of the three axes (x , y and z) may be any point on the plane of the body (it may even lie outside the body). This point may or may not be the centre of mass of the body.

Moment of Inertia of a Thin Rod About an Axis Through One End

Let I' be the moment of inertia of the rod about an axis LM passing through the end A and perpendicular to its length (see figure). The axis LM is at a distance $L/2$ from the axis YY' through the centre of mass of the rod. Therefore, according to the theorem of parallel axis, the moment of inertia of the rod about the axis LM is given by

$$I' = I + M \left(\frac{L}{2} \right)^2 = \frac{1}{12} ML^2 + \frac{1}{4} ML^2$$

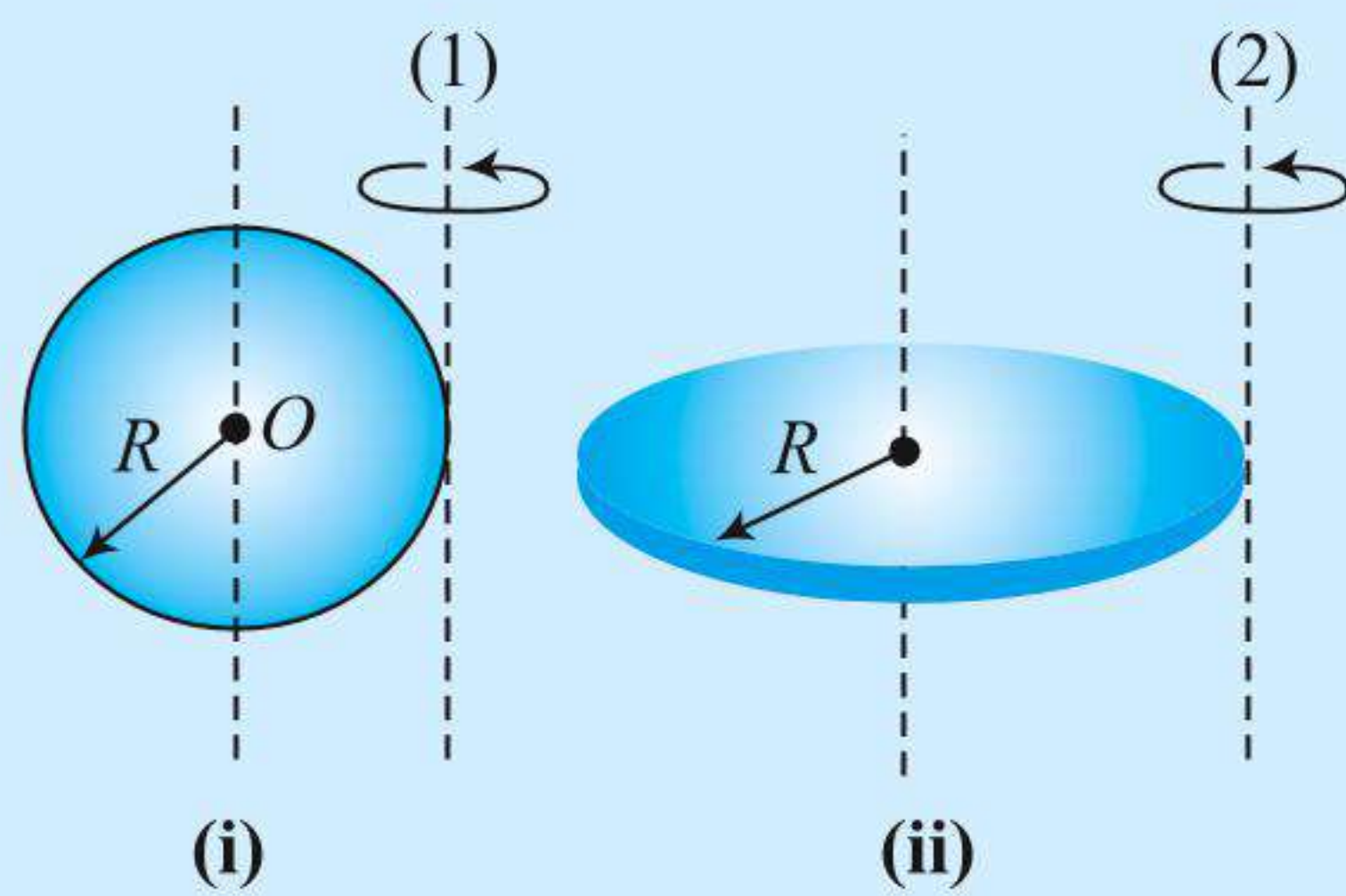


$$\text{or } I' = \frac{1}{3} ML^2$$

ILLUSTRATION 3.16

Find out the moment of inertia of a disc having uniform mass distribution (mass M and radius R) about an axis which is tangent to the disc if

- (a) the tangent is in the plane of the disc (Fig. (i)),
 (b) the tangent is perpendicular to the plane of the disc (Fig. (ii)).



Sol. The moment of inertia of a disc about an axis passing through centre and perpendicular to plane of ring is $I_0 = MR^2/2$ because of symmetry we can say $I_x = I_y$ and using perpendicular axis theorem,

$$I_z = I_x + I_y \Rightarrow \frac{MR^2}{2} = 2I_y$$

$$\Rightarrow I_y = \frac{MR^2}{4} = \text{M.I. about diameter axis [in Fig. (i)]}$$

For case (a): Using parallel axis theorem,

$$I_1 = I_c + MR^2$$

As the diameter axis passes through the centre of mass, so here

$$\text{we can write } I_y = I_c = \frac{MR^2}{4}$$

$$\text{Hence, } I_1 = I_c + MR^2 = \frac{MR^2}{4} + MR^2 = \frac{5}{4} MR^2$$

For case (b): Using parallel axis theorem,

$$I_2 = I_z + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2} MR^2$$

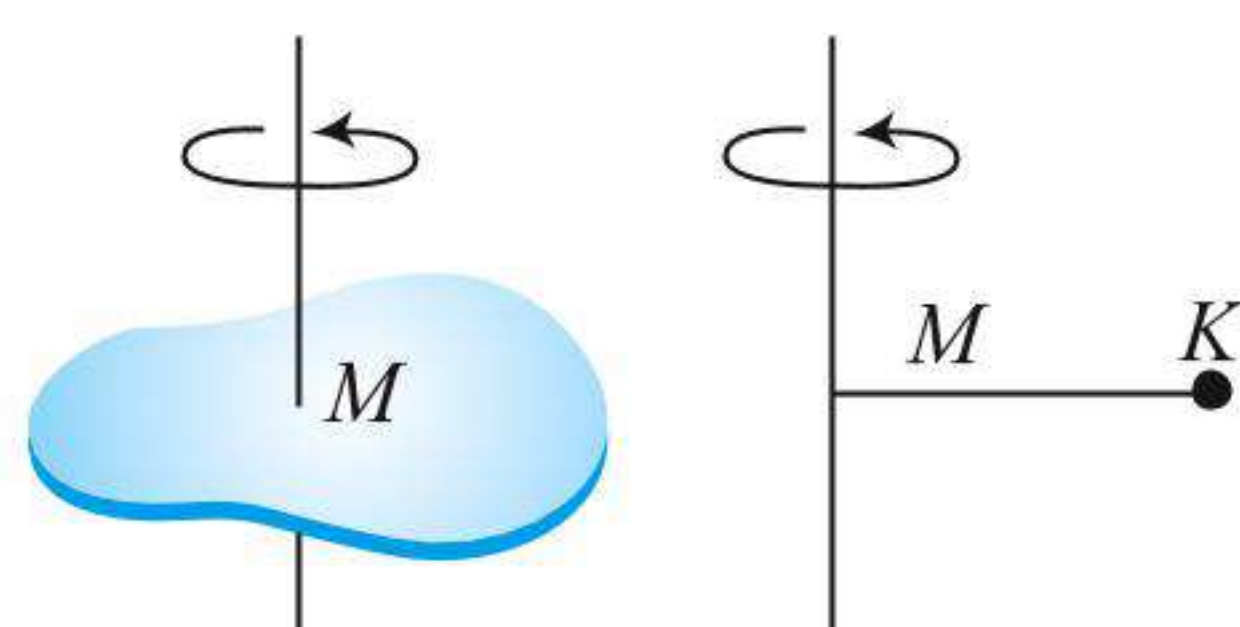
Note: Similarly, the moment of inertia of a solid sphere about a tangent is

$$I_{\text{tangent}} = I = I_c + MR^2 = \frac{2}{5} MR^2 + MR^2 = \frac{7}{5} MR^2$$

RADIUS OF GYRATION

Radius of gyration (K) of a body about an axis is the effective distance from the axis where the whole mass can be assumed to be concentrated so that the moment of inertia remains the same. Thus,

$$I = MK^2 \text{ or } K = \sqrt{\frac{I}{M}}$$



Radius of gyration of a disc about an axis perpendicular to its plane and passing through its centre of mass is

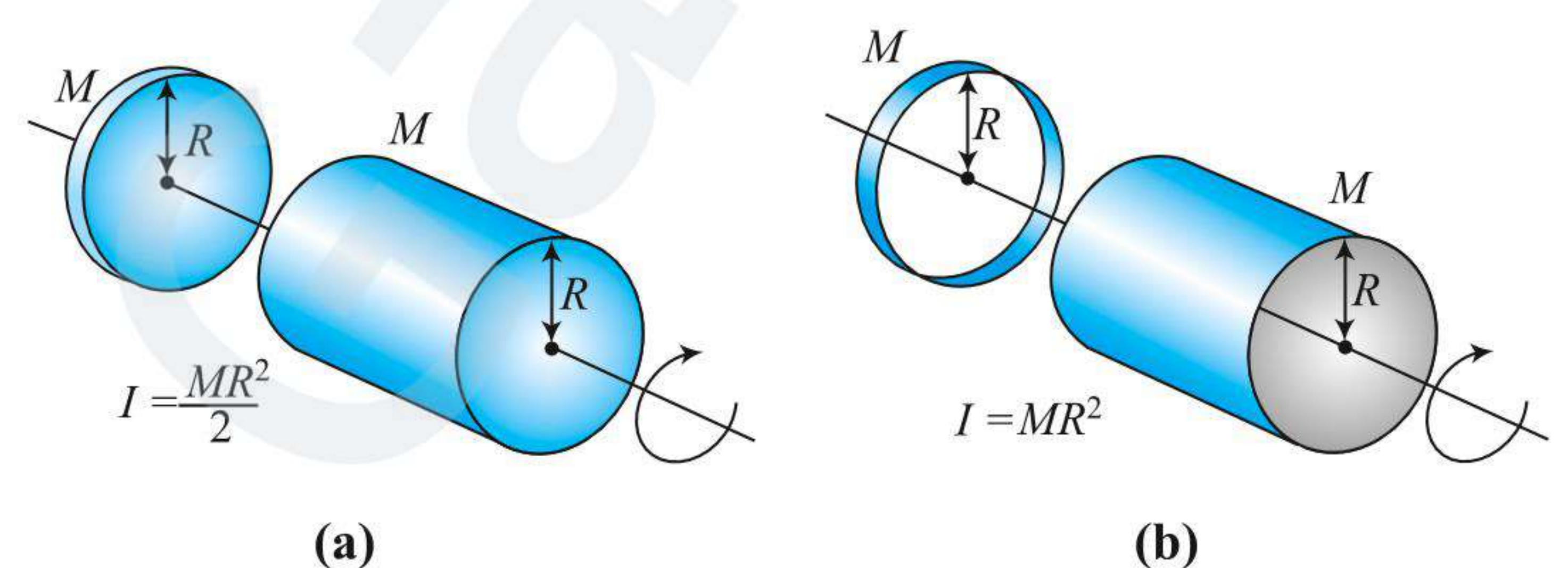
$$K = \sqrt{\frac{\frac{1}{2} MR^2}{M}} = \frac{R}{\sqrt{2}}$$

- Radius of gyration of a solid sphere, $K = \sqrt{\frac{2}{5}} R$
- Radius of gyration of a hollow sphere, $K = \sqrt{\frac{2}{3}} R$

MASS DISTRIBUTION AND MOMENT OF INERTIA

We have read that moment of inertia of an object depends on its mass as well as its mass distribution about the axis of rotation. There are several objects which can be derived from the basic objects by changing their dimensions.

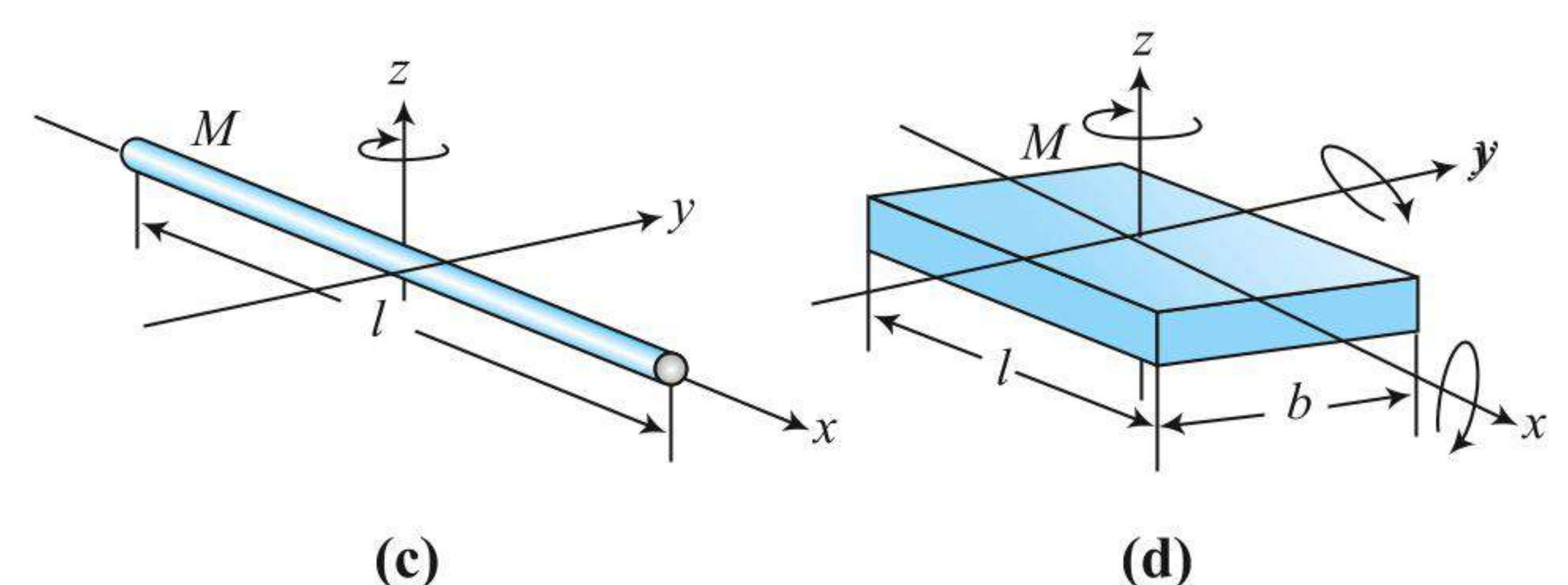
For example, if we increase the thickness of a disc it becomes a cylinder. The expression for moment of inertia for the derived object will also remain same as shown in Fig. (a).



Similarly, if we think about a hollow cylinder shown in figure, it is derived from a ring. Its moment of inertia can be given directly as MR^2 as shown in Fig. (b). It means in changing the dimension parallel to axis of rotation, the expression for moment of inertia will not change.

For example, consider a thin rod of length l , shown in Fig. (c). The moment of inertia of the rectangular plate can also be given by the expression used for rod as

$$I = \frac{mL^2}{12}$$



If we increase the thickness of this rod parallel to x -axis without changing its mass, it becomes a rectangular plate. Now consider a rectangular plate shown in Fig. (d), with dimension $l \times b$. If this plate is rotated about x -axis and y -axis, the respective moment of inertias can be given as

$$I_x = \frac{Mb^2}{12} \text{ and } I_y = \frac{Ml^2}{12}$$

Now if this plate is rotated about the axis passing through its centre and perpendicular to its plane as shown in figure. The moment of inertia can be given by the sum of the above two moment of inertias according to perpendicular axes theorem. So, the moment of inertia of plate about z -axis is given by

$$I_z = I_x + I_y = \frac{Ml^2}{12} + \frac{Mb^2}{12} = \frac{M}{12} (l^2 + b^2)$$

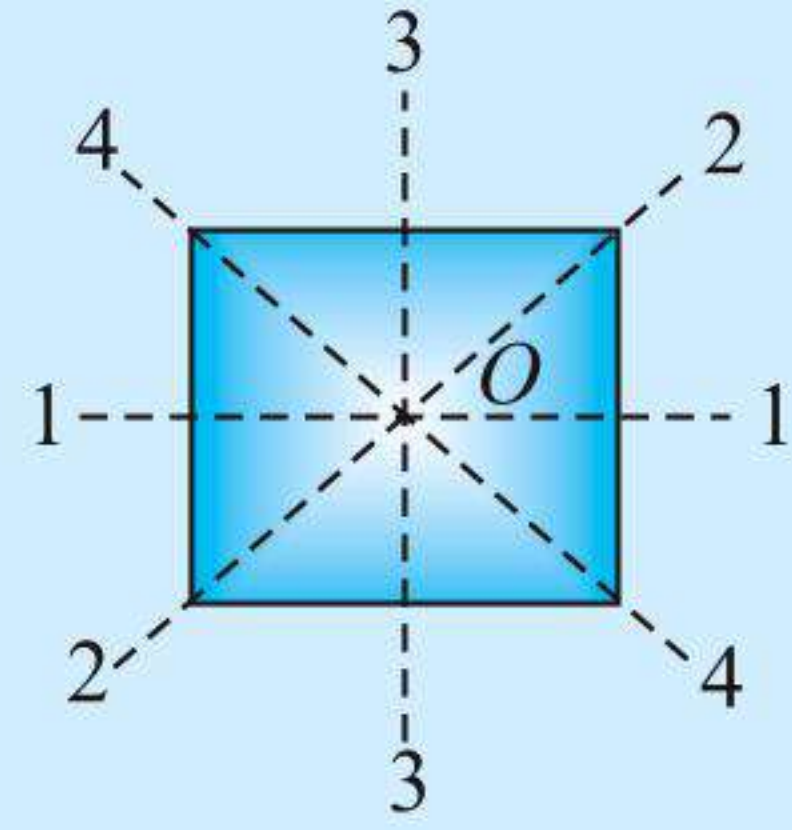
Now we can say:

“If size of a body in any or all dimensions is increased in such a way that its mass distribution about given axis of rotation remain same then expression of moment of inertia also remain same.”

ILLUSTRATION 3.17

Find the moment of inertia of a square lamina of mass M and side l about following axis:

- (i) $I_{1,1}$ (ii) $I_{2,2}$ (iii) $I_{3,3}$ (iv) $I_{4,4}$ and (v) an axis passing through O and perpendicular to the plane of the square (I_{oz}) (as shown in figure).



Sol. We know the moment of inertia of a body is independent of its size parallel to axis of rotation. Hence

$$I_{1,1} = I_{3,3} = \frac{Ml^2}{12}$$

Using perpendicular axis theorem

$$I_{oz} = I_{1,1} + I_{3,3} = 2 \times \frac{Ml^2}{12} \Rightarrow I_{oz} = \frac{Ml^2}{6}$$

As mass distribution is symmetrical about diagonal axis, hence

$$I_{2,2} = I_{4,4}$$

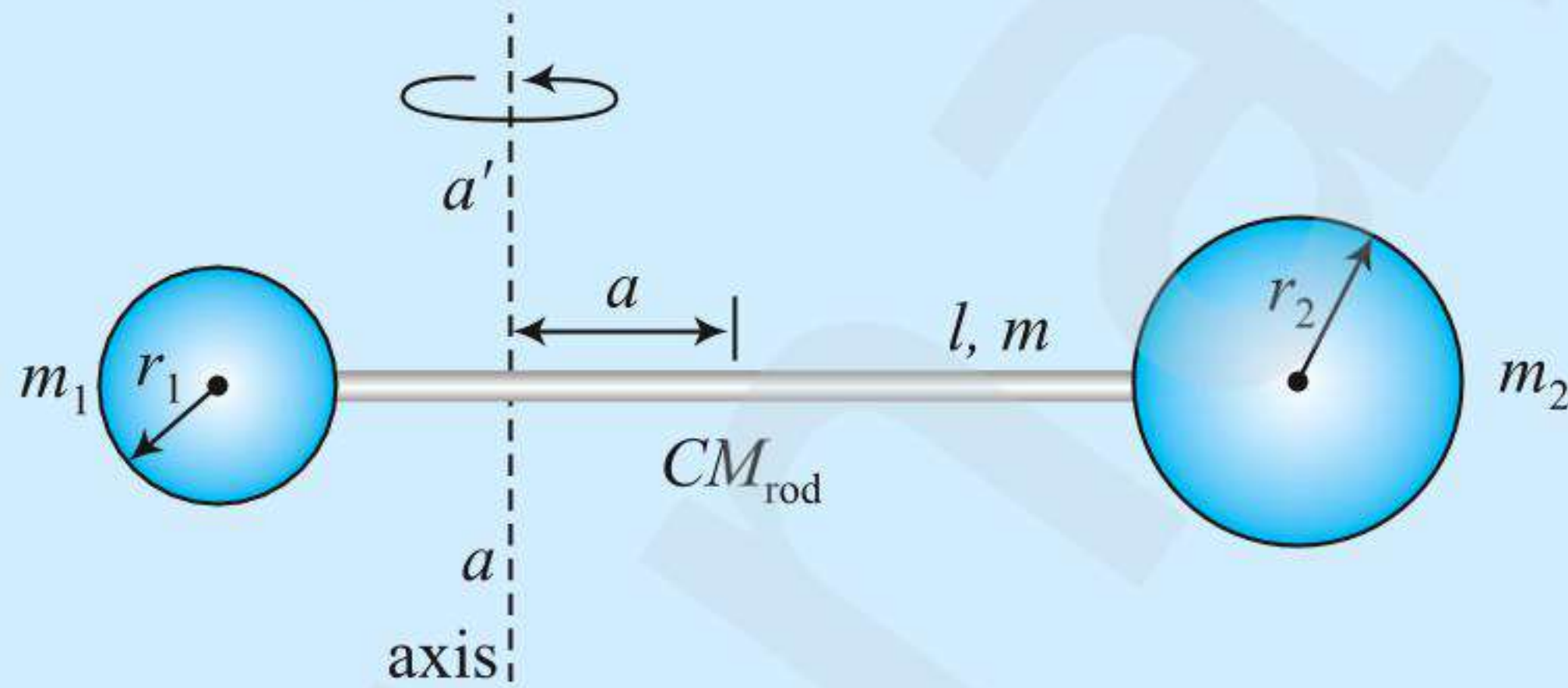
Again, from perpendicular axis theorem, $I_{oz} = I_{2,2} + I_{4,4}$

$$\text{or } \frac{Ml^2}{6} = 2I_{2,2} \Rightarrow I_{2,2} = \frac{Ml^2}{12}$$

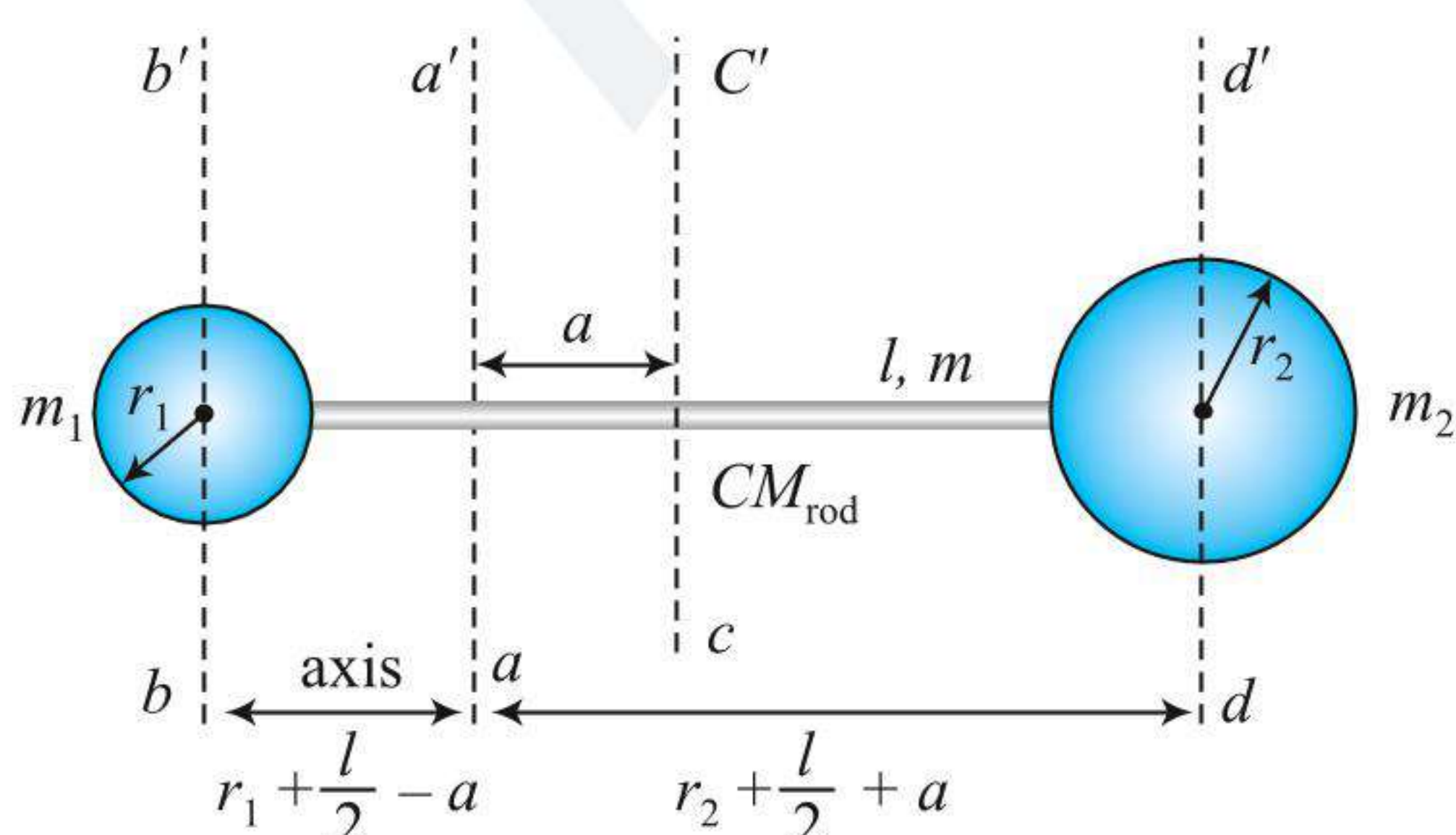
$$\text{It means } I_{2,2} = I_{4,4} = \frac{Ml^2}{12}$$

ILLUSTRATION 3.18

Two uniform solid spheres of masses m_1 and m_2 and radii r_1 and r_2 , respectively, are connected at the ends of a uniform rod of length l and mass m . Find the moment of inertia of the system about an axis perpendicular to the rod and passing through a point at a distance a from the centre of mass of the rod as shown in figure.



Sol. Draw the axes bb' , dd' and cc' passing through the centers of the spheres and the centre of mass of the rod as shown in figure.



$$\text{For } m_1: I_{bb'} = \frac{2}{5} m_1 r_1^2$$

$$\text{and } (I_{aa'})_1 = I_{bb'} + m_1 \left(r_1 + \frac{l}{2} - a \right)^2 = \frac{2}{5} m_1 r_1^2 + m_1 \left(r_1 + \frac{l}{2} - a \right)^2$$

[From parallel axis theorem]

$$\text{For } m: I_{cc'} = \frac{ml^2}{12}; (I_{aa'})_2 = I_{cc'} + ma^2 = \frac{ml^2}{12} + ma^2$$

$$\text{For } m_2: I_{dd'} = \frac{2}{5} m_2 r_2^2$$

$$\Rightarrow (I_{aa'})_3 = I_{dd'} + m_2 \left(r_2 + \frac{l}{2} + a \right)^2$$

$$= \frac{2}{5} m_2 r_2^2 + m_2 \left(r_2 + \frac{l}{2} + a \right)^2$$

The moment of inertia of the given system about aa' ,

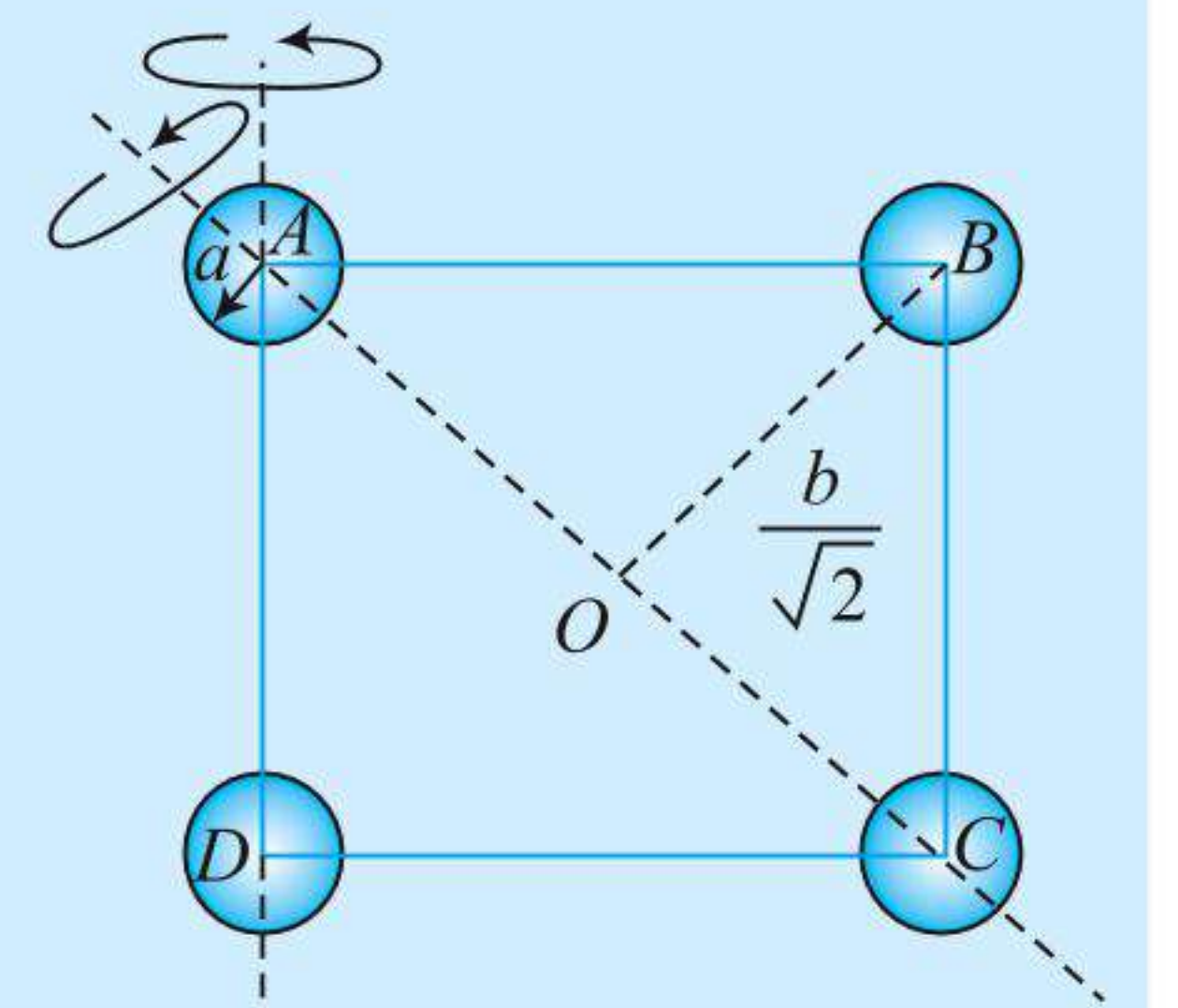
$$I_{aa'} = (I_{aa'})_1 + (I_{aa'})_2 + (I_{aa'})_3$$

$$= \left\{ \frac{2}{5} m_1 r_1^2 + m_1 \left(r_1 + \frac{l}{2} - a \right)^2 \right\} + \frac{ml^2}{12} + ma^2 + \left\{ \frac{2}{5} m_2 r_2^2 + m_2 \left(r_2 + \frac{l}{2} + a \right)^2 \right\}$$

ILLUSTRATION 3.19

Four spheres, each of radius a and mass m , are placed with their centres on four corners of a square of side b . Calculate the moment of inertia of the arrangement about any

- (a) diagonal of the square and
(b) any side of the square.



Sol.

- (a) Moment of inertia of the arrangement about the diagonal AC :

The moment of inertia of each of spheres A and C about their common diameter $AC = (2/5)ma^2$. The moment of inertia of each of spheres B and D about an axis passing through their centres and parallel to AC is $(2/5)ma^2$. The distance between the axis and AC is $b/\sqrt{2}$, b being side of the square. Therefore, the moment of inertia of each spheres B and D about AC by the theorem of parallel axis is

$$\frac{2}{5} ma^2 + m \left(\frac{b}{\sqrt{2}} \right)^2 = \frac{2}{5} ma^2 + \frac{mb^2}{2}$$

Therefore, the moment of inertia of all the four spheres about diagonal AC is

$$I = 2 \left(\frac{2}{5} ma^2 \right) + 2 \left[\frac{2}{5} ma^2 + \frac{mb^2}{2} \right] = \frac{8}{5} ma^2 + mb^2$$

- (b) The moment of inertia of each of spheres A and D about side AD is $(2/5)ma^2$.

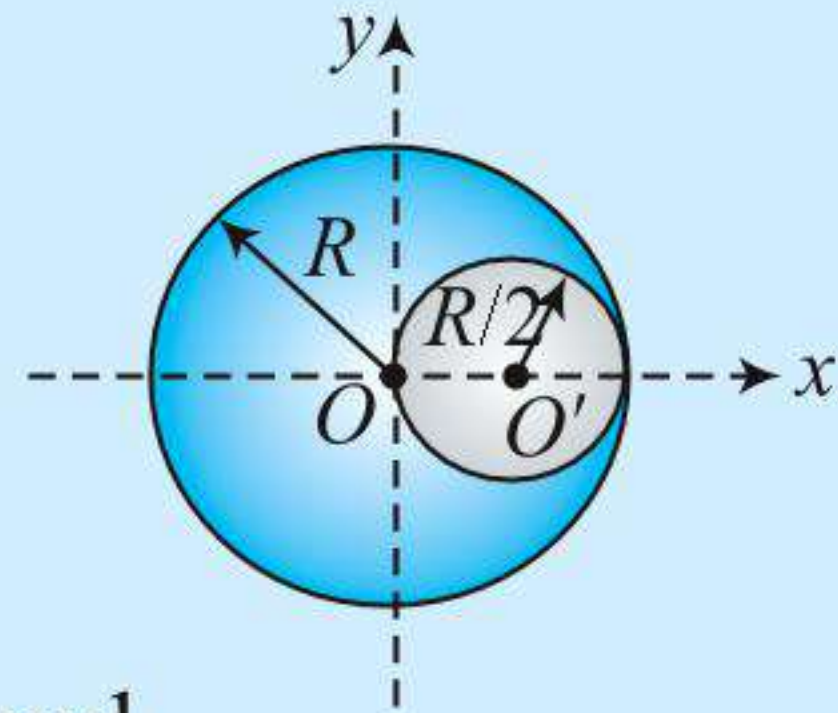
The moment of inertia of each of spheres B and C about AD is $(2/5)ma^2 + mb^2$.

Therefore, the moment of inertia of the whole arrangement about side AD is

$$2 \left[\frac{2}{5} ma^2 + \left(\frac{2}{5} ma^2 + mb^2 \right) \right] = \frac{2m}{5} [4a^2 + 5b^2]$$

ILLUSTRATION 3.20

A circular hole of radius $R/2$ is cut from a circular disc of radius R . The disc lies in the x - y plane and its centre coincides with the origin. If the remaining mass of the disc is M , then



- (a) determine the initial mass of the disc, and
(b) determine its moment of inertia about the z -axis.

Sol.

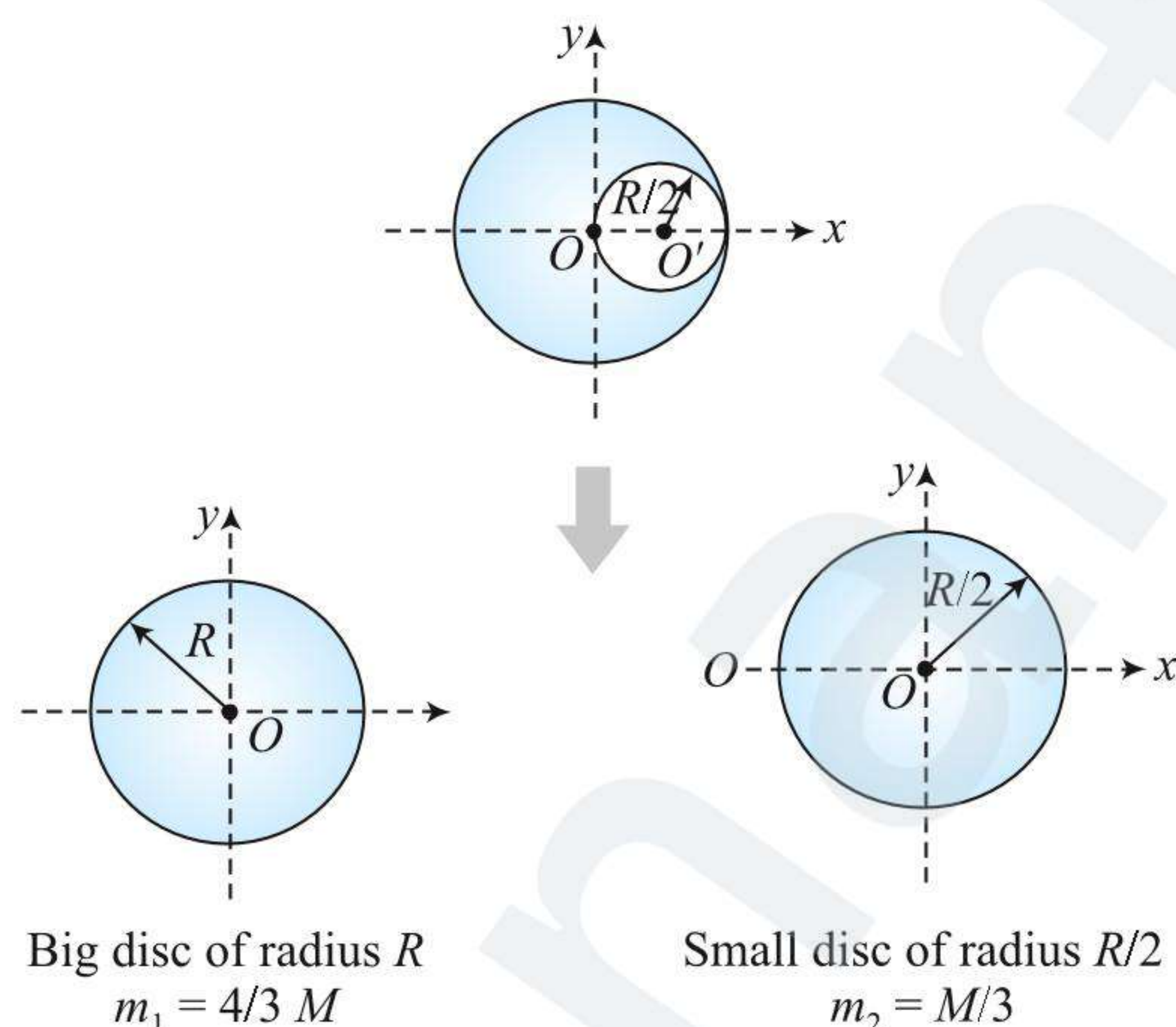
- (a) Let σ be the surface mass density of the plate given by

$$\sigma = \frac{M}{\pi R^2 - \pi \left(\frac{R}{2} \right)^2} = \frac{4}{3} \frac{M}{\pi R^2}$$

Now, initial mass of the plate, $m_1 = \sigma(\pi R^2) = \frac{4}{3} M$

Mass of the plate removed, $m_2 = \sigma \left(\frac{\pi R^2}{4} \right) = \frac{M}{3}$

- (b) The given arrangement may be considered as a combination of a circular plate of mass $(4/3) M$ and circular plate of negative mass $(M/3)$.



The moment of inertia of the big disc of radius R and mass m_1 about O and parallel to the z -axis is

$$I_{1o} = \frac{m_1 R^2}{2}$$

The moment of inertia of the small disc of mass m_2 and radius $R/2$ about O' and parallel to the z -axis is

$$I = \frac{m_2 (R/2)^2}{2} = \frac{m_2 R^2}{8}$$

The moment of inertia of the small disc about O (using parallel axis theorem)

$$I_{2o} = \frac{m_2 R^2}{8} + \frac{m_2 R^2}{4} = \frac{3m_2 R^2}{8}$$

As the part of the disc of mass m_2 is removed from disc m_1 , hence the net moment of inertia of structure is

$$I_o = I_{1o} - I_{2o} \Rightarrow I_o = \frac{m_1 R^2}{2} - \frac{3m_2 R^2}{8}$$

Substituting the values of m_1 and m_2 , we get

$$I_o = \frac{13}{24} MR^2$$

ILLUSTRATION 3.21

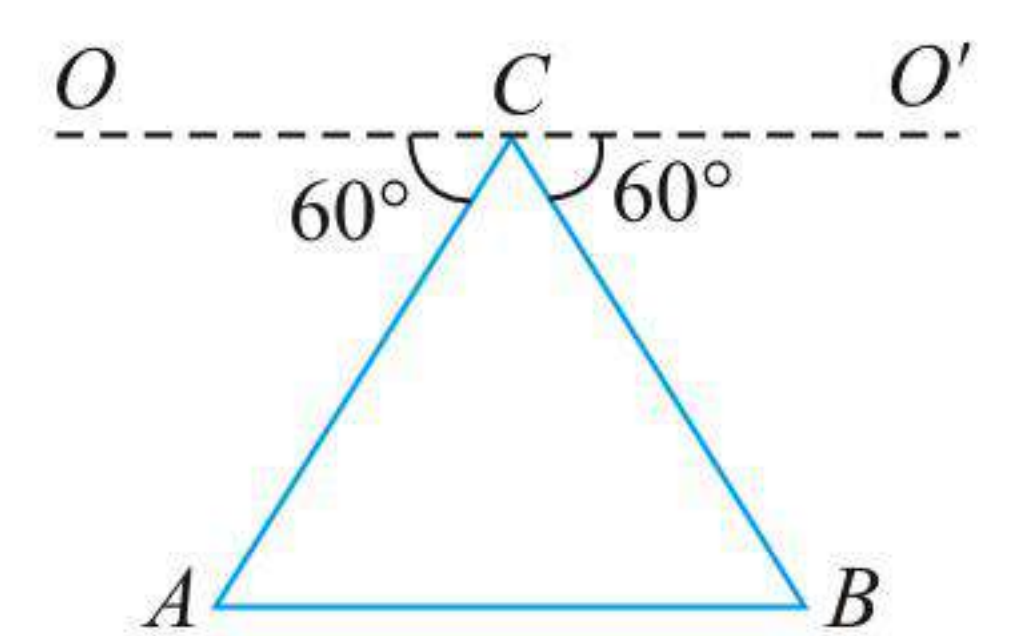
Three identical thin rods, each of mass m and length l , are joined to form an equilateral triangular frame. Find the moment of inertia of the frame about an axis parallel to its one side and passing through the opposite vertex. Also find its radius of gyration about the given axis.

Sol. Moment of inertia of each of rod AC and BC about the given axis OO' is

$$I_{AC} = I_{BC} = \frac{ml^2}{3} \sin^2 60^\circ = \frac{ml^2}{4}$$

and MI of rod AB about the given axis OO' is

$$I_{AB} = m \left(\frac{l\sqrt{3}}{2} \right)^2 = \frac{3}{4} ml^2$$

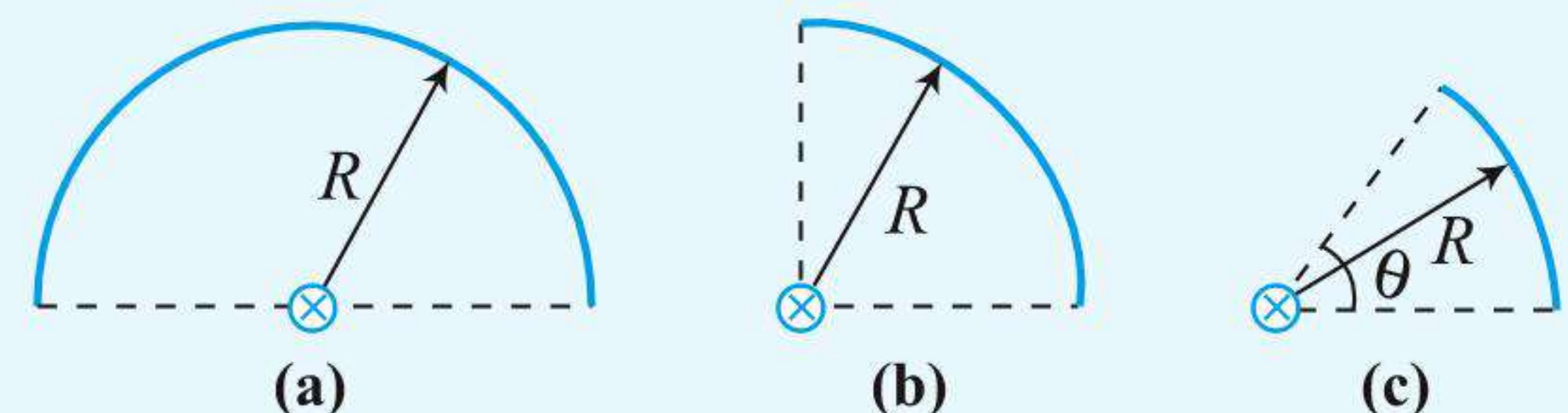


$$\text{Hence, } I = I_{AC} + I_{BC} + I_{AB} = \frac{ml^2}{4} + \frac{ml^2}{4} + \frac{3}{4} ml^2 = \frac{5}{4} ml^2$$

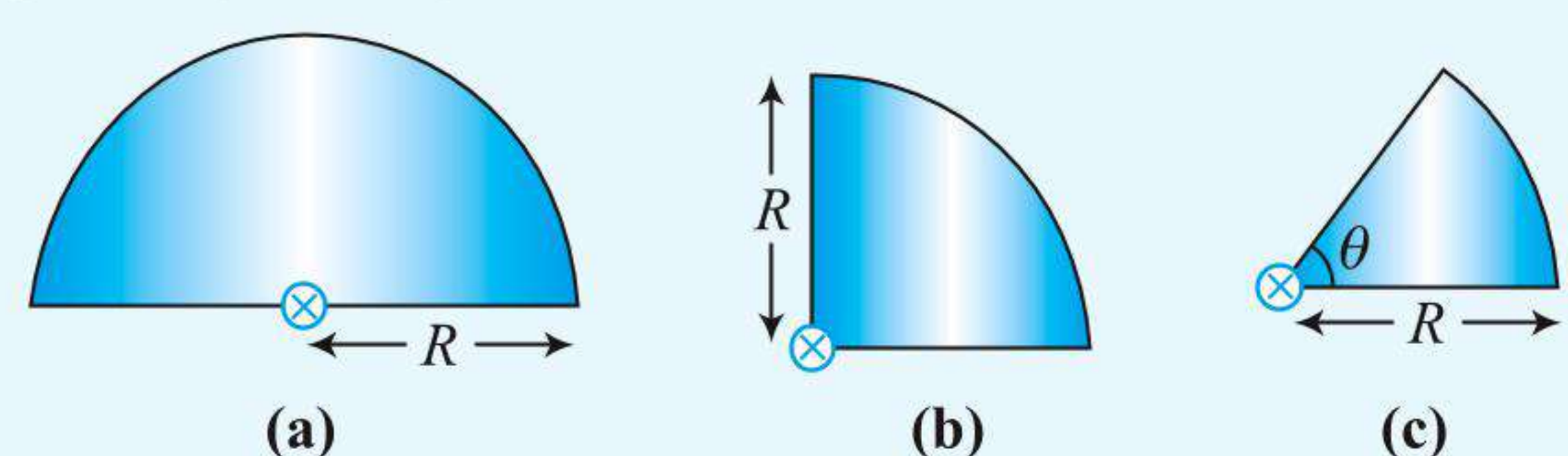
$$\text{Now, } I = MK^2 \Rightarrow K = \sqrt{\frac{I}{M}} = \sqrt{\frac{\frac{5}{4} ml^2}{3m}} = l \sqrt{\frac{5}{12}}$$

CONCEPT APPLICATION EXERCISE 3.3

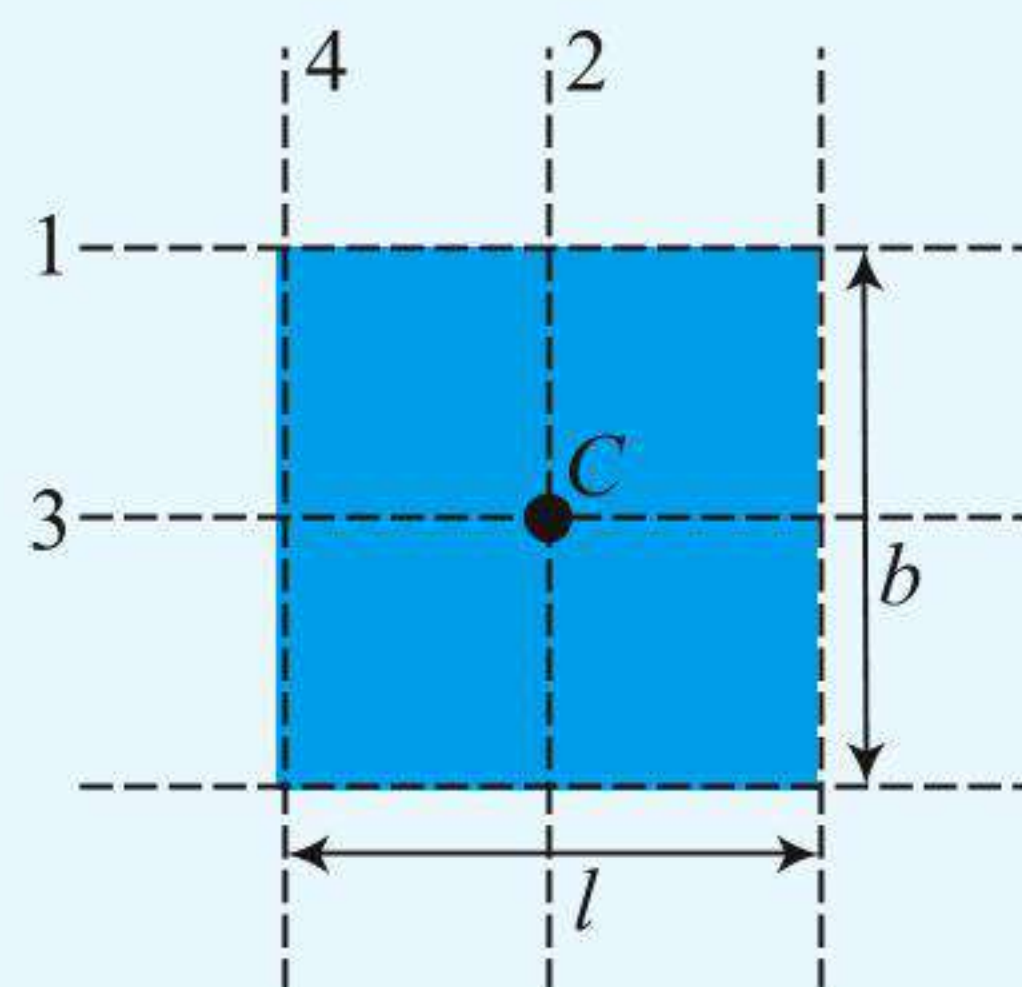
1. Find out the moment of inertia of the circular arcs shown, each having mass M , radius R and having uniform mass distribution about an axis passing through the centre and perpendicular to the plane.



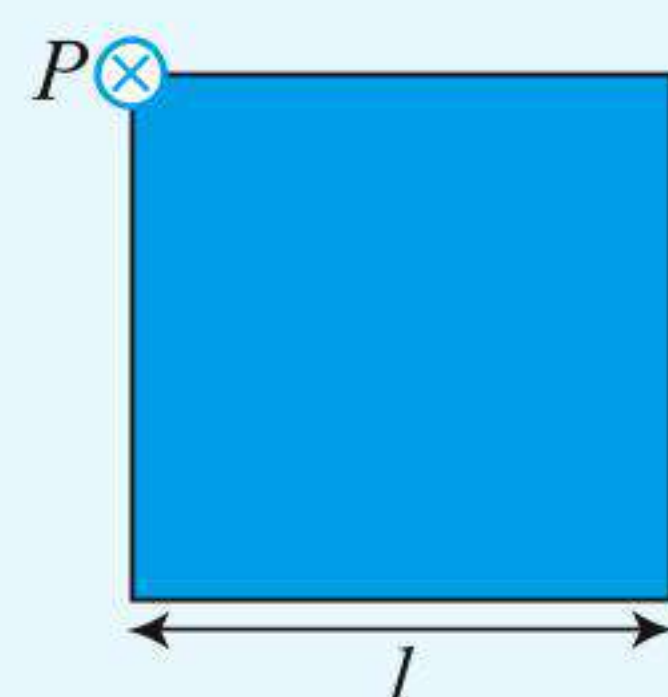
2. Calculate the moments of inertia of the figures shown, each having mass M , radius R and having uniform mass distribution about an axis perpendicular to the plane and passing through the centre.



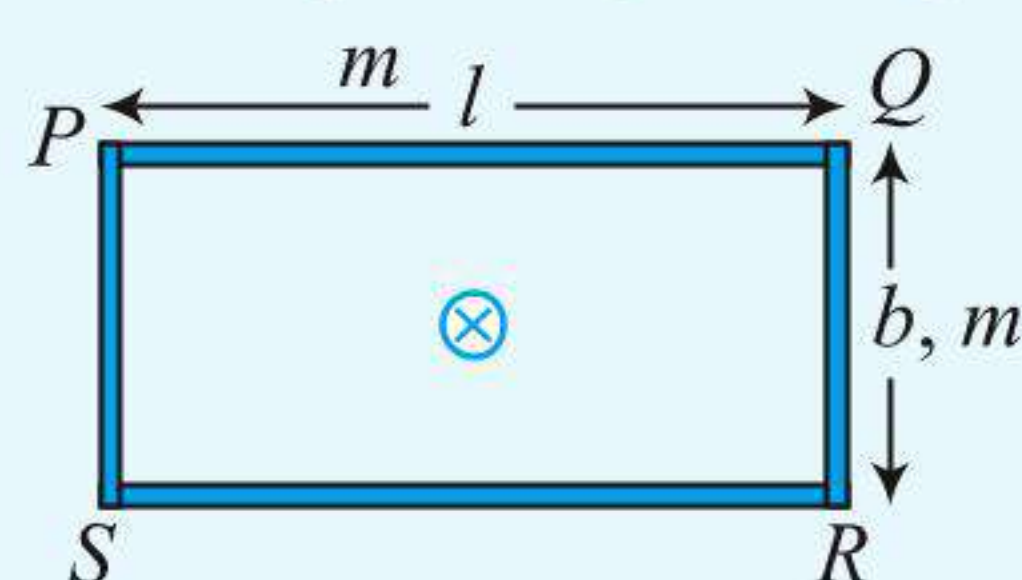
3. In figure, find moment of inertia of a plate having mass M , length l and width b about axes 1, 2, 3 and 4. Assume that C is the centre and mass is uniformly distributed.



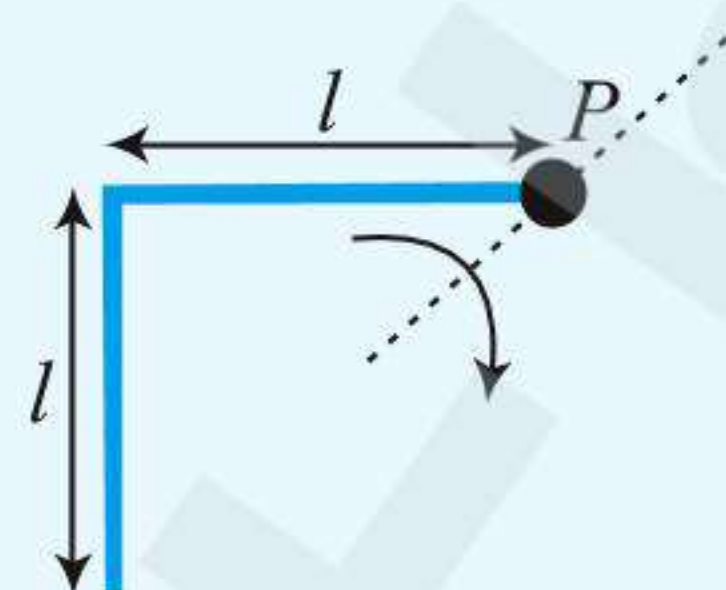
4. Find the moment of inertia of a uniform square plate of mass M and edge of length ' l ' about its axis passing through P and perpendicular to it.



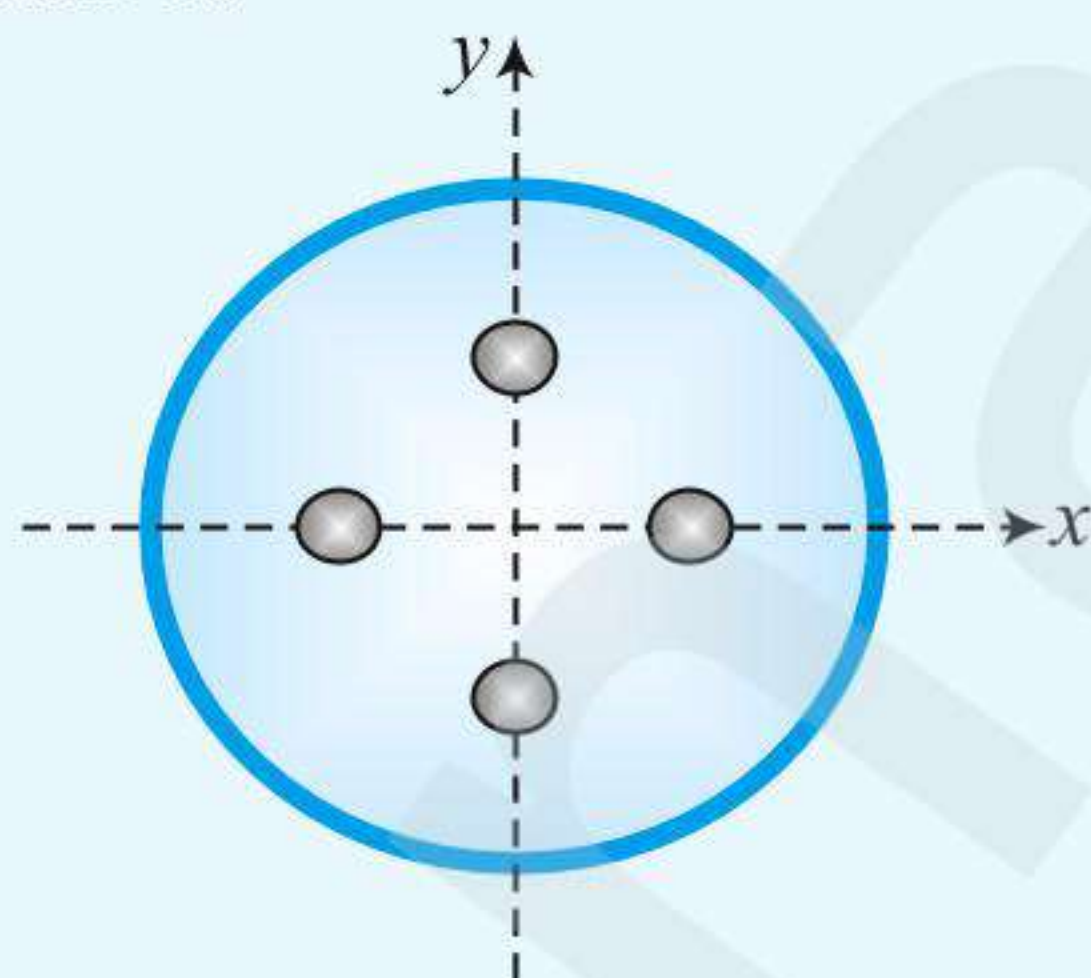
5. Calculate the moment of inertia of a rectangular frame formed by uniform rods having mass m each as shown in figure about an axis passing through its centre and perpendicular to the plane of frame. Also find moment of inertia about an axis passing through PQ ?



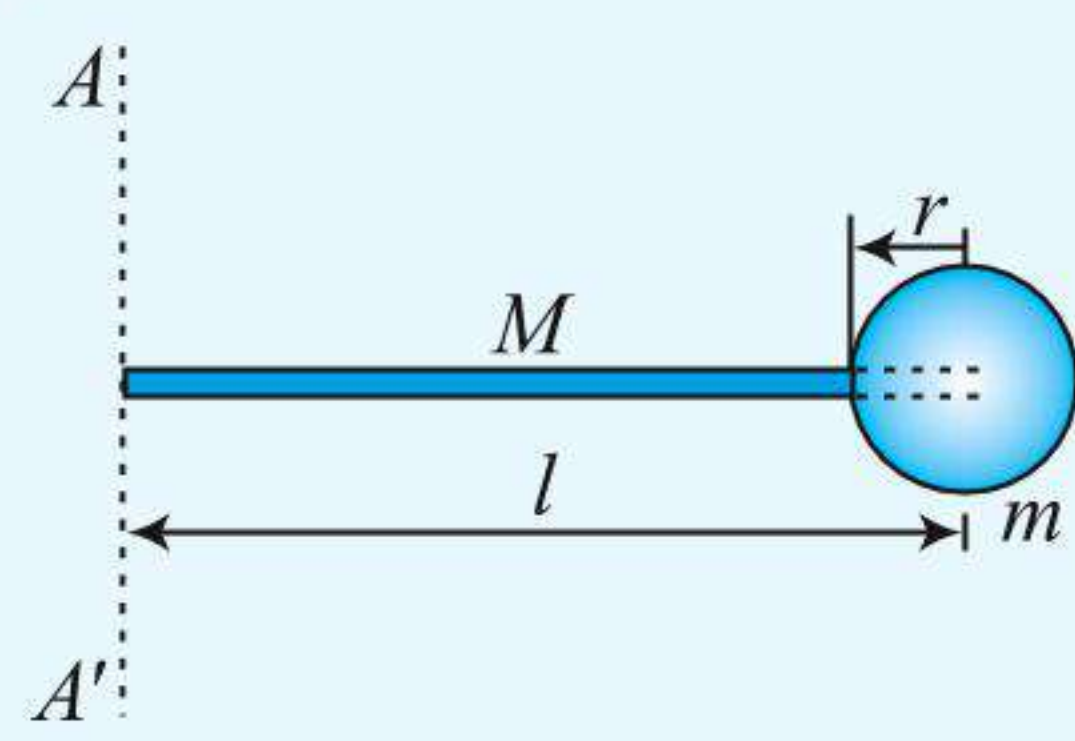
6. Find the moment of inertia of the two uniform joint rods about point P as shown in figure. Use parallel axis theorem. Mass of each rod is M .



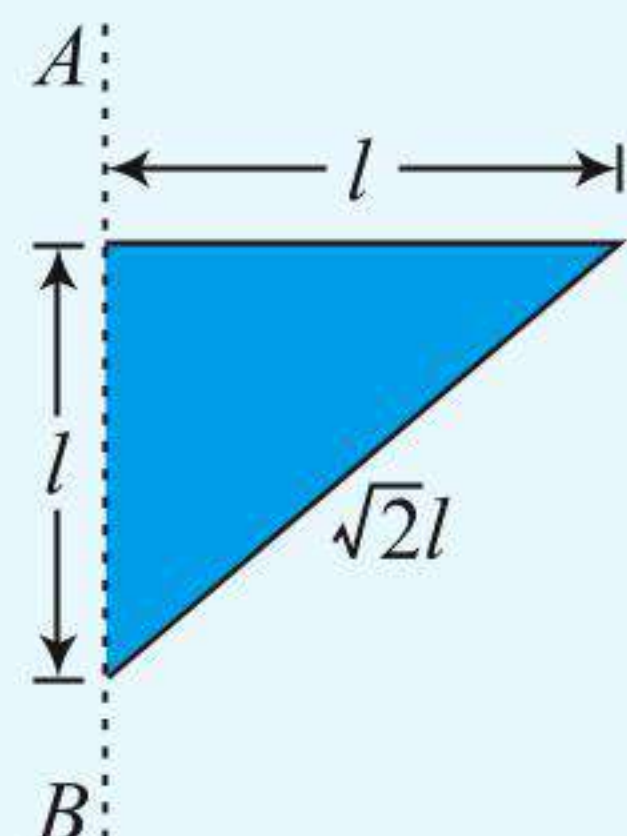
7. A uniform disc of mass m and radius R has an additional rim of mass m as well as four symmetrically placed masses, each of mass $m/4$, tied at positions $R/2$ from the centre as shown in figure. What is the total moment of inertia of the disc about an axis perpendicular to the disc through its centre?



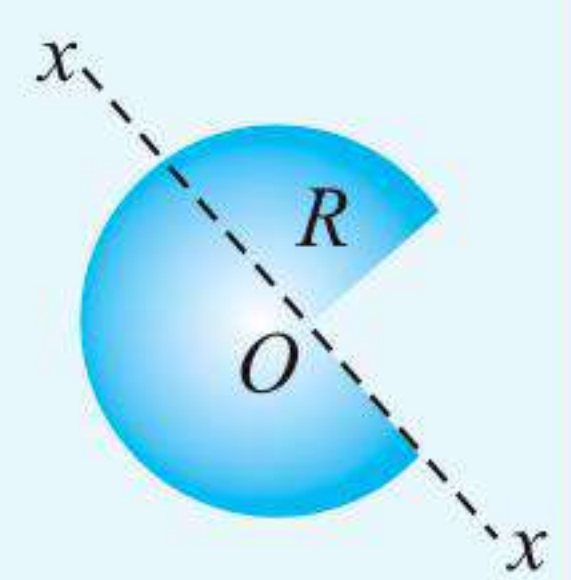
8. Find the moment of inertia of a spherical ball of mass m and radius r attached at the end of a straight rod of mass M and length l , if this system is free to rotate about an axis passing through the end of the rod (end of the rod opposite to sphere).



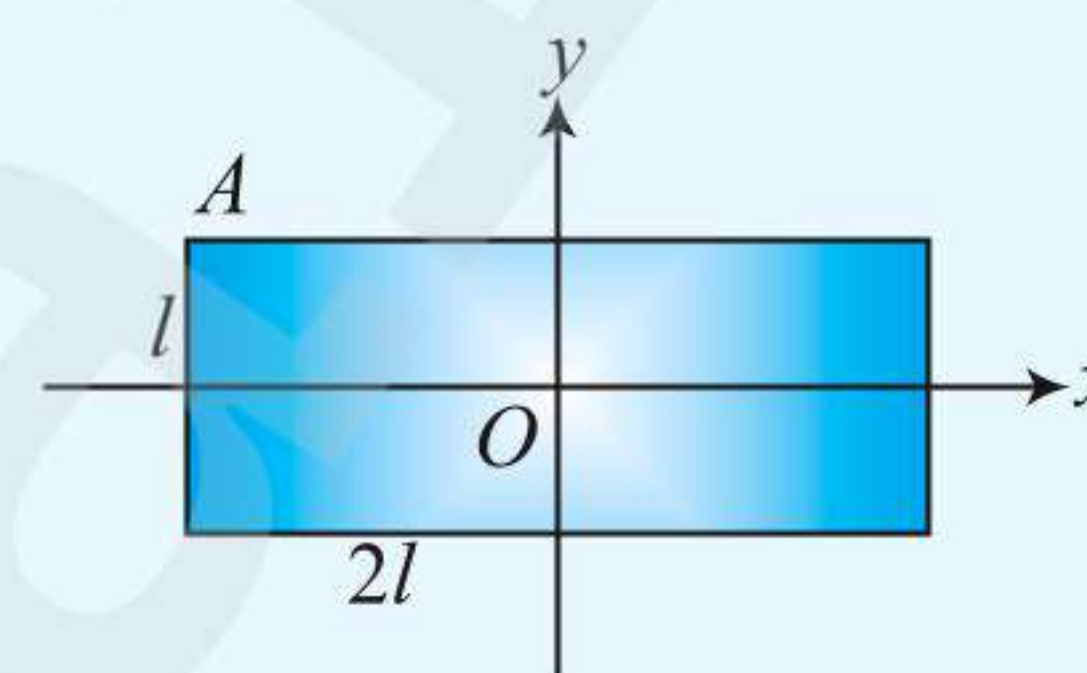
9. Find MI of a triangular lamina of mass M about the axis of rotation AB shown in figure.



10. A uniform circular disc of radius R has a sector of angle 90° removed from it. Mass of the remaining disc is M . Write the moment of inertia of the remaining disc about the axis xx shown in figure.



11. A uniform rectangular plate has side length l and $2l$. The plate lies in x - y plane with its centre at origin and sides parallel to x - and y -axes. The moment of inertia of the plate about an axis passing through a vertex (say A) perpendicular to the plane of the figure is I_0 . Now the axis is shifted parallel to itself so that moment of inertia about it remains I_0 . Write the locus of point of intersection of the axis with x - y plane.



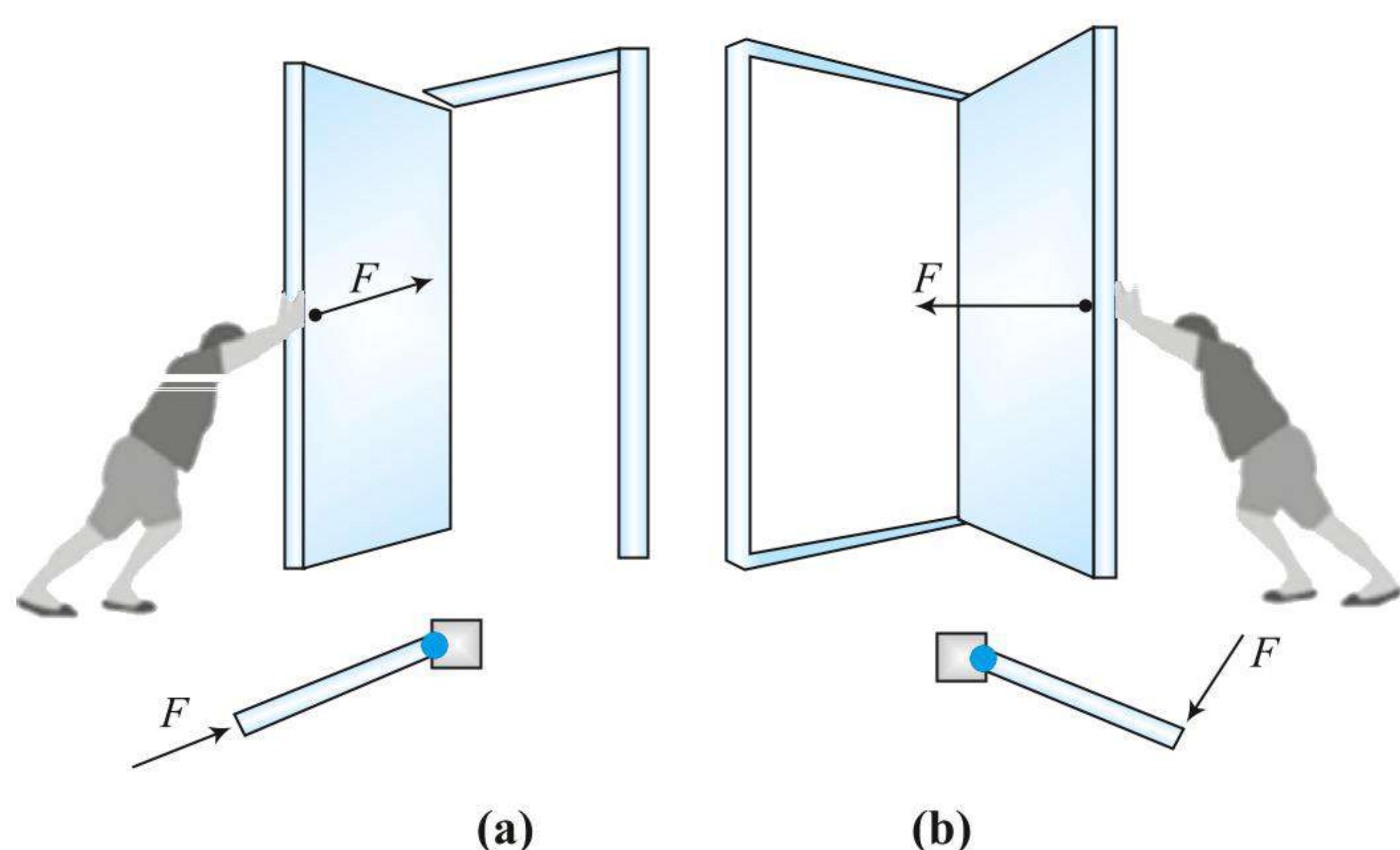
ANSWERS

3. $\frac{Mb^2}{3}, \frac{Ml^2}{12}, \frac{Ml^2}{12}, \frac{Mb^2}{3}$ 4. $\frac{2Ml^2}{3}$
 5. $\frac{2}{3}m(l^2 + b^2), \frac{5}{3}mb^2$ 6. $\frac{5}{3}Ml^2$ 7. $\frac{7mR^2}{4}$
 8. $\frac{Ml^2}{3} + \left(\frac{2}{5}mr^2 + ml^2\right)$ 9. $\frac{Ml^2}{6}$ 10. $\frac{MR^2}{4}$
 11. $x^2 + y^2 = 5l^2/4$

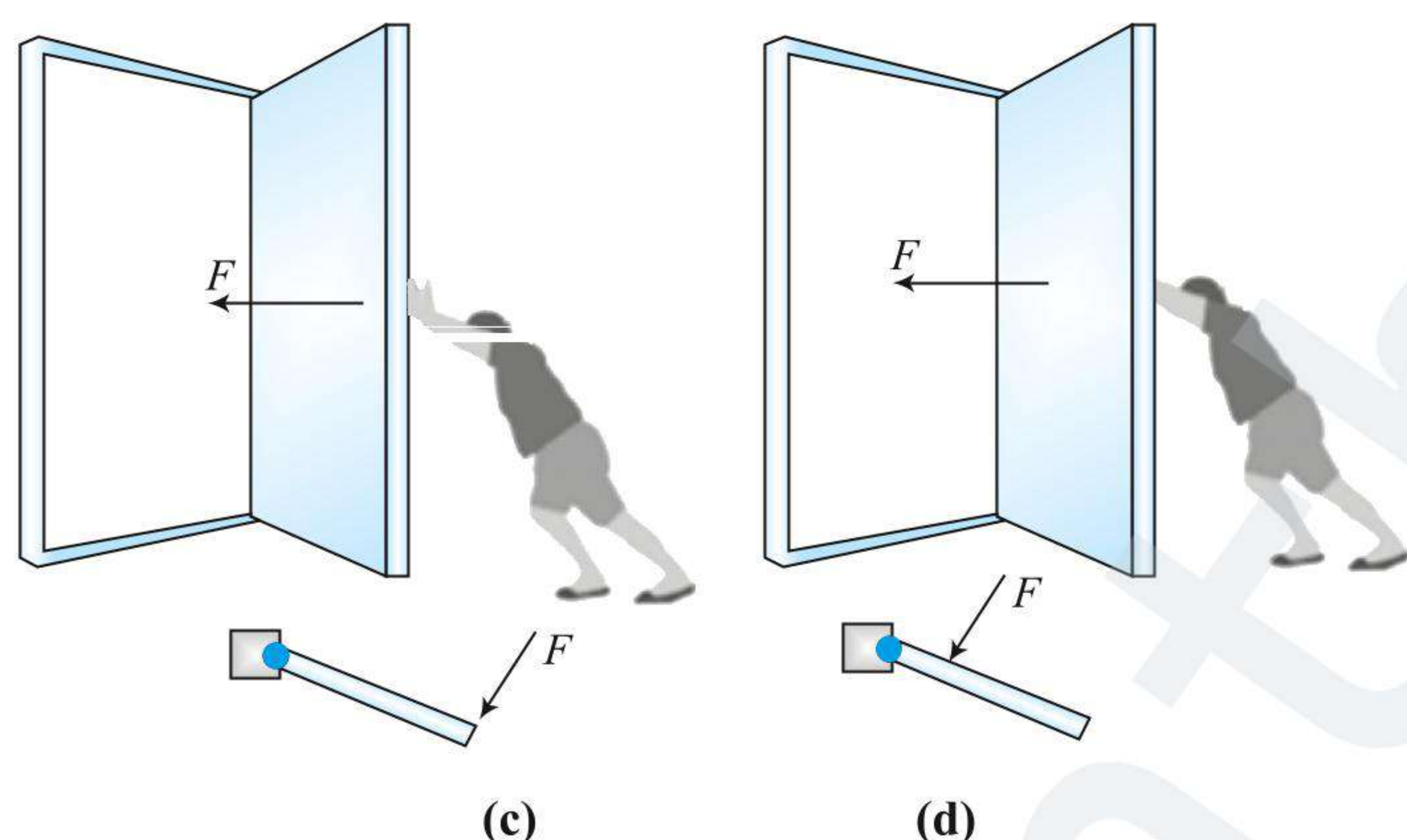
TORQUE

We have learned in particle dynamics, how a net force affects linear motion by causing an object to accelerate. Now in this section let us now analyze how a rigid object can also have an angular acceleration. A net external force causes linear motion to change, but what causes rotational motion to change? Is it simply the net force which produces angular acceleration? No, it is the torque which is responsible for angular motion. When a force is exerted on a rigid object pivoted about an axis, the object tends to rotate about that axis. The tendency of a force to rotate an object about some axis is measured by a quantity called **torque** $\vec{\tau}$ (Greek letter tau). The word 'Torque', which comes from the Latin word meaning "to twist," may be loosely identified as the turning or twisting action of the force $\vec{F}$. When you apply a force to an object—such as a screwdriver or torque wrench—with the purpose of turning that object, you are applying a torque.

If you want to open a heavy door, you must certainly apply a force, but that is not enough. Where you apply that force and in what direction you push are also important. When you push on a door with a force $\vec{F}$, as in Fig. (a), it is not possible to rotate the door, you will have a hard time in opening the door, because the torque is nearly zero. If you push door perpendicular to the surface of door as shown in Fig. (b), now you can rotate the door with ease.



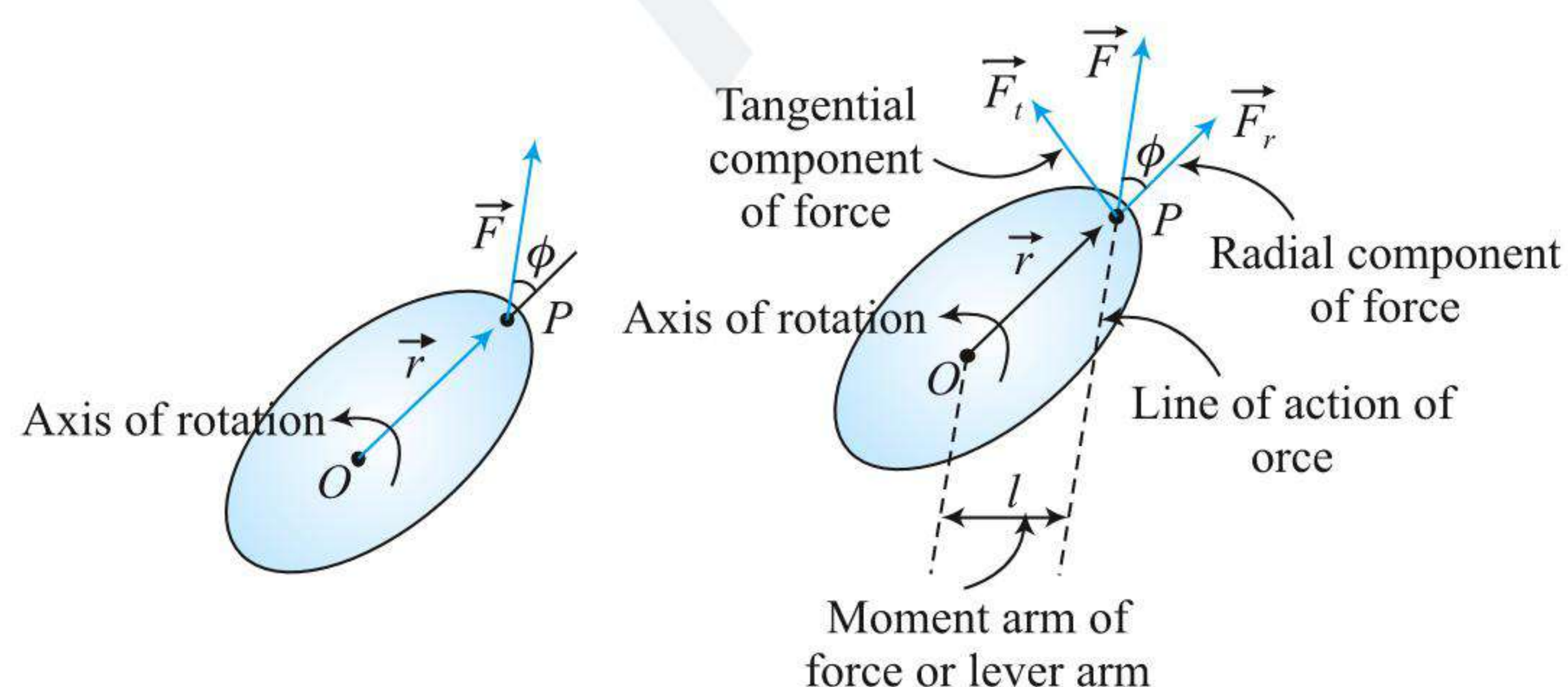
Now apply the force of larger magnitude Fig. (c), the door opens more quickly. In case of Fig. (c), other things being same as in case of Fig. (b), a larger force generates a larger torque. However, the door does not open as quickly if you apply the same force at a point closer to the hinge, as shown in Fig. (d), because the force now produces less torque. In summary, the torque depends on the magnitude of the force, on the point where the force is applied relative to the axis of rotation and on the direction of the force.



Two concepts that are important in the definition of torque, line of action of the force and lever arm. The **line of action** is an extended line drawn collinear with the force. The **lever arm** is the distance l between the line of action and the axis of rotation, measured on a line that is perpendicular to both.

CALCULATION OF TORQUE

Let us take a body which is free to rotate about an axis passing through O . Considering the cross-section of the body as shown in Fig. (a). A force $\vec{F}$ is applied at point P , whose position relative to O is defined by a position vector $\vec{r}$. The force $\vec{F}$ is in the plane of the page, the directions of vectors $\vec{F}$ and $\vec{r}$ make an angle ϕ with each other.



To determine how force ($\vec{F}$) results in a rotation of the body around the rotation axis, we resolve the force ($\vec{F}$) into two components as shown in Fig. (b). One component which acts along a line that extends through O and points along $\vec{r}$ is called the radial component has magnitude $F_r = F \cos \phi$. This component does not cause rotation. The other component of force ($\vec{F}$), called the *tangential component* F_t , is perpendicular to $\vec{r}$ and has magnitude $F_t = F \sin \phi$. This component *does* cause rotation.

The ability of $\vec{F}$ to rotate the body depends not only on the magnitude of its tangential component F_t , but also on how far from O the force is applied. To include both these factors, we define a quantity called **torque** τ as the product of the two factors and write it as

$$\tau = (r)(F \sin \phi) \quad \dots(i)$$

Two equivalent ways of computing the torque are:

$$\tau = (r)(F \sin \phi) = rF_t \quad \dots(ii)$$

$$\text{and } \tau = (r \sin \phi)(F) = lF = Fl \quad \dots(iii)$$

where l is the perpendicular distance between the rotation axis at O and an extended line running through the vector $\vec{F}$ as shown in Fig.(b), this extended line is called the **line of action** of $\vec{F}$, and l is called the **moment arm or lever arm** of $\vec{F}$.

Thus, Torque = Force $\times$ Perpendicular distance of line of action of force from the axis of rotation = Force $\times$ Moment arm

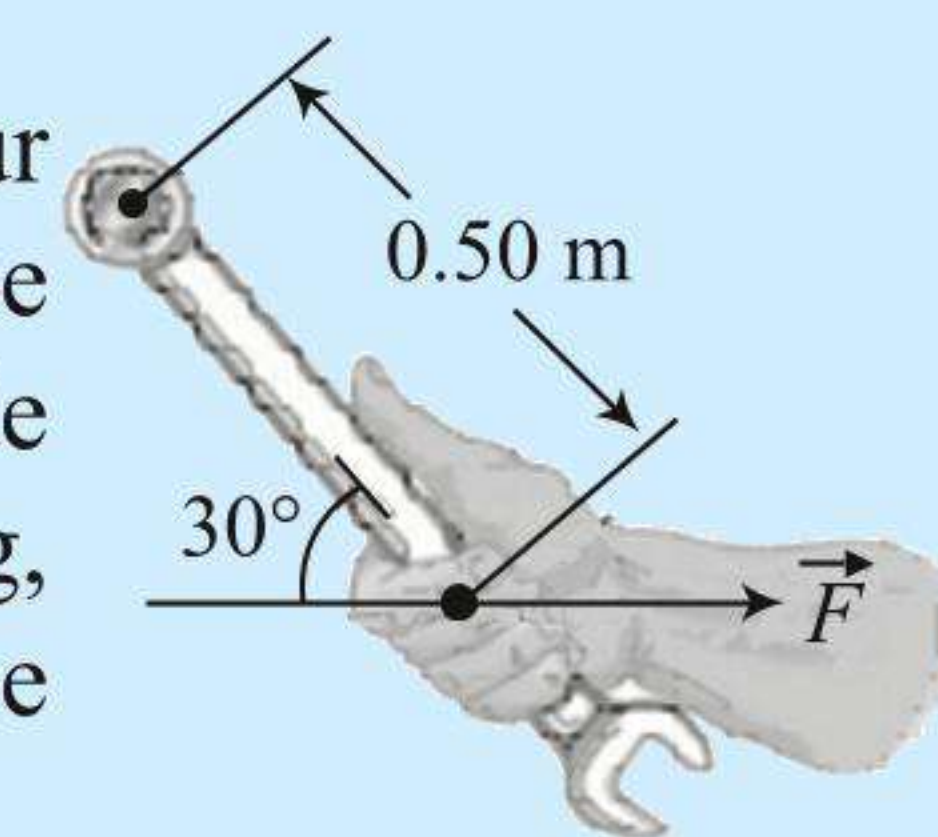
Torque is a vector quantity (rotational or axial vector) and its unit is newton-metre (N m) in SI. Its dimensional formula is $[L] \times [MLT^{-2}] = [ML^2T^{-2}]$, this dimensional formula is the same as for work. But the torque and work both are different physical quantities.

Though torque or moment of force is a vector quantity, but when we deal with rotation around a single axis, we can use only an algebraic sign. The torque or moment of the force is taken as positive, if it tends to rotate in anticlockwise direction and as negative, if it tends to rotate in clockwise direction.

Torques obey the superposition principle, when several torques act on a body, the net torque (or resultant torque) is the sum of the individual torques. The symbol for net torque is τ_{net} .

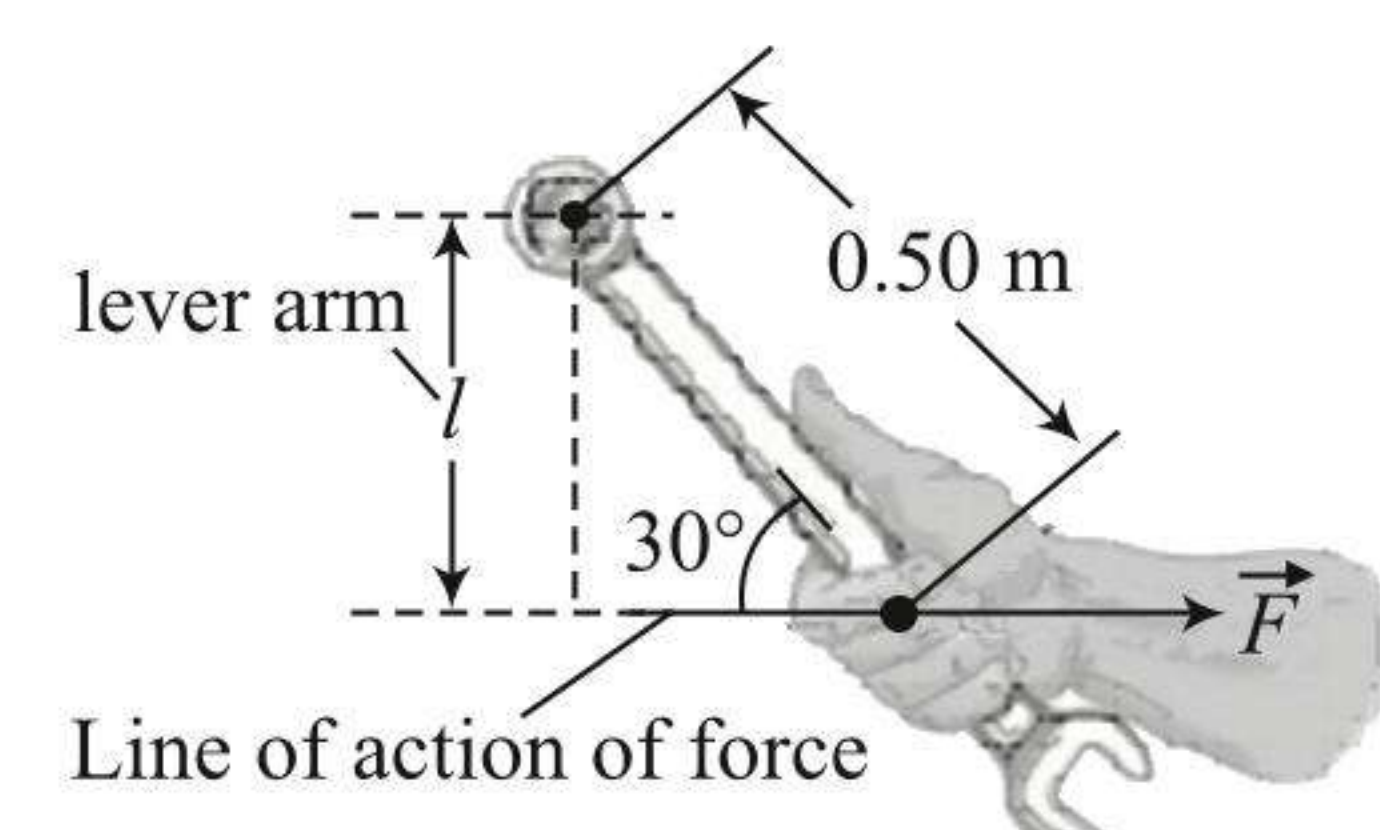
ILLUSTRATION 3.22

You are installing a new spark plug in your car, and the manual specifies that it will be tightened to a torque that has a magnitude of 50 N m. Using the data in the drawing, determine the magnitude F of the force that you must exert on the wrench.



Sol. The magnitude of torque applied to spark plug, $\tau = Fl$, where F is the magnitude of the applied force and l is the lever arm.

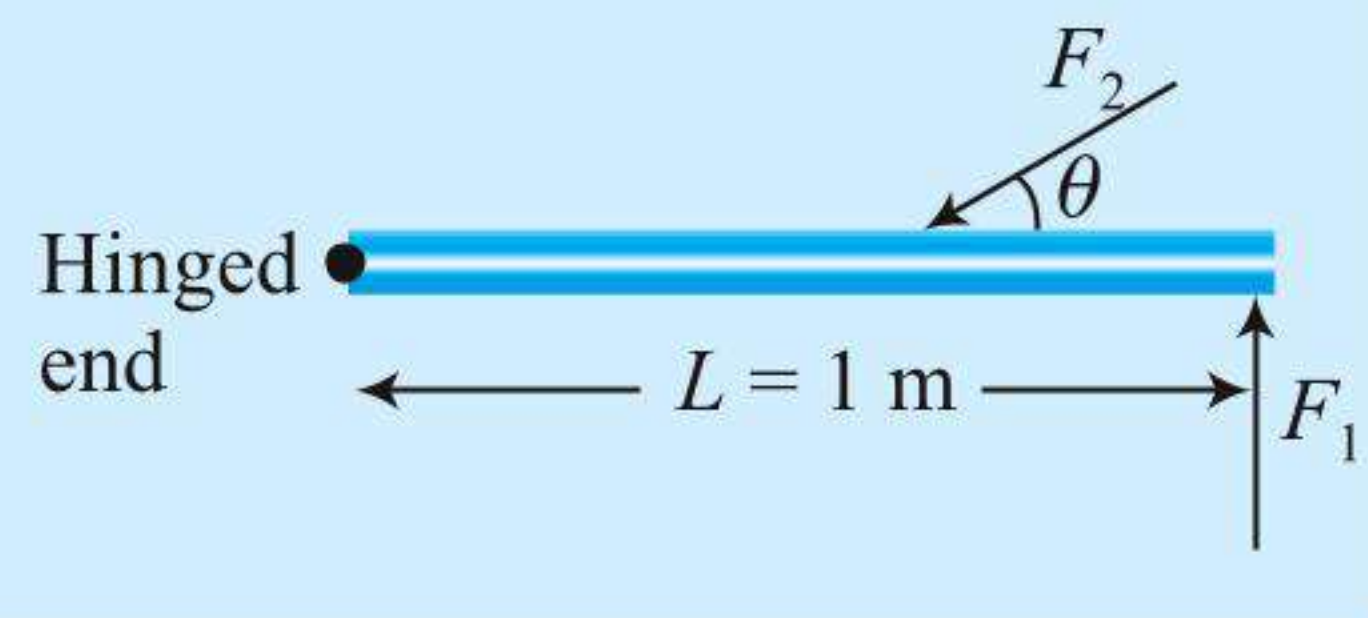
From the figure given, the lever arm is, $l = (0.50 \text{ m}) \sin 30^\circ$. Since both the magnitude of the torque and l are known, now solving for F .



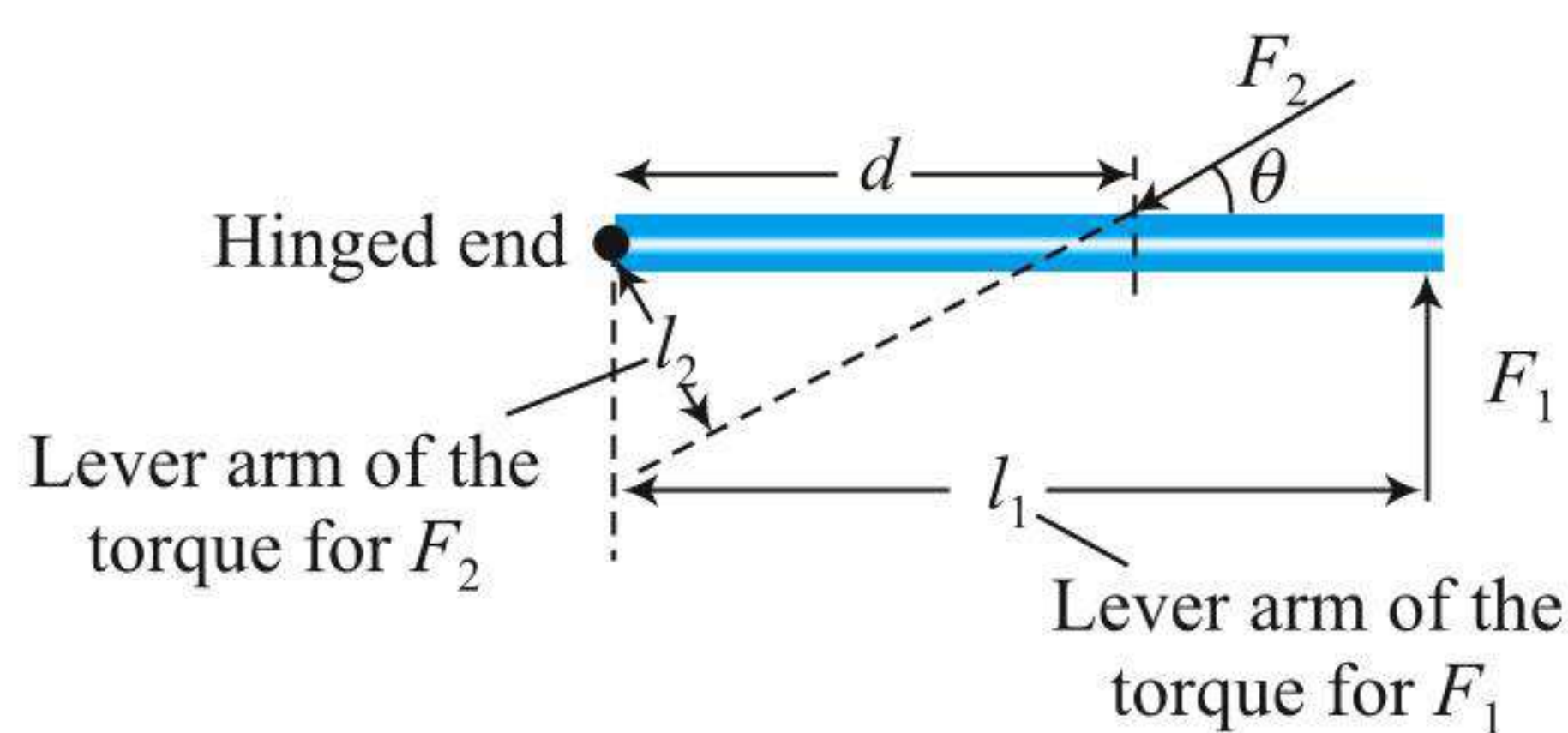
$$F = \frac{\text{Magnitude of torque}}{l} = \frac{50 \text{ N} \cdot \text{m}}{(0.50 \text{ m}) \sin 30^\circ} = 200 \text{ N}$$

ILLUSTRATION 3.23

A meter stick is placed on smooth table. One end of the meter stick is hinged to a table, so the stick can rotate freely in a plane parallel to the tabletop. Two forces are applied to the stick in such a way that the net torque is zero. The first force has a magnitude of $F_1 = 2$ N and is applied perpendicular to the length of the stick at the free end. The second force has a magnitude of $F_2 = 5$ N and acts at a 30° angle with respect to the length of the stick. Where along the stick is the 5 N force applied? Express this distance with respect to the end of the stick that is hinged.



Sol. If the torques generated by the two applied forces sum to zero, for this one of the forces must tend to produce a clockwise rotation about the hinged end of the meter stick, and the other must tend to produce a counterclockwise rotation about the hinged end. We can observe the first force, applied perpendicular to the length of the meter stick at its free end, tends to produce a counterclockwise rotation. We take counterclockwise torque as positive.



For net torque to be zero:

$$F_1 l_1 - F_2 l_2 = 0 \quad \text{or} \quad F_1 l_1 = F_2 l_2 \quad \dots(i)$$

From the figure that the lever arm l_1 of the force F_1 is equal to the length of the meter stick: $L = 1$ m. The force F_2 is applied a distance d from the hinged end of the meter stick.

The lever arm l_2 of the second force is,

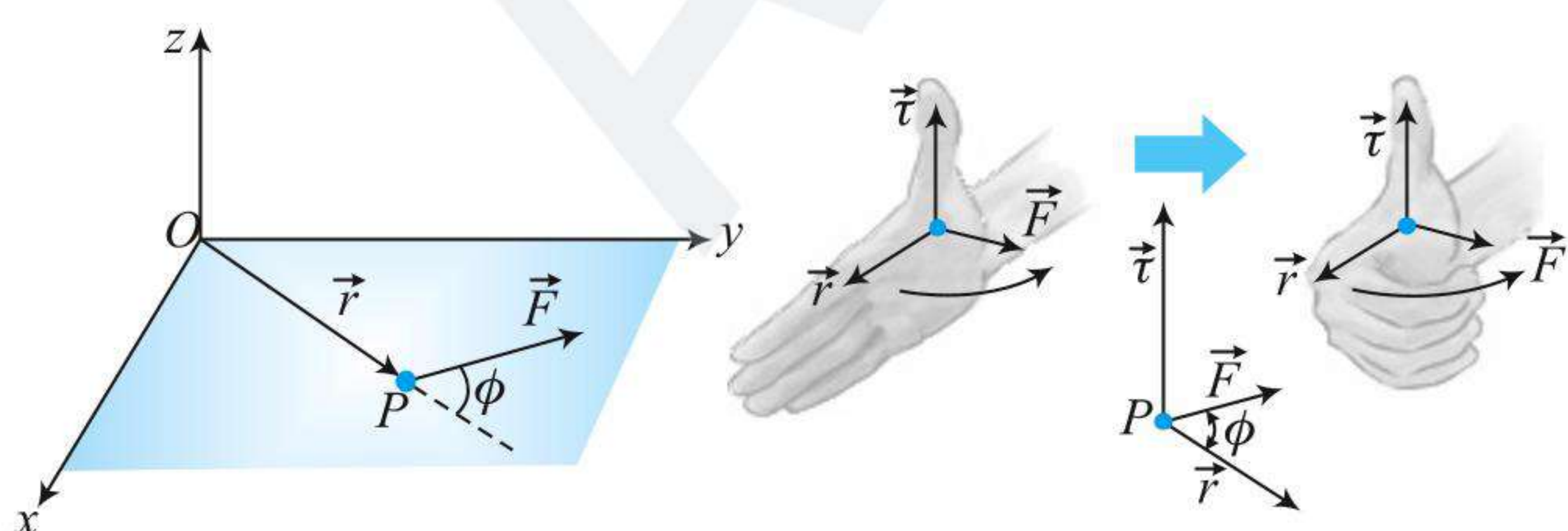
$$l_2 = d \sin \theta \quad \dots(ii)$$

Substituting Eq. (ii) into Eq. (i) and solving for the distance d , we obtain

$$F_1 l_1 = F_2 (d \sin \theta) \quad \text{or} \quad d = \frac{F_1 l_1}{F_2 \sin \theta} = \frac{(2\text{N})(1\text{m})}{(5\text{N}) \sin 30^\circ} = 0.8 \text{ m}$$

Expressing Torque in Vector Form

Figure (a) shows a particle at point P in an x - y plane. A single force $\vec{F}$ in that plane acts on the particle, and the particle's position relative to the origin O is given by position vector $\vec{r}$



The torque $\vec{\tau}$ acting on the particle relative to the fixed-point O is a vector quantity defined as

$$\vec{\tau} = \vec{r} \times \vec{F} \quad \dots(i)$$

To find the direction of $\vec{\tau}$, we use the right-hand rule in Fig. (b), sweeping the fingers of the right hand from $\vec{r}$ (the first vector in the product) into $\vec{F}$ (the second vector). The outstretched right thumb then gives the direction of $\vec{\tau}$. In Fig. (b), it is in the positive direction of the z axis.

$$\text{If } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \text{ and } \vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

The cross product of any two vectors $\vec{A}$ and $\vec{B}$ can be expressed in the following determinant form:

$$\vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} y & z \\ F_y & F_z \end{vmatrix} \hat{i} - \begin{vmatrix} x & z \\ F_x & F_z \end{vmatrix} \hat{j} + \begin{vmatrix} x & y \\ F_x & F_y \end{vmatrix} \hat{k}$$

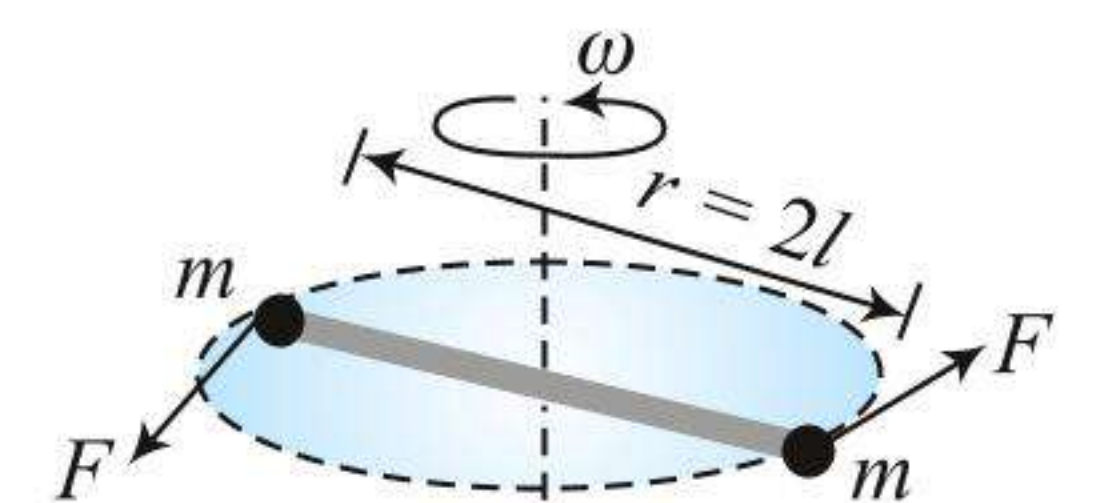
Expanding these determinants gives the result,

$$\vec{\tau} = \vec{r} \times \vec{F} = (yF_z - zF_y)\hat{i} - (xF_z - zF_x)\hat{j} + (xF_y - yF_x)\hat{k}$$

COUPLE

Two equal and opposite forces whose lines of action do not coincide make a couple.

Moment of a couple is given by $\tau = r_\perp F$, where $r_\perp$ is the separation of the lines of action of the forces and F is the magnitude of either of the forces.



Two balls each of mass m are fixed at the ends of a rigid body of length $2l$

EQUILIBRIUM OF RIGID BODIES

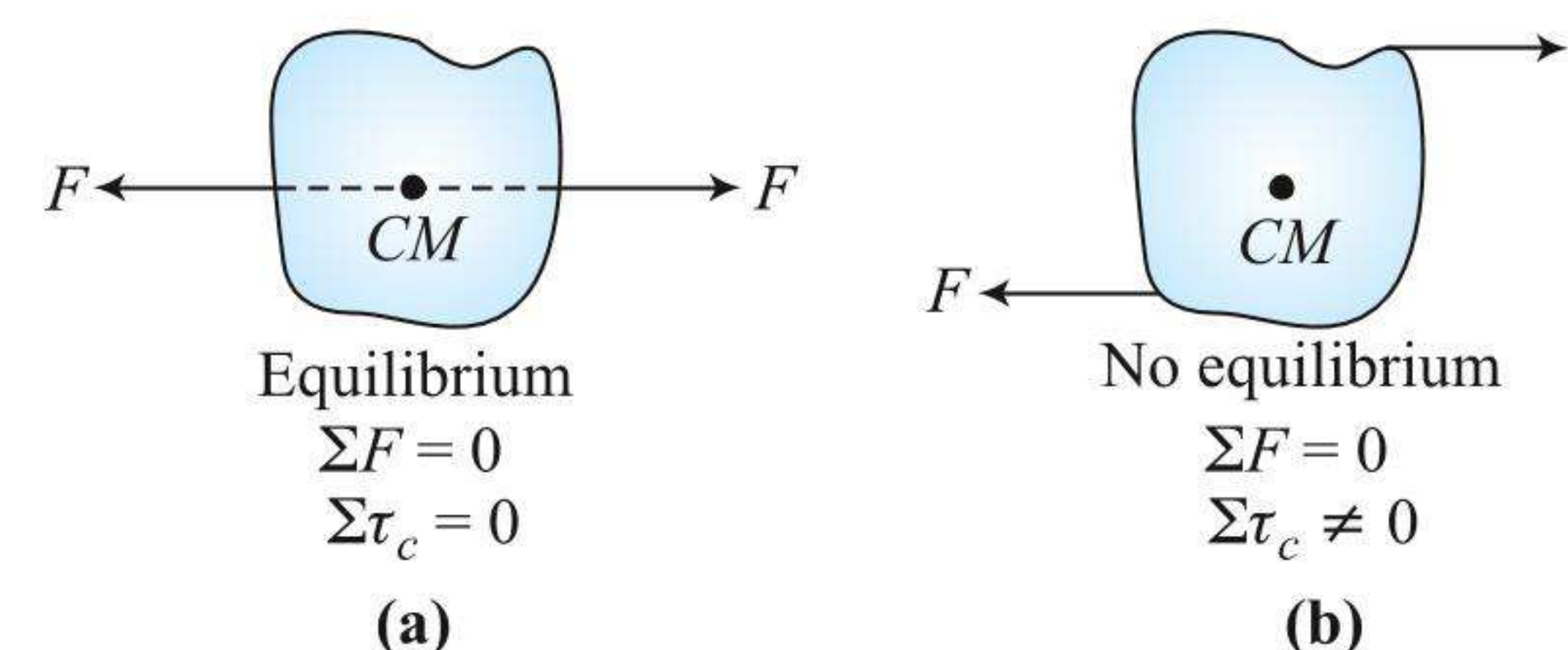
The conditions of equilibrium for a rigid body are different from that of a particle. Unlike particles, rigid bodies have tendency to rotate, therefore, one must consider the rotational equilibrium also, in addition to the translational equilibrium.

The equations of equilibrium (in two dimensions) for a rigid body are stated as:

$$\text{Translational equilibrium: } \sum \vec{F} = 0$$

$$\text{Rotational equilibrium: } \sum \vec{\tau}_c = 0$$

Note that for rotational equilibrium of a rigid body, the net torque about its centre of mass must be zero. The body have both translational and rotational equilibria.



Action of two forces on a particle and a rigid body is shown in Figs. (a) and (b).

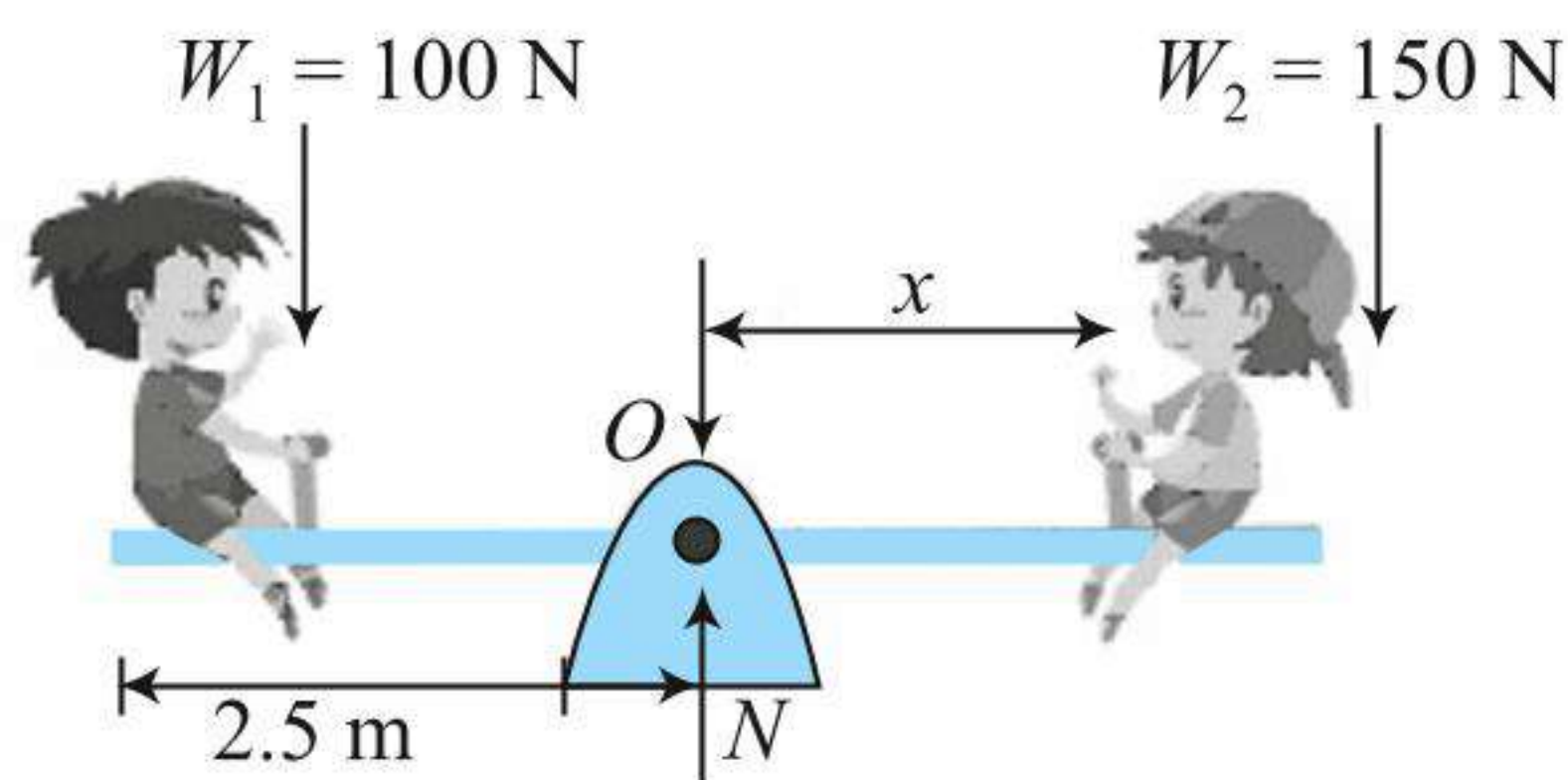
A particle is in equilibrium under the action of two equal and opposite forces, but a rigid body may or may not be in equilibrium under the action of two equal and opposite forces.

The body is in translational equilibrium but not necessarily in rotational equilibrium.

ILLUSTRATION 3.24

Two small kids weighing 10 kg and 15 kg, respectively, are trying to balance a seesaw of total length 5.0 m, with the fulcrum at the centre. If one of the kids is sitting at an end, where should the other sit?

Sol. If the kid of mass 15 kg sits at the end, he will produce maximum torque about fulcrum. The seesaw will not be balance for any position of kid of mass 10 kg. The 10 kg kid should sit at the end and the 15 kg kid should sit closer to the centre. Suppose his distance from the centre is x .



As the kids are in equilibrium, the normal force between a kid and the seesaw equals the weight of that kid. Considering the rotational equilibrium of the seesaw, the torque of the forces acting on it should add to zero. The forces are as follows:

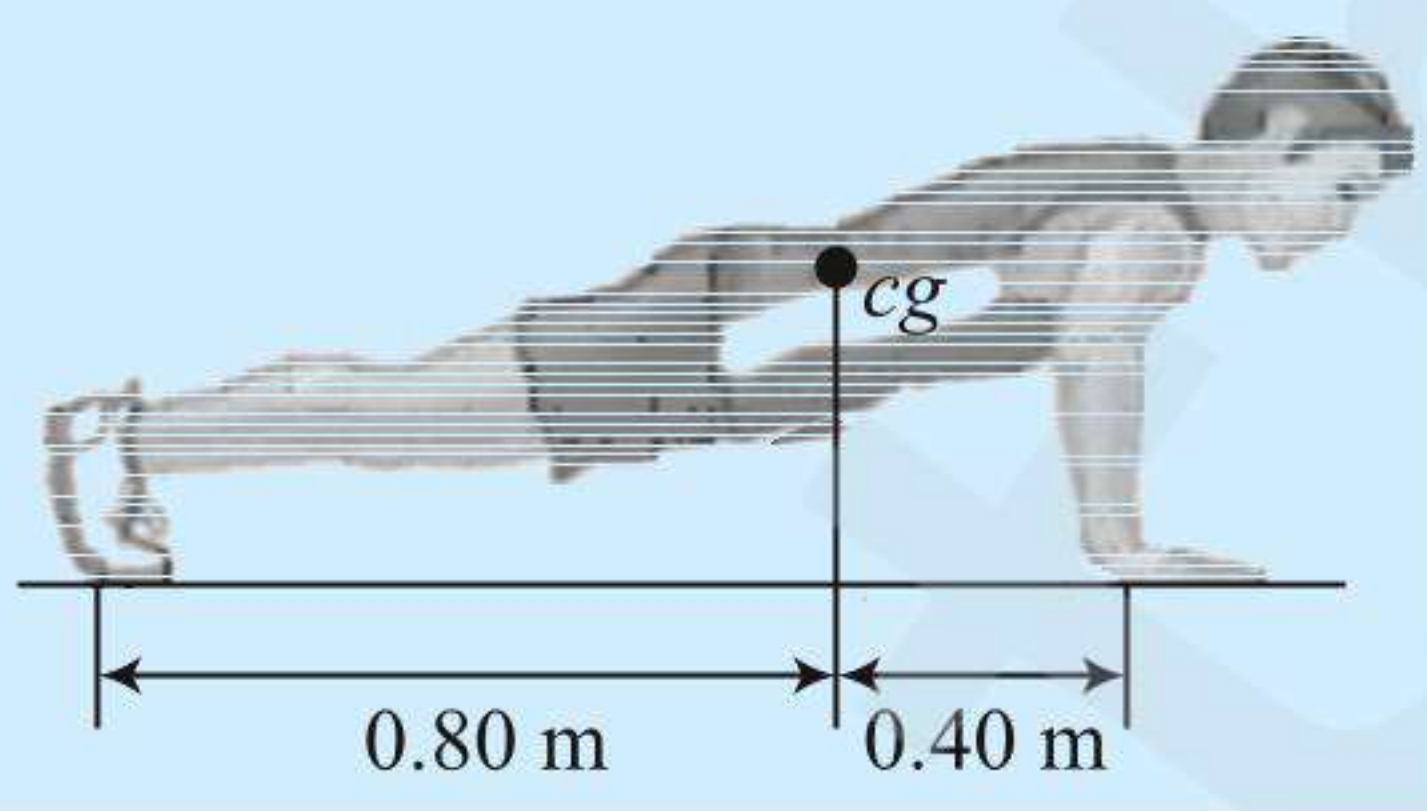
- (a) $(15 \text{ kg})g = 150 \text{ N}$ downwards by the 15 kg kid,
- (b) $(10 \text{ kg})g = 100 \text{ N}$ downwards by the 10 kg kid,
- (c) weight of the seesaw, and
- (d) the normal force by the fulcrum.

Taking torques about the fulcrum,

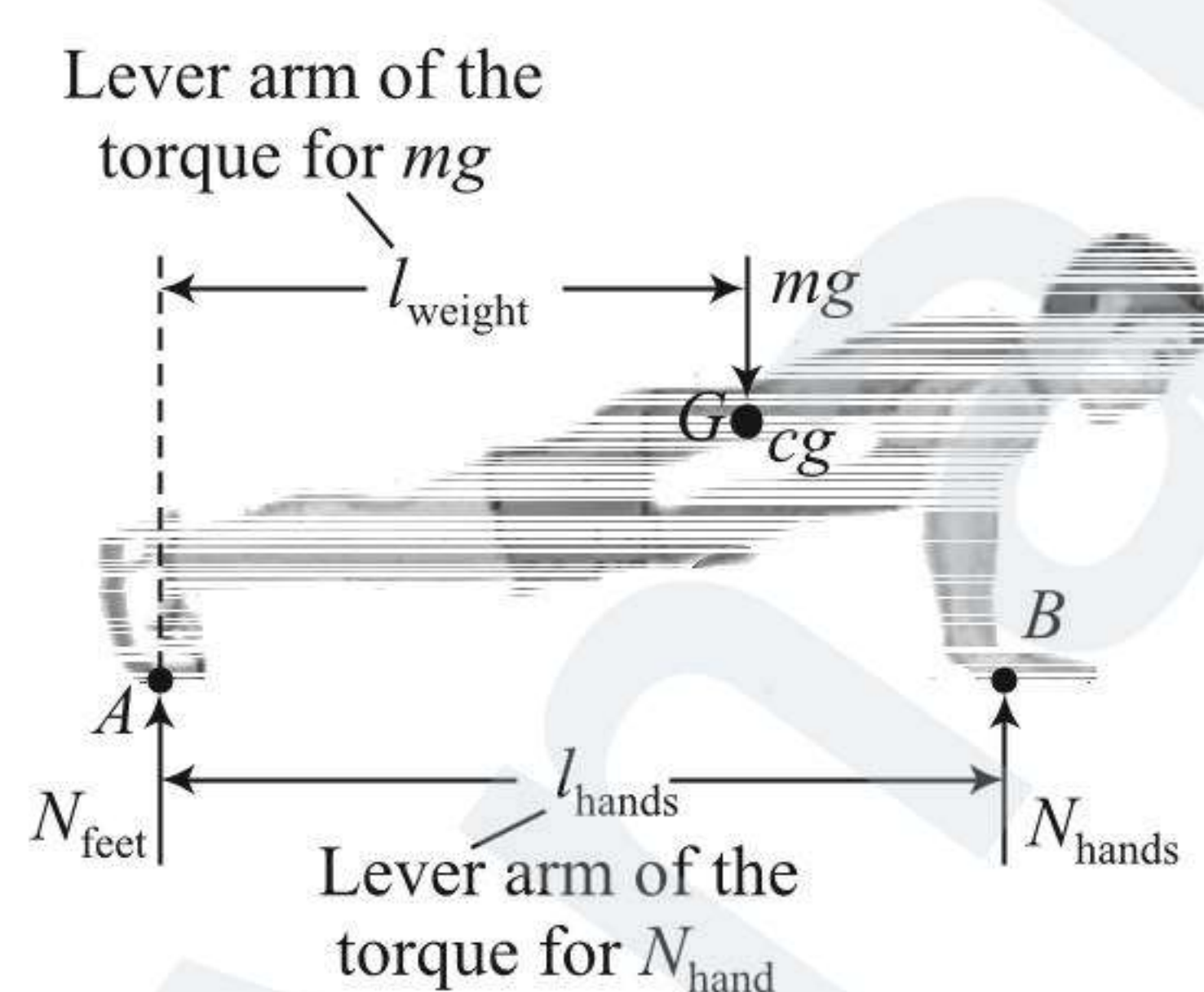
$$150 \times x = 100 \times 2.5 \text{ or } x = 1.7 \text{ m}$$

ILLUSTRATION 3.25

In the given figure, a man of mass 60 kg is doing push-ups. Find the normal force exerted by the floor on each hand and each foot, if the man holds this position.



Sol. The man is in equilibrium, the sum of the forces acting on the man must be zero.



From the free body diagram of man, we have,

$$N_{\text{Feet}} + N_{\text{Hands}} = mg \quad \dots(i)$$

Let us consider a rotational axis perpendicular to floor and passing through point A where the man's toes touch the floor. The lever arms for the torques are shown in figure. As the man is in equilibrium, we know that the sum of the torques about any axis should also be zero.

$$\text{We have, } mg \times l_{\text{Weight}} - N_{\text{Hands}} l_{\text{Hands}} = 0$$

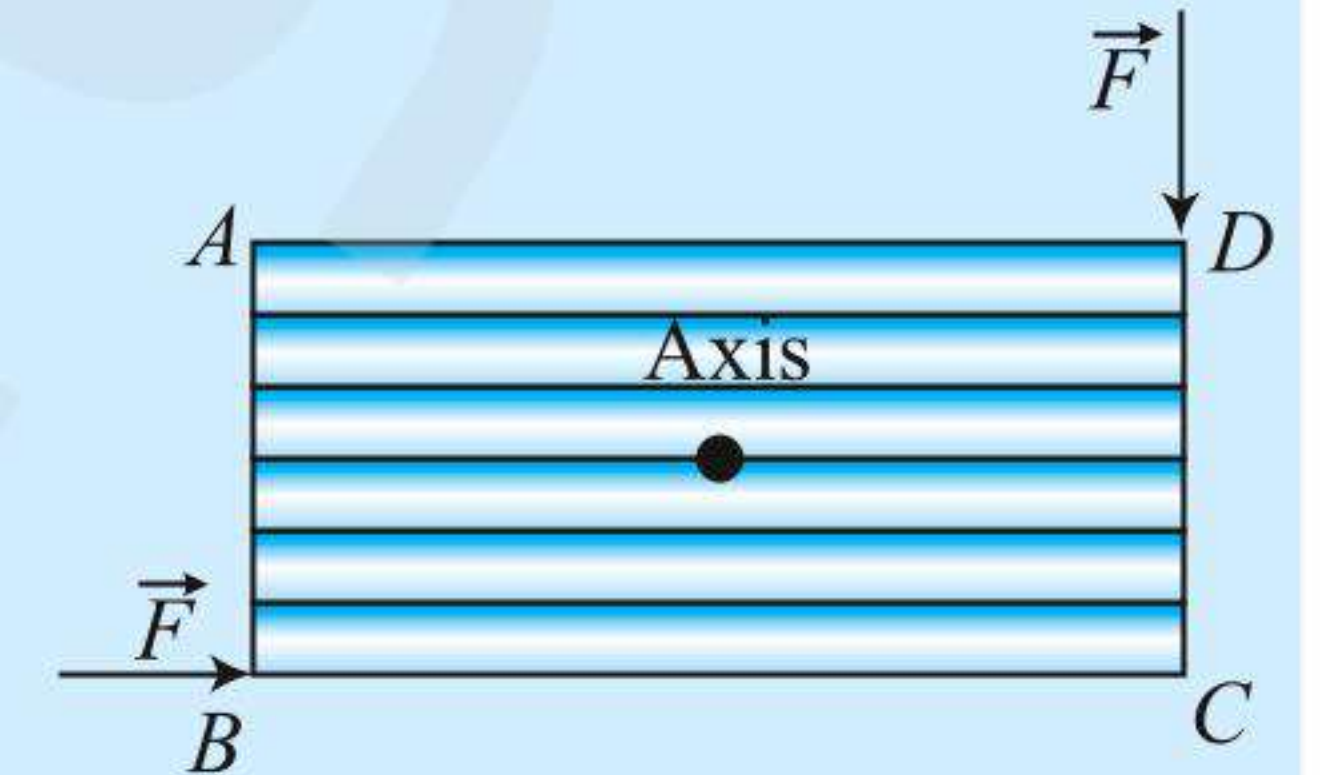
$$\Rightarrow N_{\text{Hands}} = \frac{mgl_{\text{Weight}}}{l_{\text{Hands}}} = \frac{(60 \times 10 \text{ N})(0.80 \text{ m})}{(0.80 \text{ m} + 0.40 \text{ m})} = 400 \text{ N}$$

$$\text{Hence, } N_{\text{Feet}} = mg - N_{\text{Hands}} = 600 \text{ N} - 400 \text{ N} = 200 \text{ N}$$

The force on each hand is half the value calculated above, i.e., 200 N. Likewise, the force on each foot is half the value calculated above, i.e., 100 N.

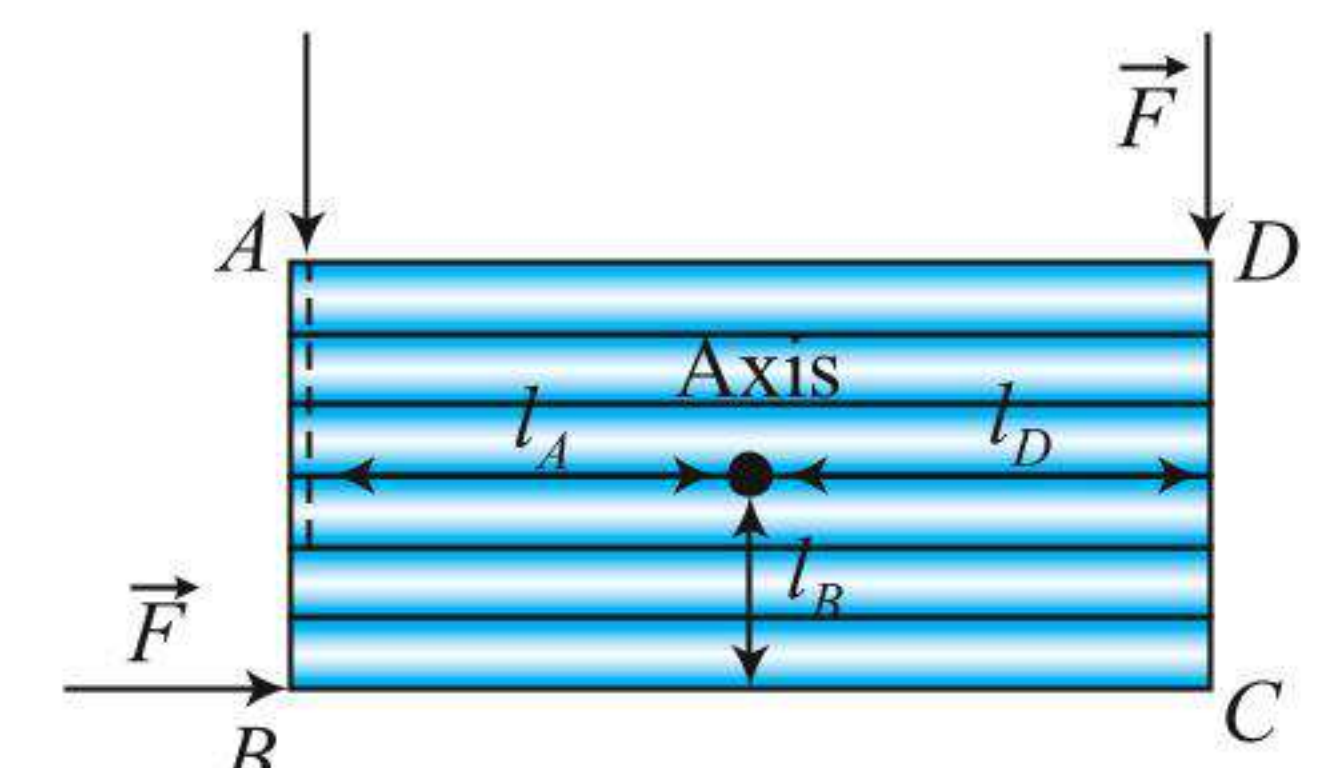
ILLUSTRATION 3.26

In the figure shown a rectangular piece of wood is pinned at its centre. The forces applied to corners B and D have the same magnitude of 20 N and are directed parallel to the long and short sides of the rectangle. The long side of the rectangle is twice the short side. A third force (not shown in the figure) is applied to corner A , directed along the short side of the rectangle (either toward B or away from B), such that the rectangle is at equilibrium. Find the magnitude and direction of the force applied to corner A .



Sol. The net torque acting on rectangle is the sum of the torques produced by the three forces acting at points A , B and D . We are considering the torque that tends to produce a counterclockwise rotation about the axis as positive torque. As the piece of wood is at equilibrium, the net torque is equal to zero.

Let us consider L be the length of the short side of the rectangle, so that the length of the long side is $2L$. The lever arms for the forces acting at A , B and D are shown in figure. The



counterclockwise torque produced by the force at corner B is $+Fl_B$, and the clockwise torque produced by the force at corner D is $-Fl_D$. Assume the force at A (directed along the short side of the rectangle) points toward corner B , the counterclockwise torque produced by this force is $+F_A l_A$. As the net torque about the axis shown should be equal to zero:

$$\Sigma \tau = F_A l_A + Fl_B - Fl_D = 0$$

$$\Rightarrow F_A L + (20 \text{ N})\left(\frac{1}{2}L\right) - (20 \text{ N})L = 0$$

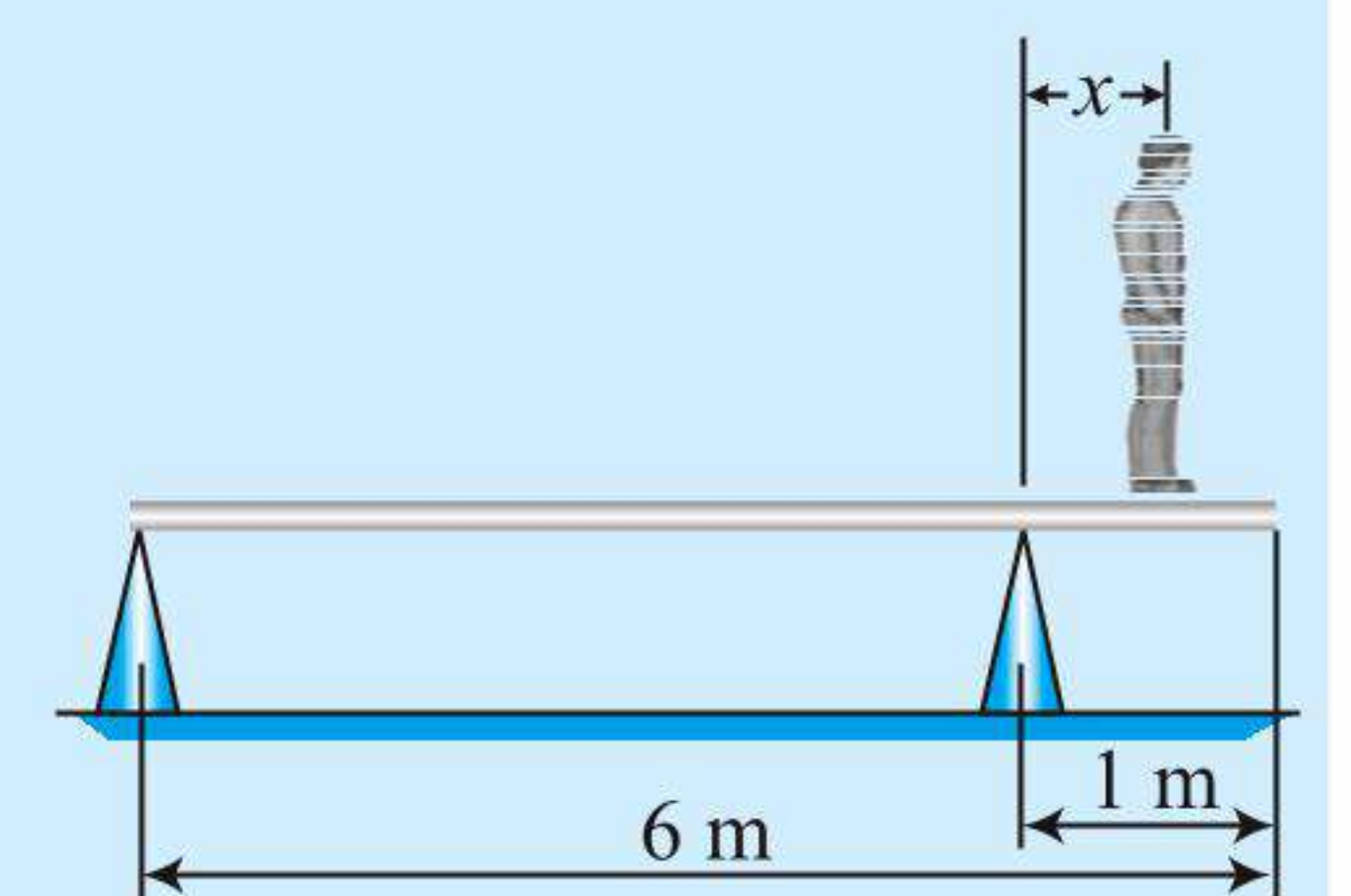
The length L can be eliminated algebraically from this result, which can then be solved for F_A :

$$F_A = -(20 \text{ N})\left(\frac{1}{2}\right) + 20 \text{ N} = 10 \text{ N}$$

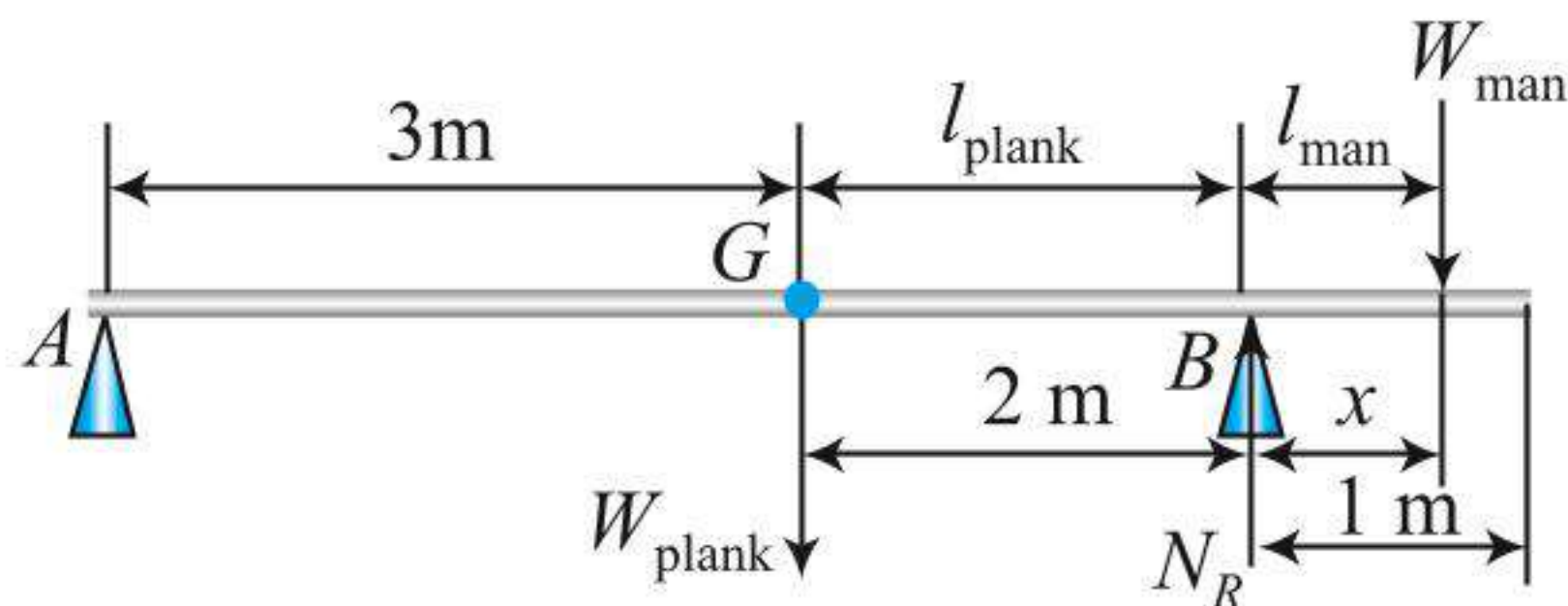
Here the value calculated for F_A is positive, our assumption that F_A points toward corner B must have been correct. Otherwise, the result for F_A would have been negative.

ILLUSTRATION 3.27

In the given figure, a uniform plank of length 6 m and weight 200 N rests horizontally on two supports, with 1 m of the plank hanging over the right support. To what distance can a person who weighs 800 N walk on the overhanging part of the plank before it just begins to tip?



Sol. At every instant before the plank begins to tip, it is in equilibrium, and the net torque on it is zero: $\Sigma \tau = 0$. When the person reaches the maximum distance x along the overhanging part, the plank is just about to rotate about the right support. At that instant, the plank loses contact with the left support, the normal reaction at this point will become zero. This leaves only three vertical forces acting on the plank: the weight W_{plank} of the plank, the normal reaction N_R due to the right support, and the weight of the man W_{man} .



Let us take torque about point B

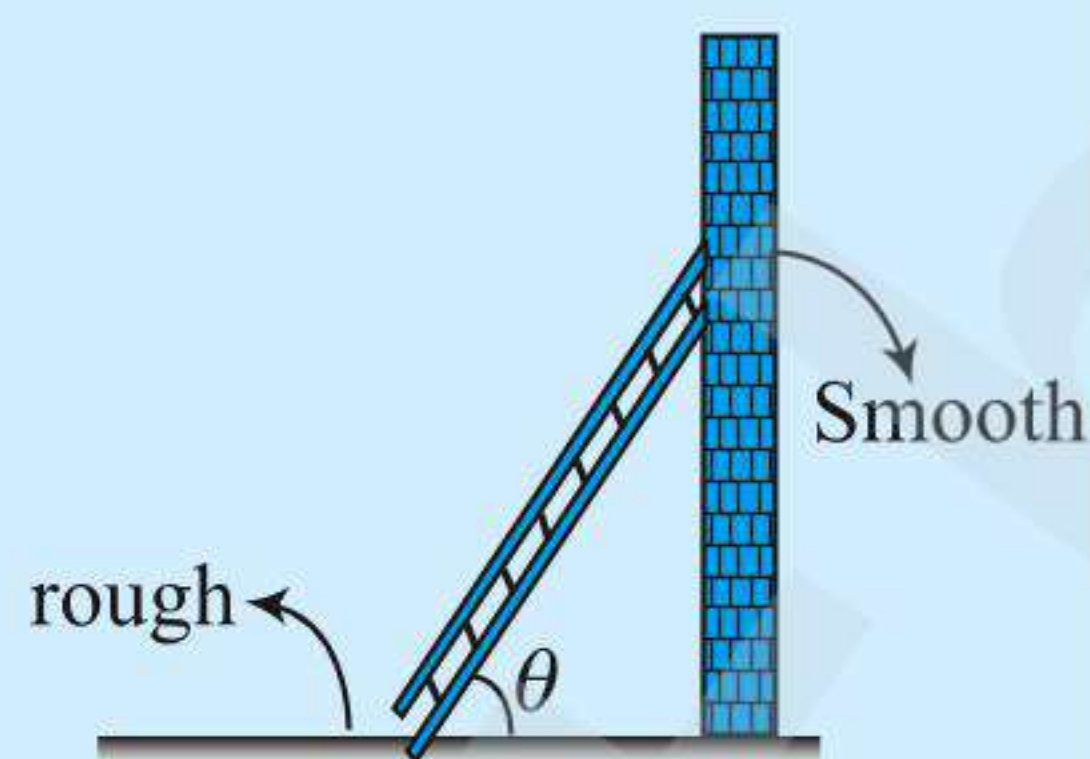
$$\Sigma \tau = W_{\text{plank}} l_{\text{plank}} - W_{\text{man}} l_{\text{man}} = 0 \quad \text{or} \quad l_{\text{man}} = x = \frac{W_{\text{plank}} l_{\text{plank}}}{W_{\text{man}}} \quad \dots(i)$$

Here, $l_{\text{plank}} + 1 = \frac{6}{2} \text{ m}$ or $l_{\text{plank}} = 2 \text{ m}$

Hence from Eq. (i) $x = \frac{(200 \text{ N})(2 \text{ m})}{800 \text{ N}} = 0.50 \text{ m}$

ILLUSTRATION 3.28

A ladder of mass M and length L is supported in equilibrium against a smooth vertical wall and a rough horizontal surface, as shown in figure. If θ be the angle of inclination of the rod with the horizontal, then calculate:

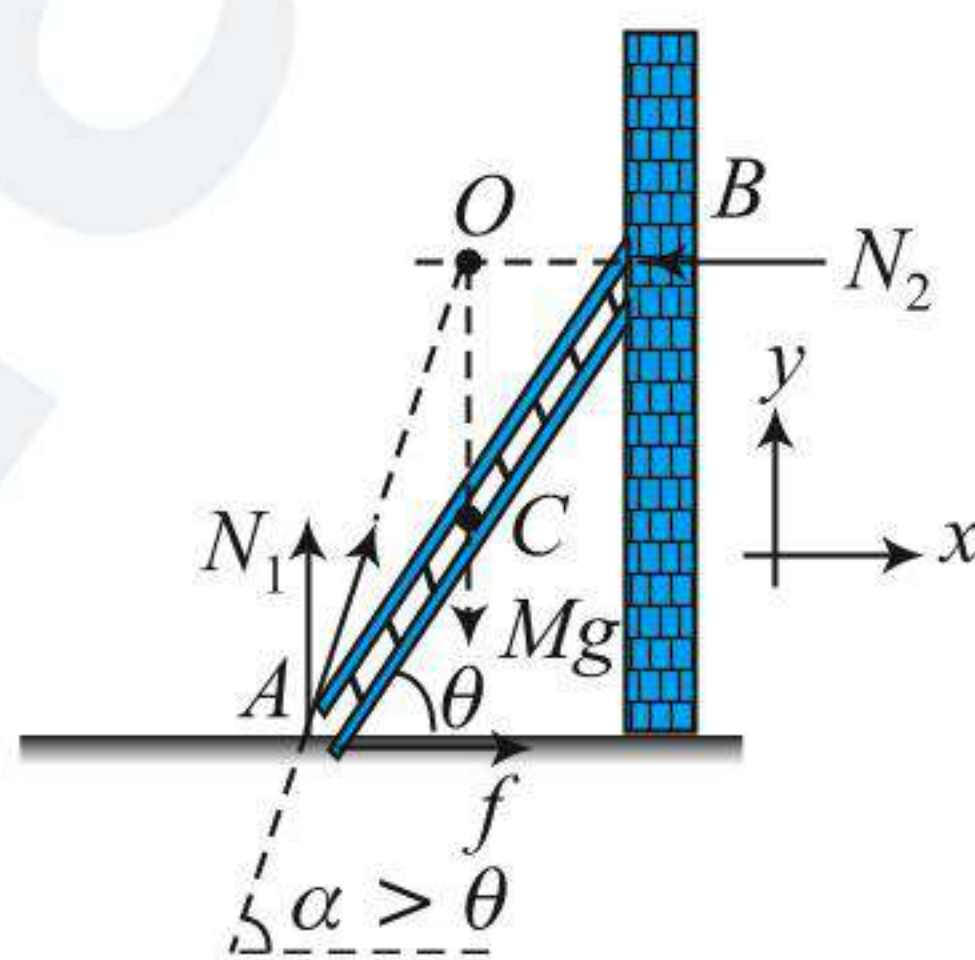


- normal reaction of the wall on the ladder,
- normal reaction of the ground on the ladder,
- friction force between the ground and the ladder and
- net force applied by the ground on the ladder.

Sol. The forces acting on the ladder are its weight, contact force of wall (normal reaction) and contact force of ground (normal reaction and friction). The free-body diagram of the ladder is shown in figure.

These forces acting on the ladder should pass through a common point O , otherwise it cannot be in equilibrium.

The total reaction force F from the horizontal surface is inclined at an angle α ($> \theta$) to the horizontal. The horizontal component of this force is friction force and its vertical component is the normal reaction applied by the ground on the ladder.



Applying the conditions of equilibrium, we get

$$\Sigma F_x = 0 \Rightarrow f - N_2 = 0$$

$$f = N_2 \quad \dots(i)$$

$$\Sigma F_y = 0 \Rightarrow N_1 - Mg = 0 \Rightarrow N_1 = Mg \quad \dots(ii)$$

Taking torque about centre of the rod 'C',

$$(N_1) \frac{L}{2} \cos \theta - (f) \frac{L}{2} \sin \theta - N_2 \frac{L}{2} \sin \theta = 0 \quad \dots(iii)$$

Substituting the value of N_1 and f from Eqs. (i) and (ii), we get

$$Mg \frac{L}{2} \cos \theta - N_2 L \sin \theta = 0$$

$$\Rightarrow N_2 = \frac{Mg}{2 \tan \theta} = \frac{1}{2} Mg \cot \theta = f \quad \dots(iv)$$

It is not compulsory to take torque about centre of mass, we can get the same result if we take torque about A.

As ladder is in equilibrium, the net torques about A will be zero. $\Sigma \tau_A = 0$, taking torque about end A of the rod.

$$Mg \left(\frac{L}{2} \cos \theta \right) - N_2 (L \sin \theta) = 0$$

$$N_2 = \frac{Mg}{2 \tan \theta} = \frac{1}{2} Mg \cot \theta = f$$

It is the same result as we get by taking torque about C.

Thus, the net force applied by ground on the ladder is

$$F = \sqrt{N_1^2 + f^2}$$

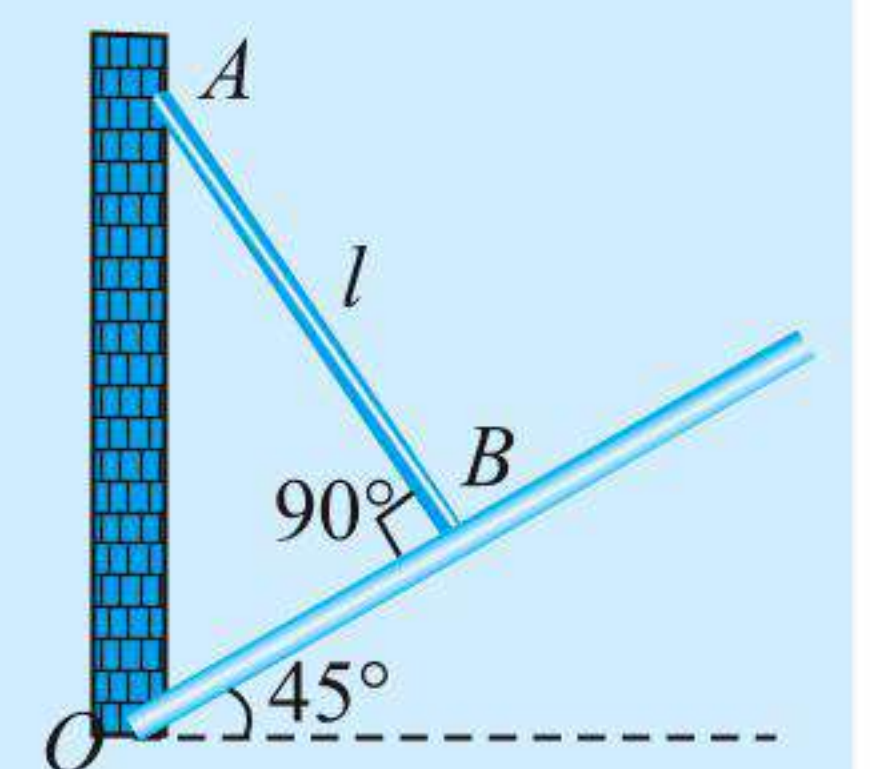
$$F = \frac{1}{2} Mg \sqrt{4 + \cot^2 \theta}$$

Results:

$$N_2 = f = \frac{1}{2} Mg \cot \theta, \quad N_1 = Mg, \quad F = \frac{1}{2} Mg \sqrt{4 + \cot^2 \theta}$$

ILLUSTRATION 3.29

At the bottom edge of a smooth wall, an inclined plane is kept at an angle of 45° . A uniform rod of length l and mass M rests on the inclined plane against the wall such that it is perpendicular to the incline. What is the minimum coefficient of friction necessary so that the rod does not slip on the incline?



Sol. For finding the direction of frictional force, let us assume there is no friction anywhere. If we observe from FBD of the rod, the normal reaction of the wall has tendency to rotate the rod in clockwise sense. It means end 'B' of the rod has tendency to slide in downward direction on the inclined plane. Hence friction should act in upward direction.

The free-body diagram of the rod is shown in figure.

Balancing torque about point A,

$$fl = Mg \left(\frac{l}{2} \sin 45^\circ \right)$$

$$\therefore f = \frac{Mg}{2\sqrt{2}} \quad \dots(i)$$

Balancing forces in the vertical direction,

$$Mg = N_2 \cos 45^\circ + f \sin 45^\circ \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$N_2 = \frac{3Mg}{2\sqrt{2}} \quad \dots(iii)$$

From Eqs. (i) and (iii), $f \leq \mu N_2$

$$\text{or} \quad \frac{Mg}{2\sqrt{2}} = \mu_{\min} \left(\frac{3Mg}{2\sqrt{2}} \right) \quad \text{or} \quad \mu_{\min} = \frac{1}{3}$$

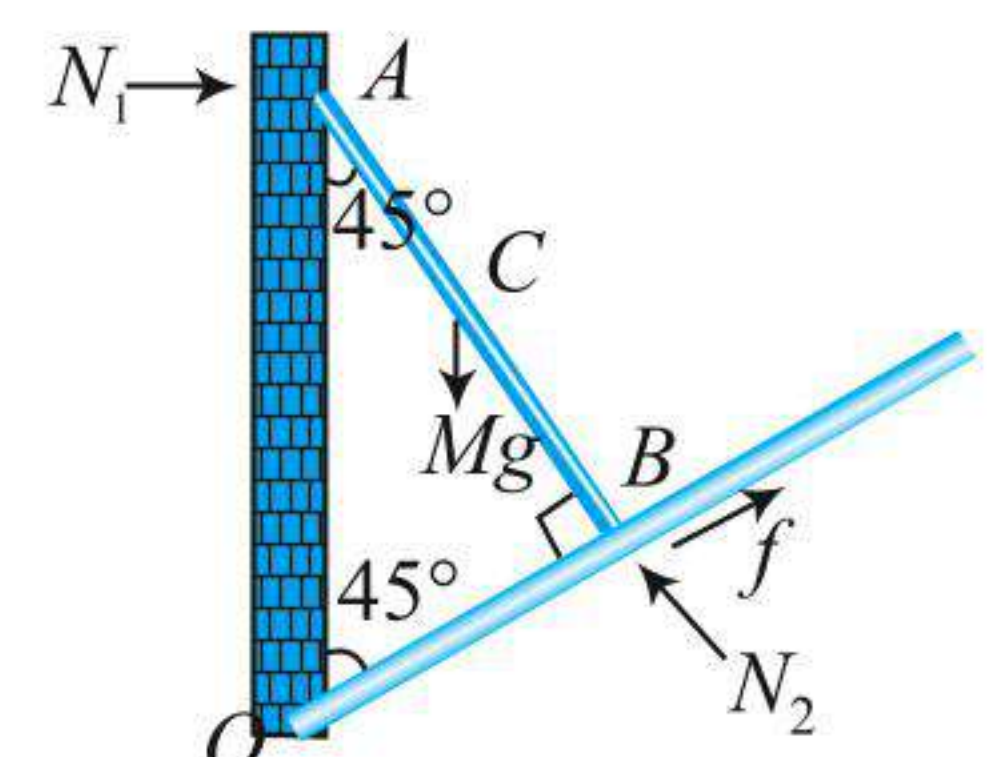
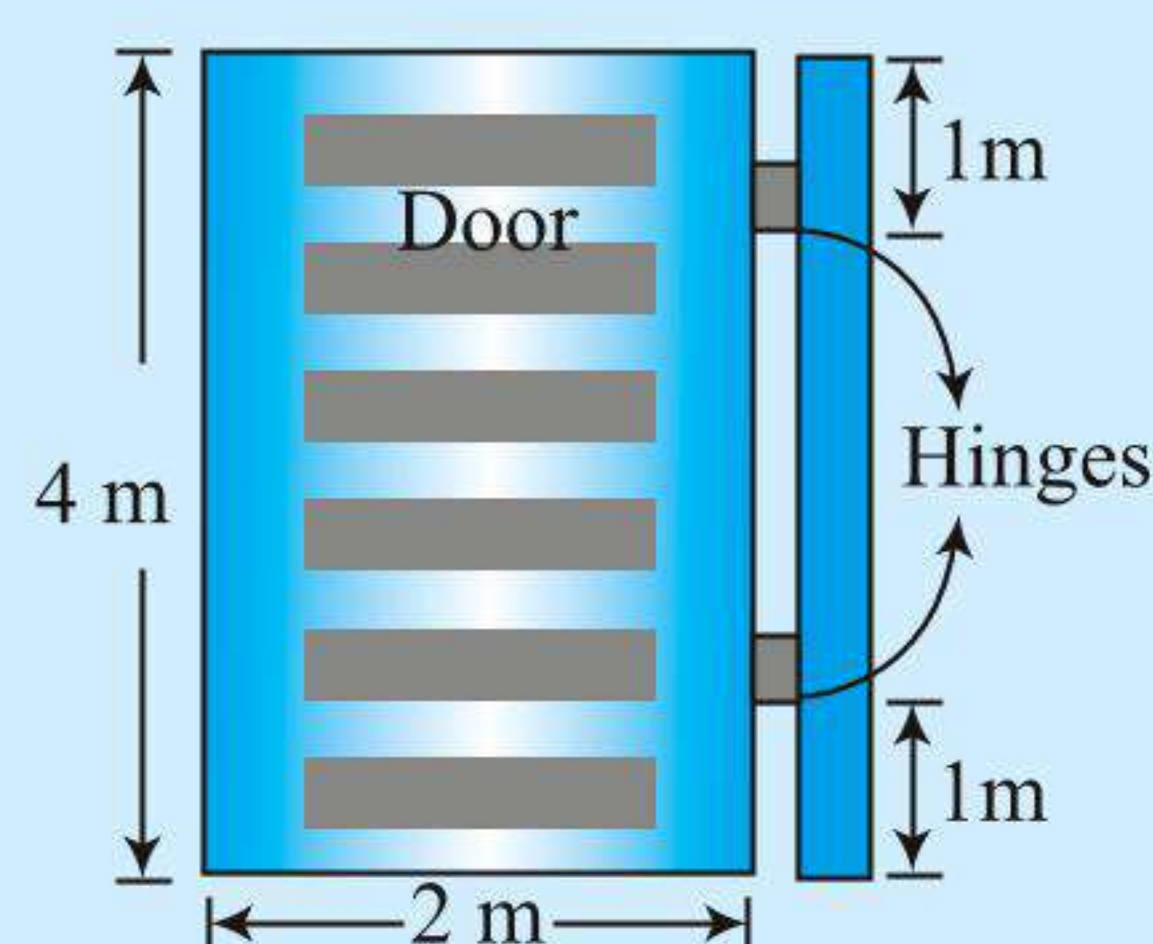
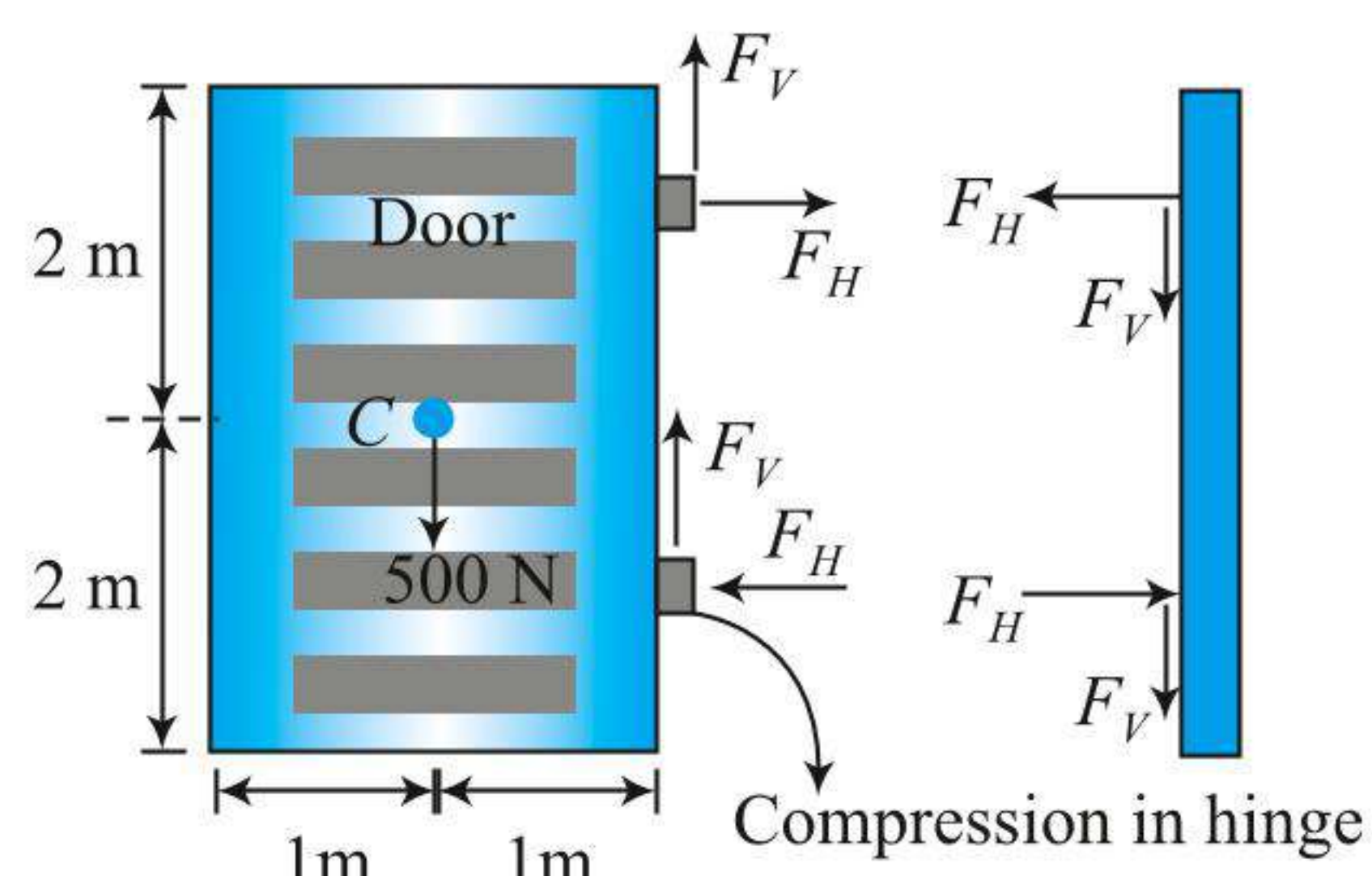


ILLUSTRATION 3.30

The door of an almirah is 4 m high, 2 m wide and weighs 50 kg. The door is supported by two hinges situated 1.0 m from the ends. If the magnitude of forces exerted by the hinges on the door is equal, find this force.



Sol. The free-body diagram of the door is shown in figure.



The forces exerted on the hinges are equal in magnitude. The door is in equilibrium in horizontal as well as in vertical direction, i.e., the horizontal and vertical components of the forces will be equal. For vertical equilibrium of the door, we have

$$2F_v = 500 \Rightarrow F_v = 250 \text{ N}$$

For rotational equilibrium of the door, we have $\sum \tau = 0$

Taking moment of all forces acting on the door about lower hinge (may be upper hinge or any other point), we get

$$500 \times 1.0 - F_H \times 2 = 0$$

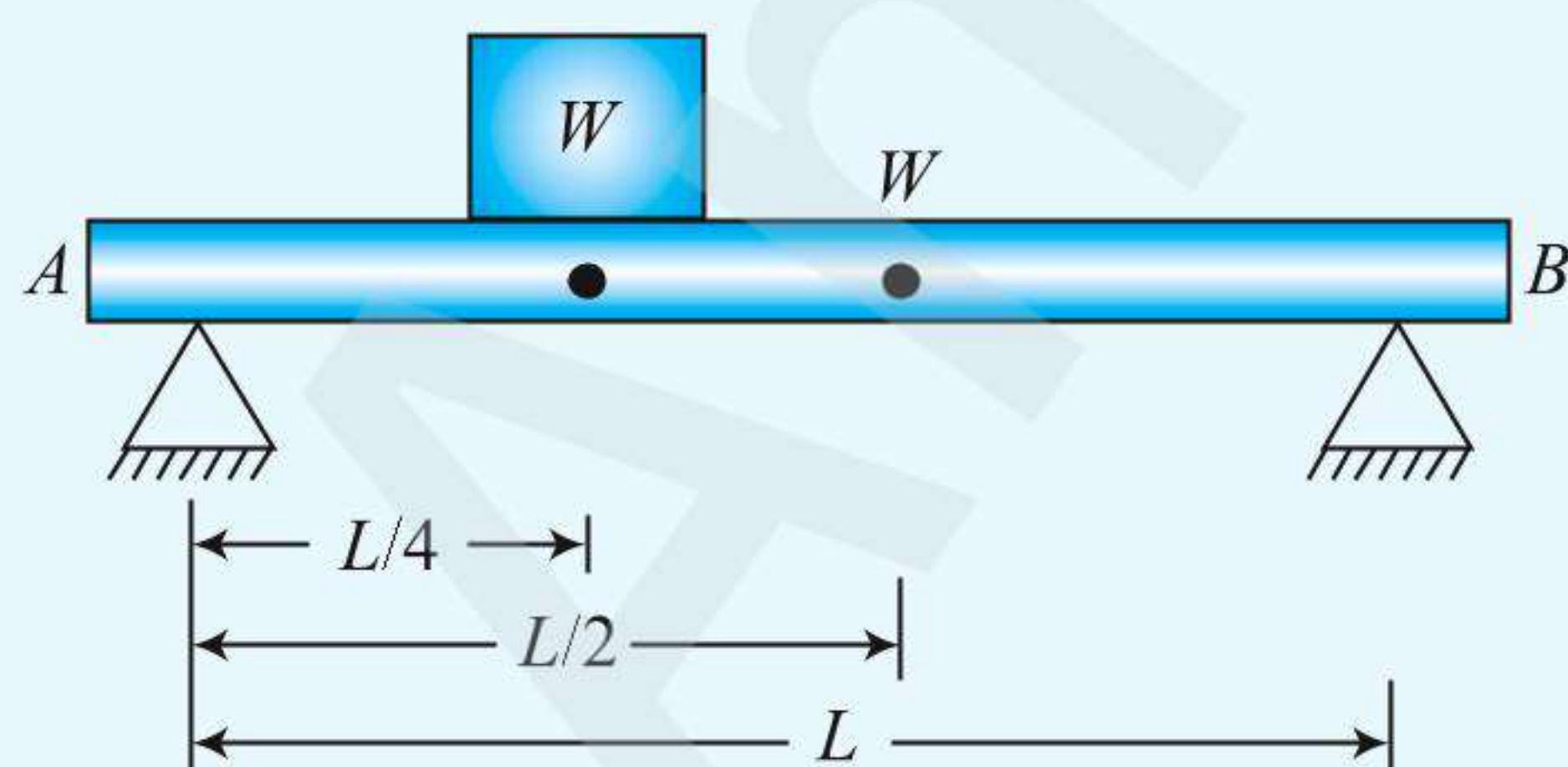
which gives, $F_H = 250 \text{ N}$

The force on the door exerted by either hinge, in magnitude, is

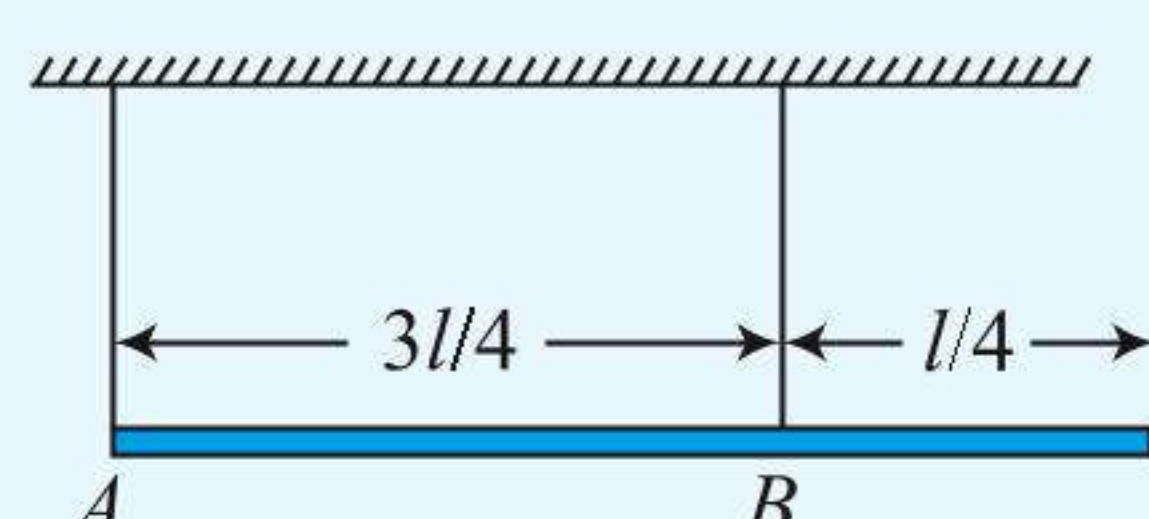
$$F = \sqrt{F_H^2 + F_v^2} = \sqrt{(250)^2 + (250)^2} = 250\sqrt{2} \text{ N}$$

CONCEPT APPLICATION EXERCISE 3.4

1. A beam of weight W supports a block of weight W . The length of the beam is L and weight is at a distance $L/4$ from the left end of the beam. The beam rests on two rigid supports at its ends. Find the reactions of the supports.

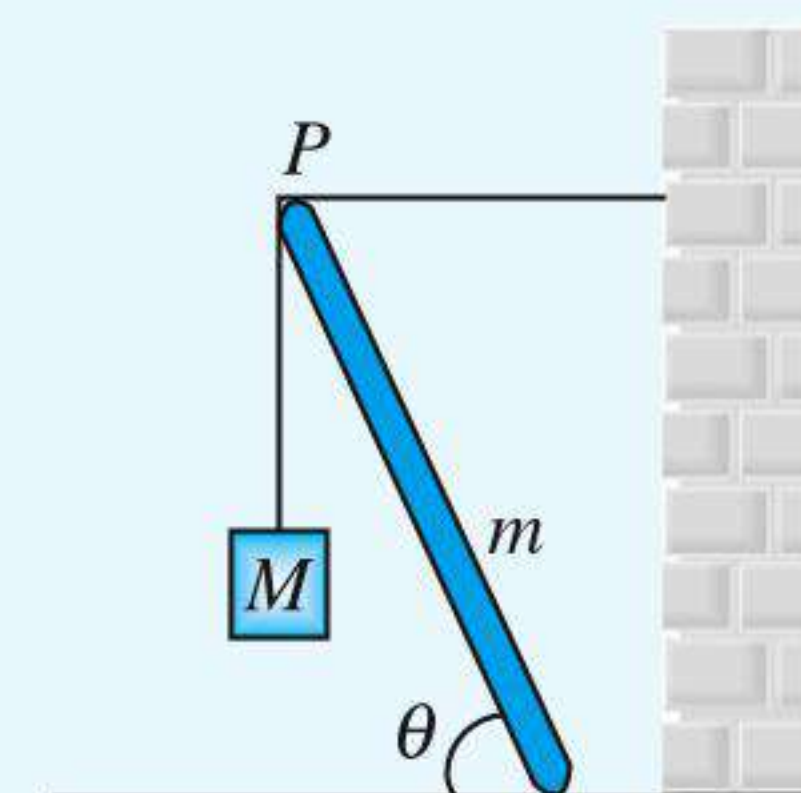


2. A uniform rod of length l and mass m is hung from two strings of equal length from a ceiling as shown in figure. Determine the tensions in the strings.

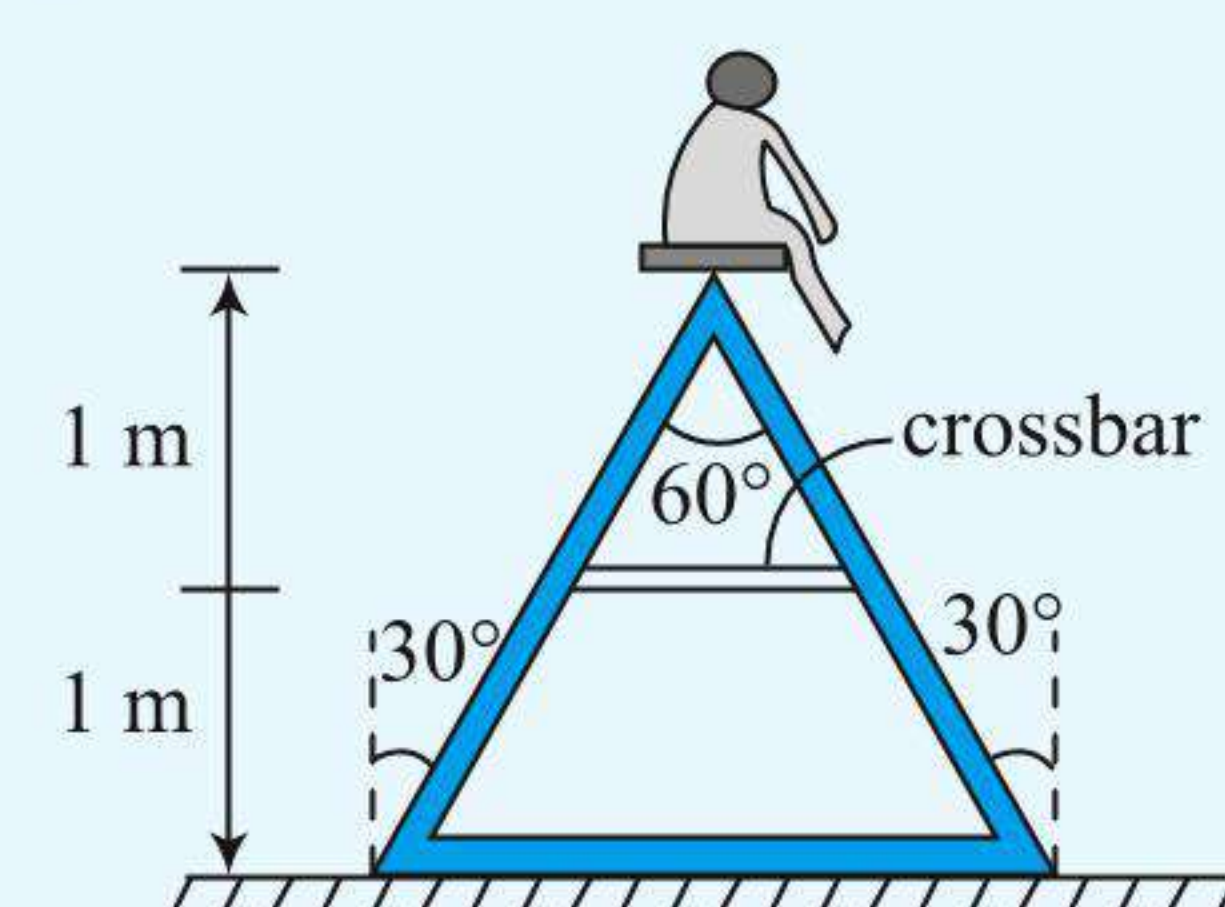


3. A uniform rod is made to lean between a rough vertical wall and the ground. The coefficient of friction between the rod and the ground is μ_1 and between the rod and the wall is μ_2 . Find the angle at which the rod can be leaned without slipping.
4. A uniform ladder of mass 10 kg leans against a smooth vertical wall making an angle of 53° with it. The other end rests on a rough horizontal floor. Find the normal force and the friction force that the floor exerts on the ladder.
5. A uniform ladder of length 10.0 m and mass 16.0 kg is resting against a vertical wall making an angle of 37° with it. The vertical wall is frictionless but the ground is rough. An electrician weighing 60.0 kg climbs up the ladder.
 - (a) If he stays on the ladder at a point 8.00 m from the lower end, what will be the normal force and the force of friction on the ladder by the ground?
 - (b) What should be the minimum coefficient of friction for the electrician to work safely?
6. A uniform ladder of length L and mass m_1 rests against a frictionless wall. The ladder makes an angle θ with the horizontal.
 - (a) Find the horizontal and vertical forces the ground exerts on the base of the ladder when a firefighter of mass m_2 is at a distance x from the bottom.
 - (b) If the ladder is just on the verge of slipping when the firefighter is a distance d from the bottom, what is the coefficient of static friction between ladder and ground?

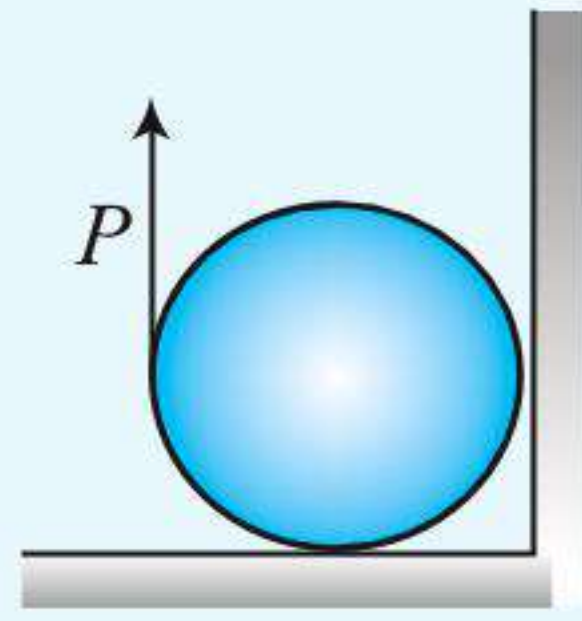
7. A uniform beam of mass m is inclined at an angle θ to the horizontal. Its upper end produces a ninety degree bend in a very rough rope tied to a wall, and its lower end rests on a rough floor (figure).



- (a) If the coefficient of static friction between beam and floor is μ_s , determine an expression for the maximum mass M that can be suspended from the top before the beam slips.
 - (b) Determine the magnitude of the reaction force at the floor and the magnitude of the force exerted by the beam on the rope at P in terms of m , M , and μ_s .
8. The ladder shown in figure has negligible mass and rests on a frictionless floor. The crossbar connects the two legs of the ladder at the middle. The angle between the two legs is 60° . The fat person sitting on the ladder has a mass of 80 kg. Find the contact forces exerted by the floor on each leg and the tension in the crossbar.

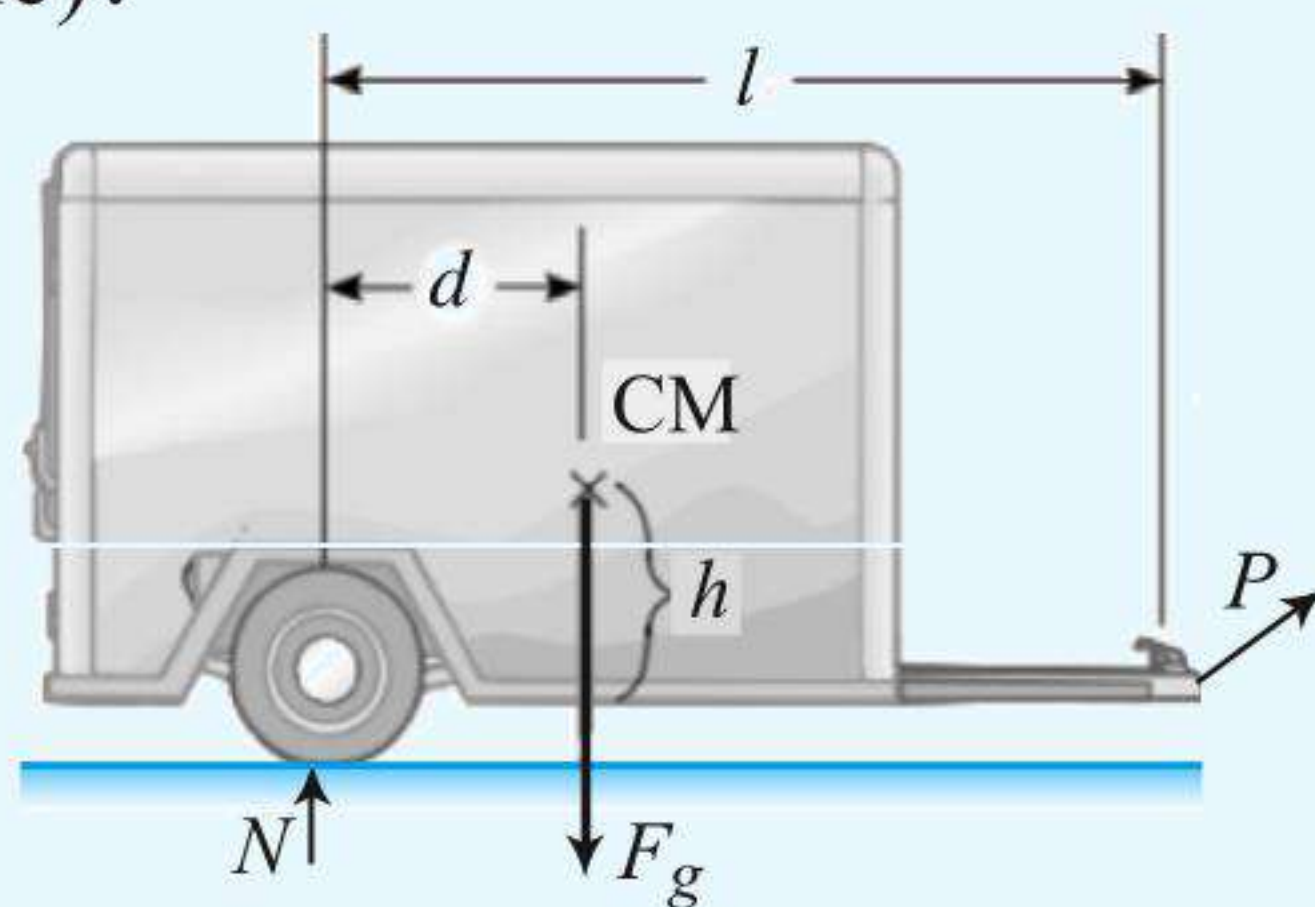


9. Figure shows a vertical force applied tangentially to a uniform cylinder of weight F_g . The coefficient of static friction between the cylinder and all surfaces is 0.500. In terms of F_g , find the maximum force P that can be applied that does not cause the cylinder to rotate.



10. A trailer with loaded weight F_g is being pulled by a vehicle with a force P , as in figure. The trailer is loaded such that its centre of mass is located as shown. Neglect the force of rolling friction and let a represent the x component of the acceleration of the trailer.

- (a) Find the vertical component of P in terms of the given parameters.
 (b) If $a = 2.00 \text{ ms}^{-2}$ and $h = 1.50 \text{ m}$, what must be the value of d in order that $P_y = 0$ (no vertical load on the vehicle)?

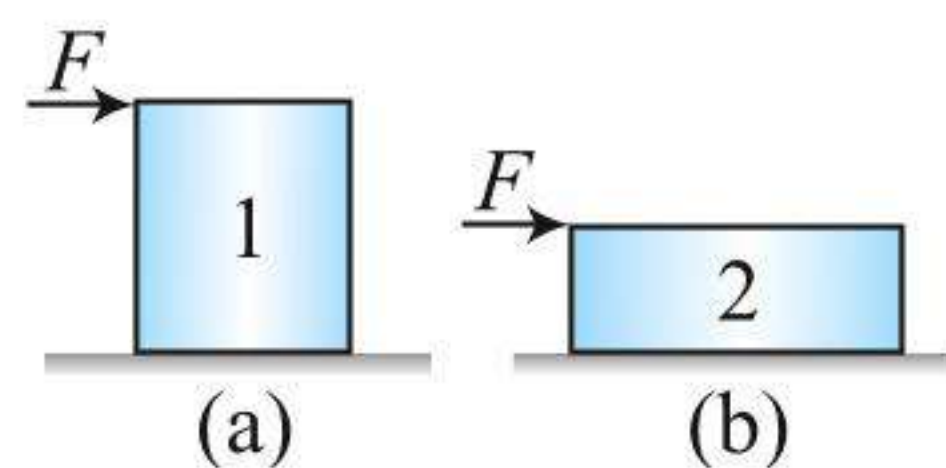


ANSWERS

1. $\frac{5W}{4}; \frac{3W}{4}$ 2. $\frac{mg}{3}; \frac{2mg}{3}$ 3. $\tan^{-1}\left(\frac{1-\mu_1\mu_2}{2\mu_1}\right)$
 4. $100 \text{ N}, \frac{200}{3} \text{ N}$ 5. (a) $760 \text{ N}, 420 \text{ N}$ (b) 0.55
 6. (a) $\left[\frac{1}{2}m_1g + \left(\frac{x}{L}\right)m_2g\right] \cot \theta, (m_1 + m_2)g$
 (b) $\frac{\left(\frac{m_1}{2} + \frac{m_2d}{L}\right) \cot \theta}{m_1 + m_2}$
 7. (a) $\frac{m}{2} \left(\frac{2\mu_s \sin \theta - \cos \theta}{\cos \theta - \mu_s \sin \theta} \right)$
 (b) $(M + m)g\sqrt{1 + \mu_s^2}, g\sqrt{M^2 + \mu_s^2(M + m)^2}$
 8. $400 \text{ N}, \frac{80}{\sqrt{3}} \text{ N}$ 9. $\frac{3}{8}F_g$
 10. (a) $\frac{F_g}{L} \left(d - \frac{ah}{g} \right)$ (b) 0.30 m

TOPPLING AND SHIFTING OF NORMAL REACTION

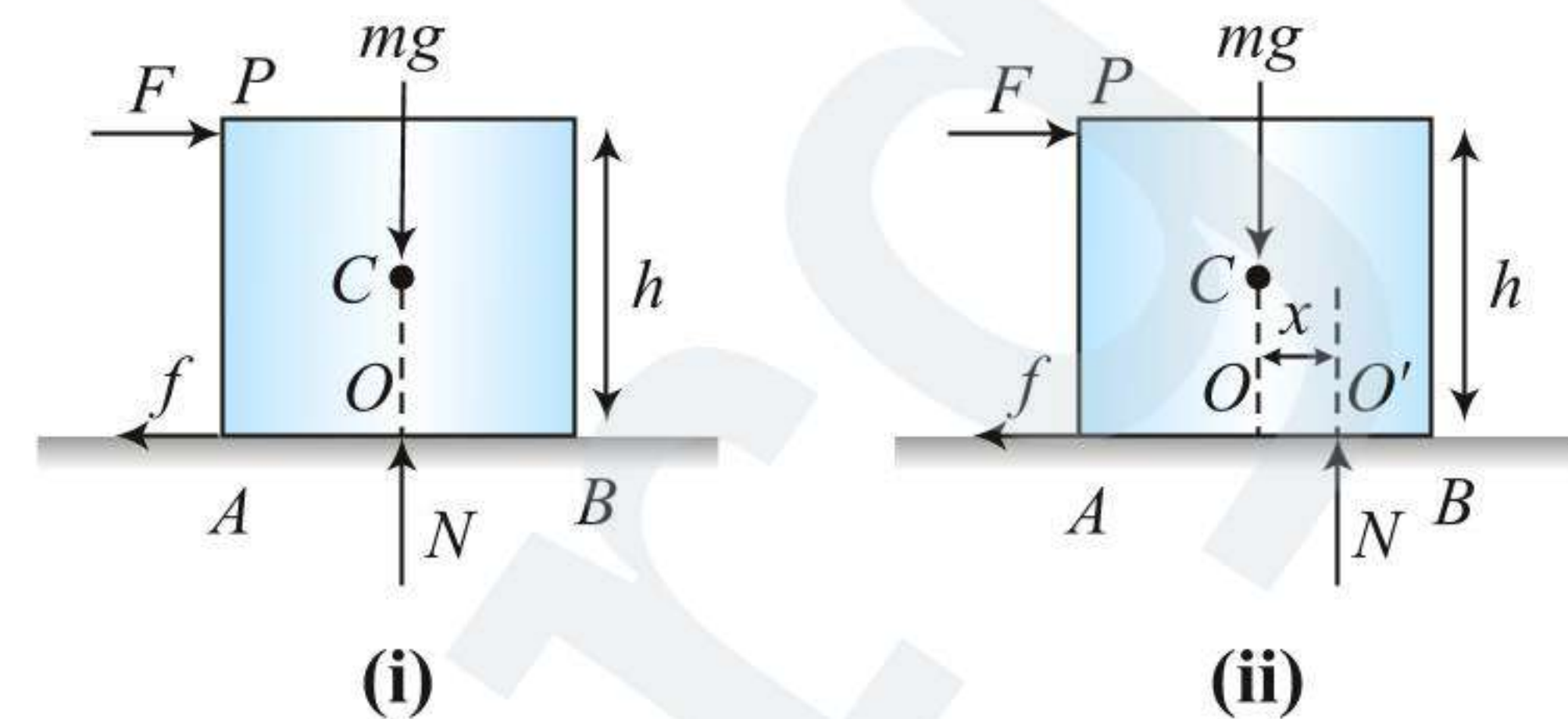
Let us take two blocks of equal masses. The block '1' shown in Fig. (a) has greater height than the block '2' shown in Fig. (b).



If same force F is applied on both blocks as shown in the figures. Block '1' is more likely to topple down before sliding. For block '2' of broader base, chances of its sliding are more

compared to its toppling. The question arises why it happens? To understand this now consider a block of mass m is placed on rough horizontal surface and a horizontal force F is applied on it as shown figure. The friction is sufficient so that the block is not moving.

It means: $\sum \vec{F} = 0$ and $\sum \vec{\tau} = 0$



If we consider the normal reaction is passing through centre of mass as shown in Fig. (i), in this case there is unbalanced torque about C in clockwise sense. To make the block in rotational equilibrium an anticlockwise torque is required, which is possible only if the normal reaction shifts towards O' as shown in Fig. (ii).

If the block is in rotational equilibrium, i.e., it is not going to topple then net torque of couples should be zero,

$$mg \times x = F \times h \Rightarrow x = \frac{Fh}{mg} \quad \dots(i)$$

The maximum value of x can be $\frac{b}{2}$ only because then the normal force will be at the extreme end. For toppling,

$$\frac{b}{2} = \frac{Fh}{mg} \Rightarrow F = \frac{bmg}{2h}$$

Now let us consider the block is kept on rough inclined plane then its FBD is as shown.

Again, if the block is in rotational equilibrium, i.e., it is not going to topple then torque about C is zero,

$$\vec{\tau}_C = Nx - f \frac{h}{2} = 0$$

$$\Rightarrow x = \frac{fh}{2N} = \frac{fh}{2mg \cos \theta}$$

If block is at rest, $f = mg \sin \theta$

$$x = \frac{mg \sin \theta h}{2mg \cos \theta} = \frac{h \tan \theta}{2} \quad \dots(i)$$

The maximum value of x can be $b/2$ only. So, for a given inclined plane and with a fixed base of the body the maximum height of a body which can be placed on it without toppling is given by putting $x = \frac{b}{2}$ in Eq. (i),

$$h = 2 \frac{b}{2 \tan \theta} = \frac{b}{\tan \theta} = b \cot \theta$$

To understand this concept in better way let us take a few illustrations.

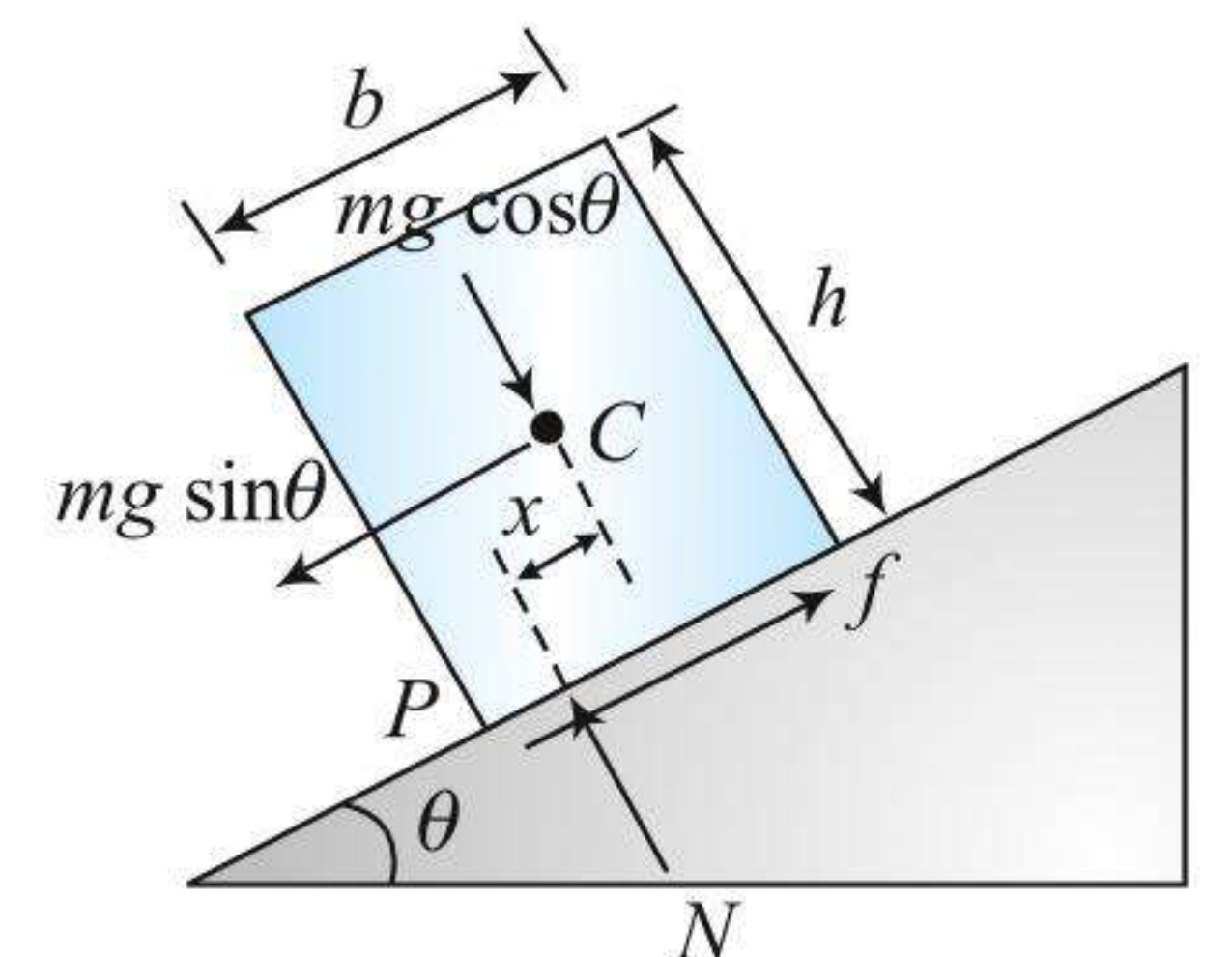
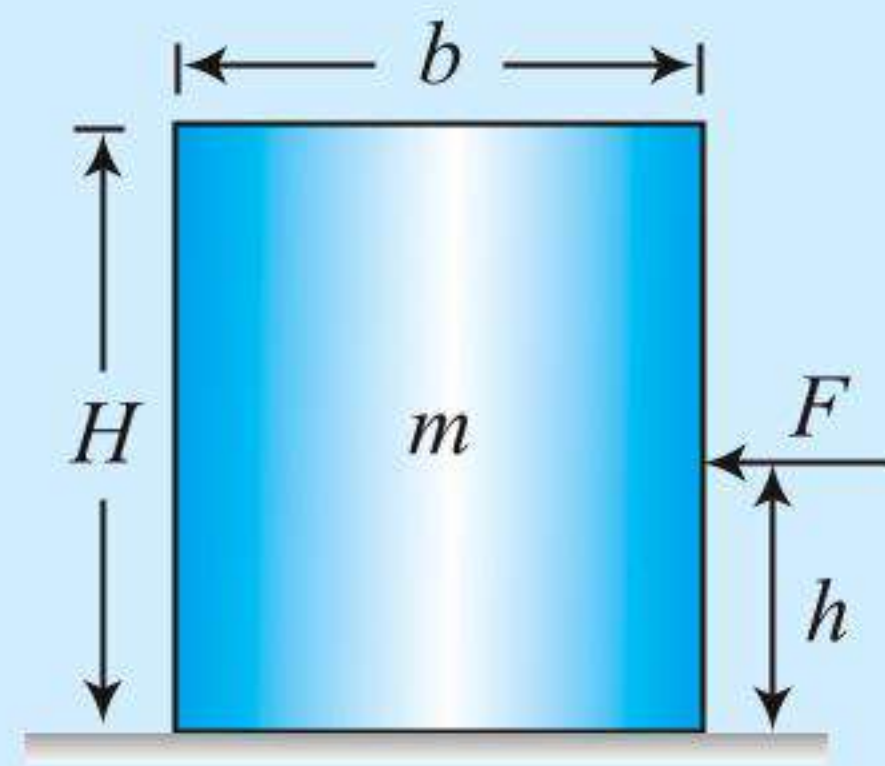


ILLUSTRATION 3.31

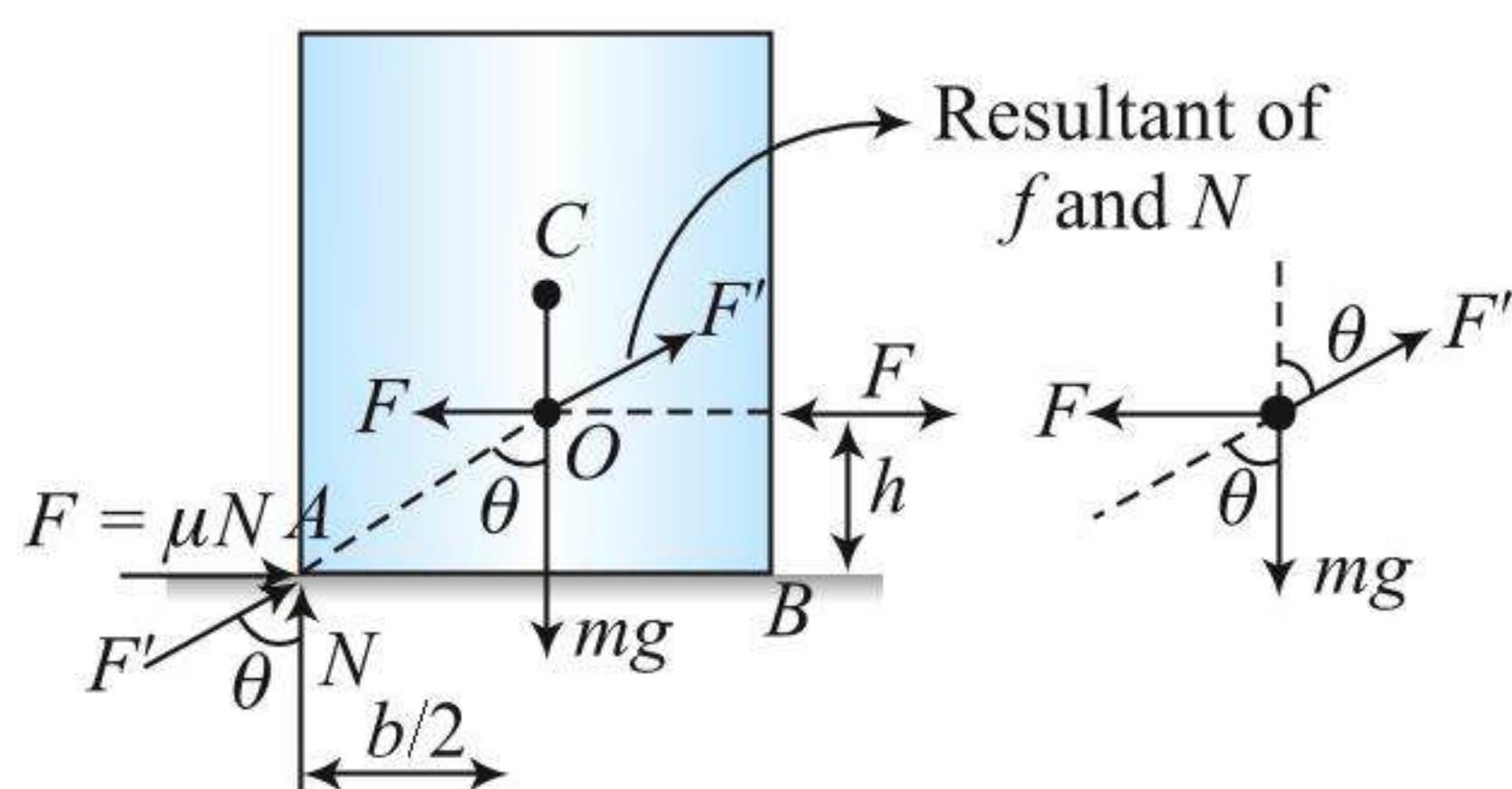
A horizontal force F is applied to a homogeneous rectangular block of mass m , width b and height H . The block moves with constant velocity; the coefficient of friction is μ_k .



- (a) What is the greater height h at which the force F can be applied so that the block will slide without tipping over?
- (b) Through which point on the bottom face of the block will the resultant of the friction and normal forces act if $h = H/2$?
- (c) If the block is at rest and coefficient of static friction is μ_s , what are the various criteria for which sliding, or tipping occurs?

Sol.

- (a) In the absence of any external force in horizontal direction the normal reaction N passes through the centre of mass of the block; when force F is applied, normal reaction shifts in the direction of applied force F (in leftward direction i.e., towards A)



Since right part of body is having tendency to lift from surface, at the instant of tipping over about the edge the normal reaction passes through edge. From the conditions of equilibrium.

In horizontal direction: $f = F = \mu_k mg$

In vertical direction: $N = mg$

Balancing torque about edge A ,

$$Fh - mg \frac{b}{2} = 0 \text{ or } h = \frac{mgb}{2F} = \frac{mgb}{2\mu_k mg} = \frac{b}{2\mu_k}$$

We can solve the problems by another approach also as the resultant of friction force ($\mu_k N$) and the normal reaction must pass through the same point through which F passes, since three coplanar forces keeping a body in equilibrium pass through a common point i.e., they should be concurrent.

From Lami's theorem:

$$\frac{F}{\sin 90^\circ} = \frac{F}{\sin(180^\circ - \theta)} = \frac{mg}{\sin(90^\circ + \theta)}$$

$$\Rightarrow \frac{F}{\sin \theta} = \frac{F}{\sin(180^\circ - \theta)} = \frac{mg}{\sin(90^\circ + \theta)}$$

From figure it is clear $\tan \theta = \frac{\mu_k N}{N} = \frac{b/2}{h}$

$$\text{or } \frac{(b/2)}{h} = \mu_k \Rightarrow h = \frac{b}{2\mu_k}$$

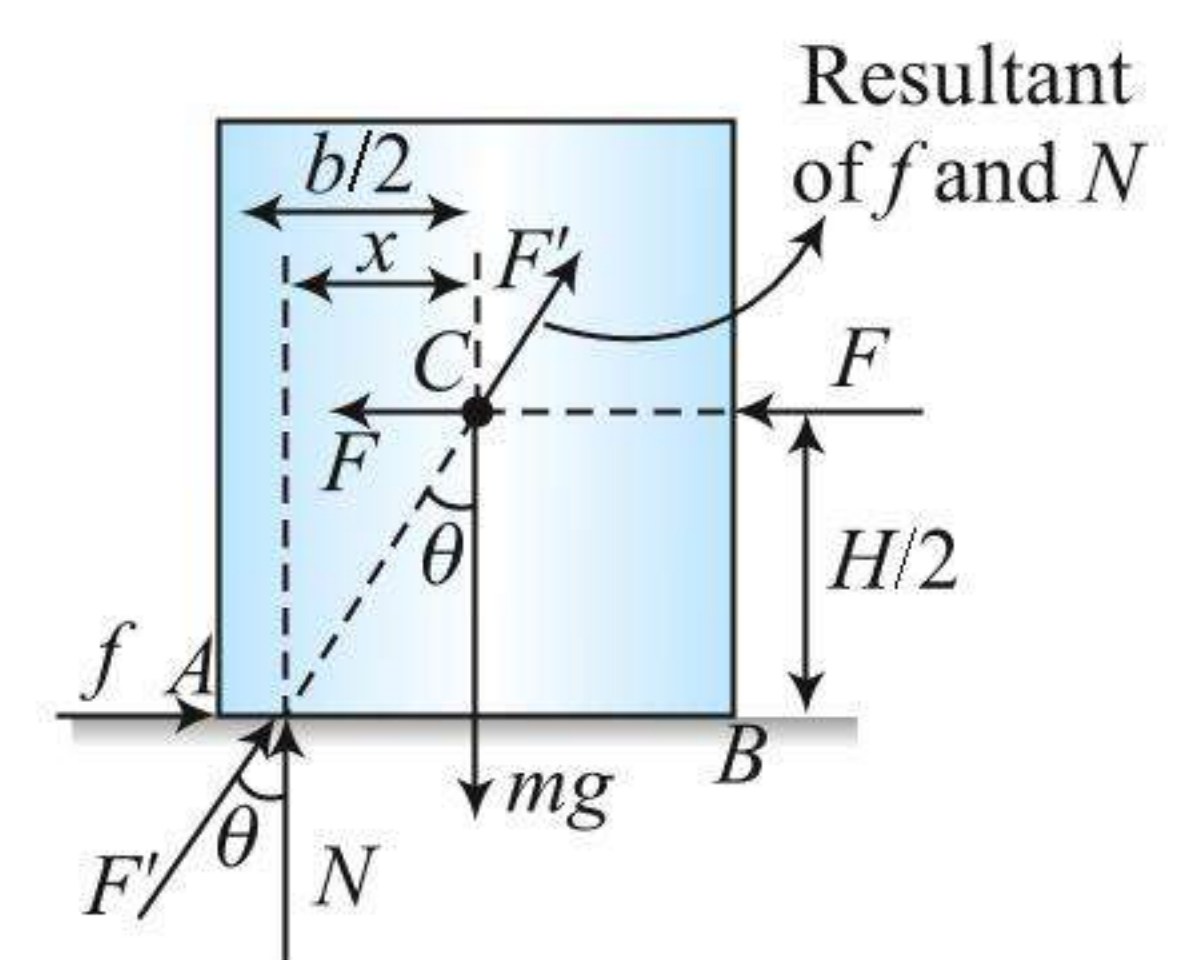
For the block slide without toppling over about edge A , h should be less than $\frac{b}{2\mu_k}$.

- (b) In this case we cannot take normal reaction at the edge. Let normal reaction acts at a distance x from the line of action of mg . Torque of all the forces about C must be zero.

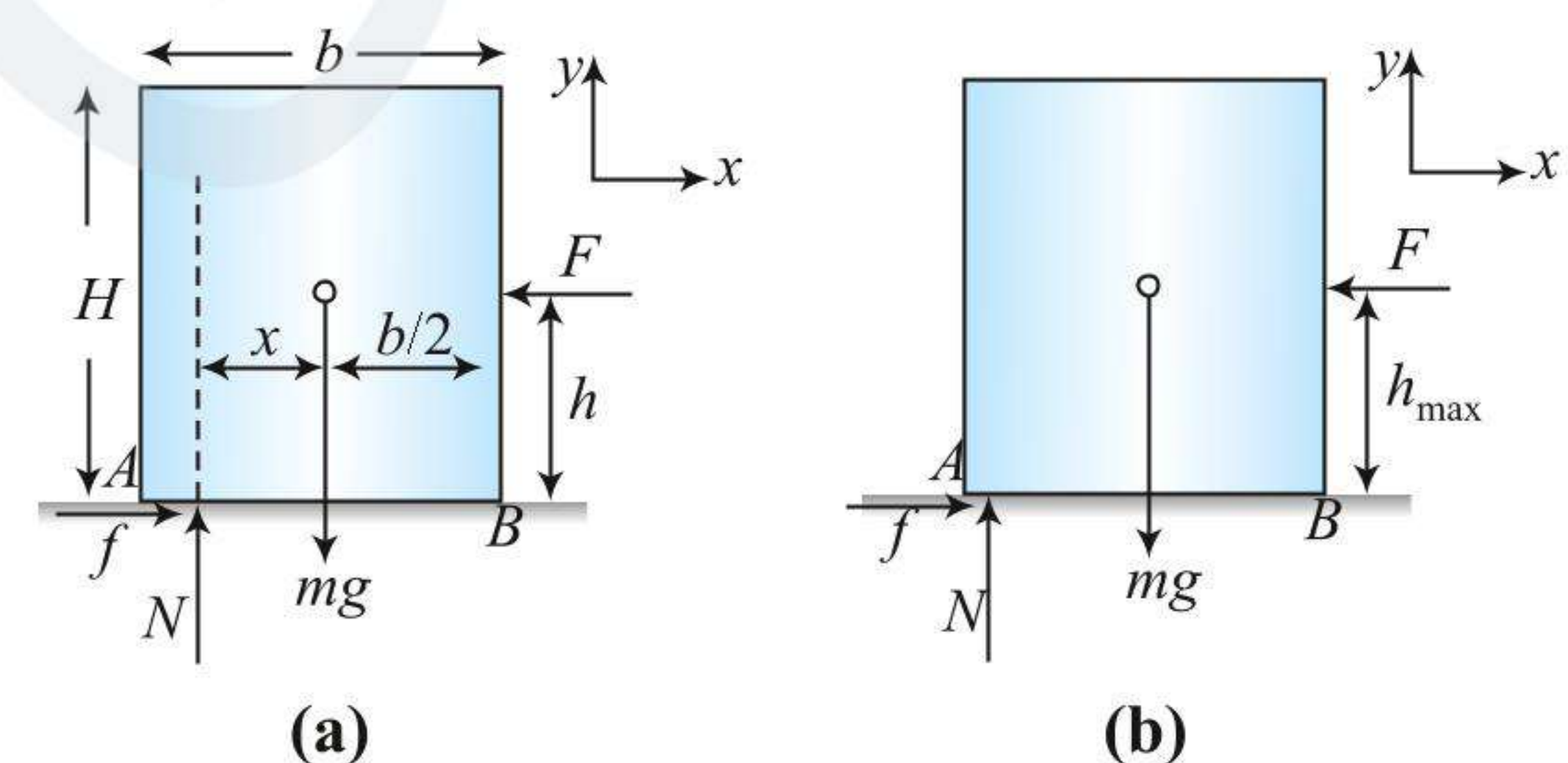
$$\left(\mu_k N \times \frac{H}{2} \right) = Nx \Rightarrow x = \frac{\mu_k H}{2}$$

Alternately: We can proceed as in part (a). Resultant of $\mu_k N$, N passes through C the intersection of F and mg . Therefore, from geometry of the figure.

$$\frac{x}{H/2} = \tan \theta = \mu_k; x = \frac{\mu_k H}{2}$$



- (c) As the point of application of force is raised higher, the location of the line of action of the normal reaction N moves towards edge A the left Fig. (a). In the limiting case (when the block is about to tip over), $x = b/2$. The normal reaction passes through edge as shown in Fig. (a).



If the block topples about edge A , then

$$Fh_{\max} = mg \frac{b}{2} \text{ or } h_{\max} = \frac{mgb}{2F} \quad \dots(i)$$

For sliding of the block $F = \mu N$

In vertical direction: $N = mg$

Hence, $F = \mu mg$

$\dots(ii)$

If sliding and tipping are equally likely to occur, we can eliminate F from Eqs. (i) and (ii) to get $h_{\max} = \frac{b}{2\mu}$ which

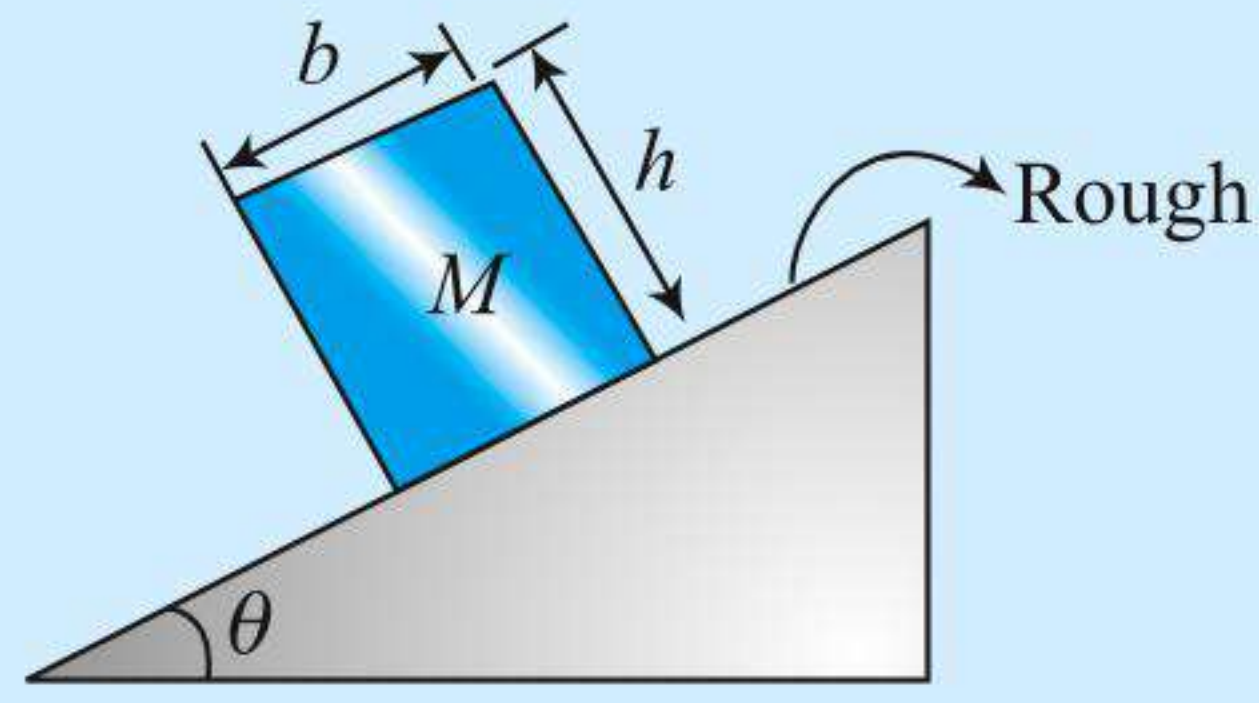
is independent of weight mg , height H of the body and applied force F .

Important conclusions from above discussion:

- If $h < h_{\max}$, $F < \mu mg$, neither toppling nor sliding occurs.
- If $h < h_{\max}$, $F = \mu mg$, no toppling, sliding motion impending.
- If $h < h_{\max}$, $F > \mu mg$, no toppling, sliding occurs.
- If $h = h_{\max}$, $F = \mu mg$, both sliding, and toppling are impending.
- If $h > h_{\max}$, the body will tip over for any value of weight of the body.
- If $\mu \geq \frac{F}{mg}$, onset of toppling occurs, and toppling edge remains stationary.
- If $\mu < \frac{F}{mg}$, onset of toppling occurs with sliding edge on surface.

ILLUSTRATION 3.32

A heavy block of length b and height h is placed at rest on a rough inclined plane of inclination θ with the horizontal, as shown in figure.

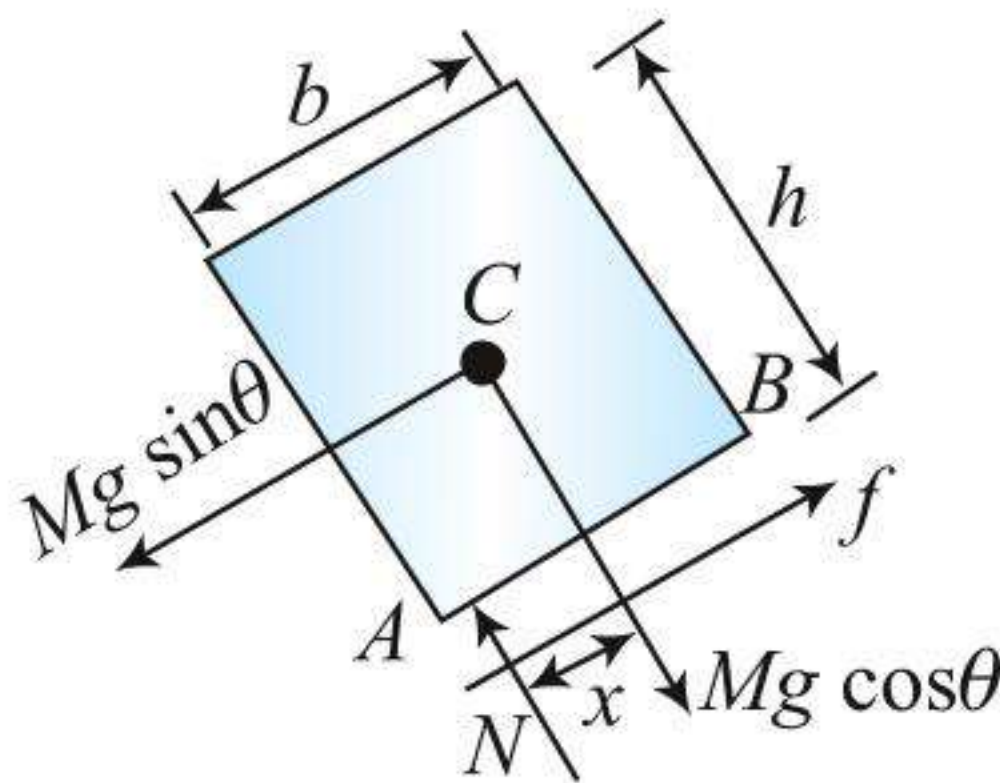


- (a) The normal reaction from the surface cannot pass through the centre mass. Why?
 (b) If μ is the coefficient of friction between the block and the inclined plane, then discuss the conditions of translational and rotational equilibrium of the block.

Sol.

- (a) The free-body diagram of the block is shown in figure.

Note that the point of application of the normal reaction N is displaced through x in the downward direction. It must produce a clockwise torque about the centre of mass which can be balanced by the anticlockwise torque produced by the friction force.



- (b) There are two tendencies of the block:
- to slide down
 - to rotate (or topple) about the point A

For translational equation $\sum \vec{F} = 0$

$$f = Mg \sin \theta \quad \dots(i)$$

$$N = Mg \cos \theta \quad \dots(ii)$$

For rotational equilibrium, $\sum \vec{\tau}_C = 0$

$$N \times x = f \times \frac{h}{2} \Rightarrow Mg \cos \theta \times x = \frac{fh}{2}$$

$$\Rightarrow f = \frac{2Mgx \cos \theta}{h}$$

If the block topples over about A, $x = \frac{b}{2}$

$$\text{Hence, (iii) } f = \frac{b Mg \cos \theta}{h} \quad \dots(iii)$$

From Eq. (i) if block do not slide down, $f < \mu N$

$$Mg \sin \theta < \mu \times Mg \cos \theta \Rightarrow \tan \theta \leq \mu$$

or we can say if the block slides $\mu < \tan \theta$

From Eq. (iii) if the block topples before sliding,

$$f > \mu(Mg \cos \theta)$$

$$\frac{b Mg \cos \theta}{h} \leq \mu Mg \cos \theta \Rightarrow \frac{b}{h} \leq \mu$$

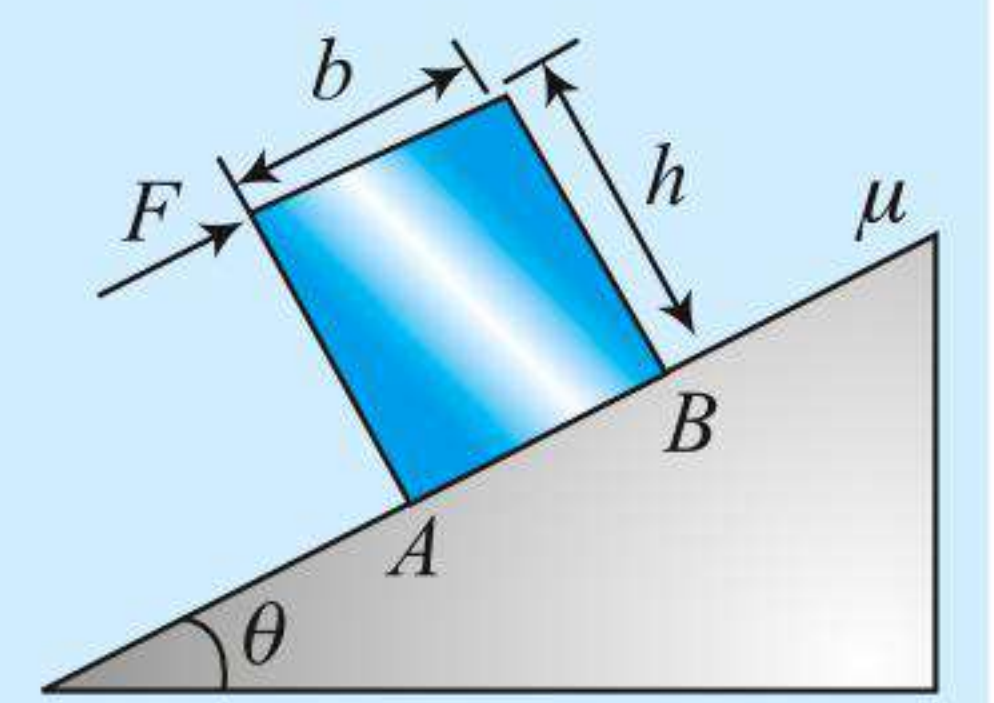
Hence, if the block topples before sliding, $\mu \geq \frac{b}{h}$

Important Points:

- The block has tendency to topple before sliding if $\mu > b/h$.
- The block slides down if $\mu < \tan \theta$, where θ is the angle of inclination of the incline with respect to the horizontal.
- The block topples over without sliding if $\mu > \tan \theta$ and $\mu > b/h$.

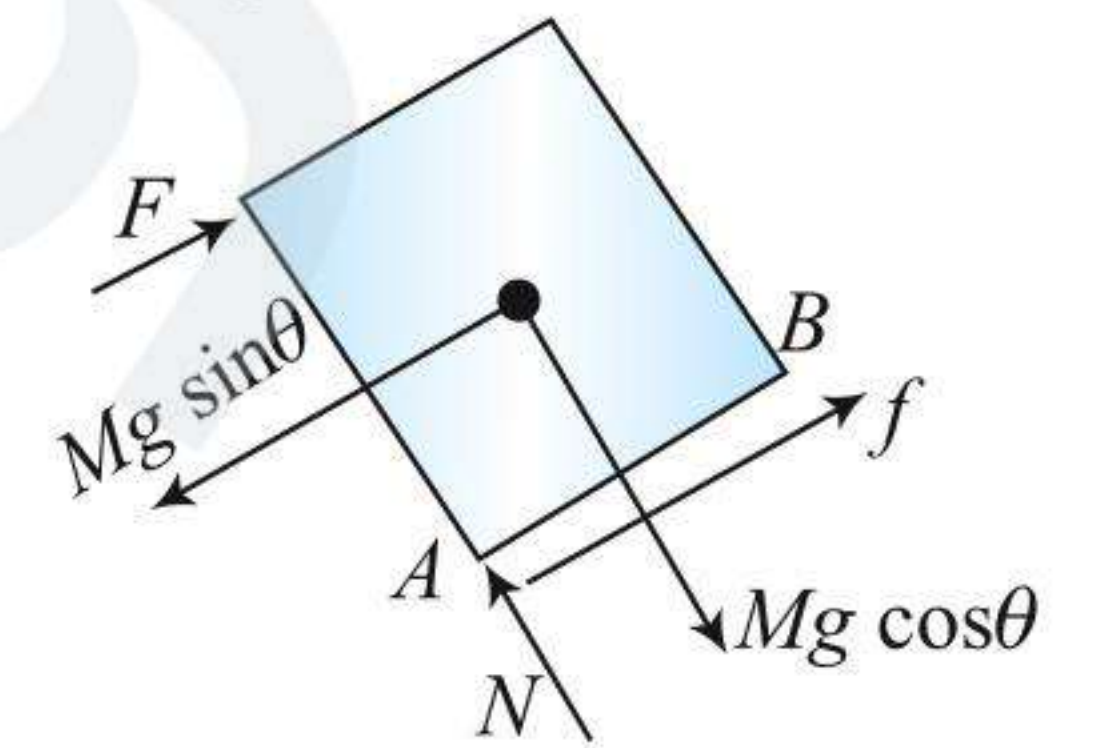
ILLUSTRATION 3.33

Determine the maximum ratio h/b for which the homogenous block will slide without toppling under the action of force F . The coefficient of static friction between the block and the incline is μ .



Sol. Block is sliding without toppling, which implies that block is in translational as well as rotational equilibrium.

When the block is on the verge of tipping about edge A, the normal reaction will pass through the edge of the block.



From conditions of equilibrium along sloping surface:

$$F + \mu N = mg \sin \theta$$

$$F = mg \sin \theta - \mu mg \cos \theta \quad \dots(i)$$

Balancing torque about point A:

$$F \times h + (mg \cos \theta) \frac{b}{2} = (mg \sin \theta) \frac{h}{2} \quad \dots(ii)$$

From Eqs. (i) and (ii)

$$(mg \sin \theta - \mu mg \cos \theta)h + mg \cos \theta \frac{b}{2} = mg \sin \theta \frac{h}{2}$$

$$(\sin \theta - \mu \cos \theta)h + \cos \theta \frac{b}{2} = \sin \theta \frac{h}{2}$$

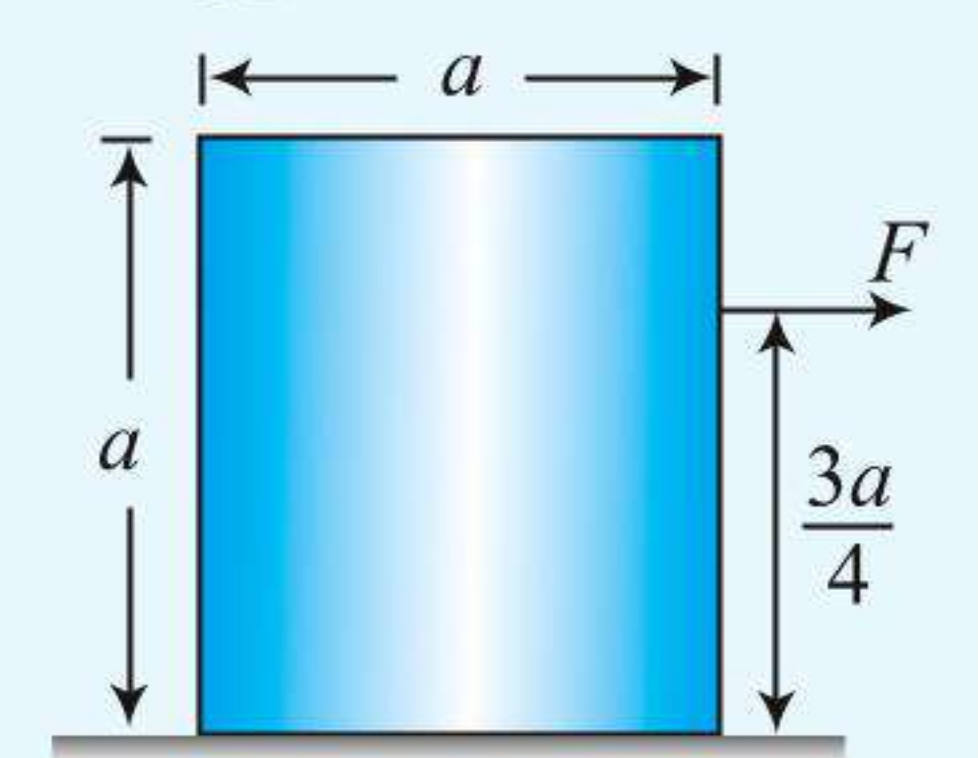
$$(2\mu \cos \theta - \sin \theta) \frac{h}{2} = \cos \theta \frac{b}{2}$$

$$\frac{h}{b} = \frac{\cos \theta}{(2\mu \cos \theta - \sin \theta)} = \frac{\cos \theta}{\cos \theta (2\mu - \tan \theta)}$$

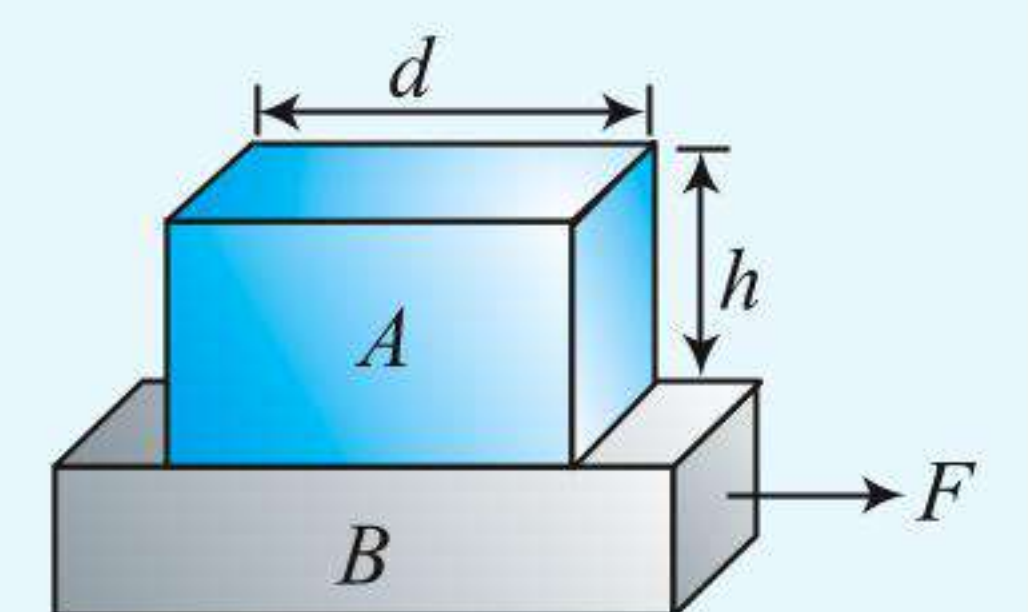
$$\Rightarrow \frac{h}{b} = \frac{1}{(2\mu - \tan \theta)}$$

CONCEPT APPLICATION EXERCISE 3.5

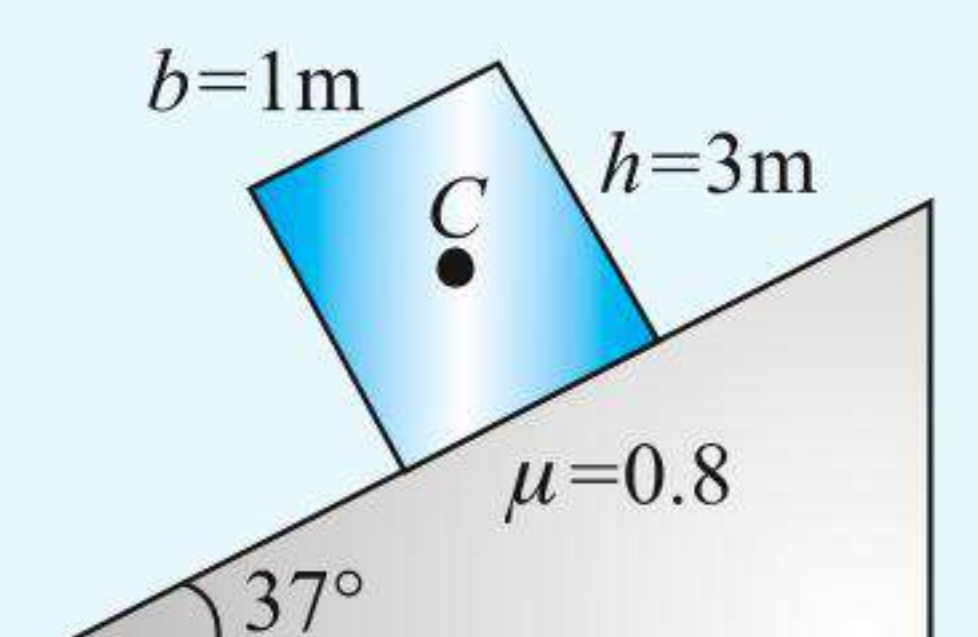
1. A uniform cube of side a and mass m rests on a rough horizontal table. A horizontal force F is applied normal to one of the faces at a point directly above the centre of the face, at a height $3a/4$ above the base. What is the minimum value of F for which the cube begins to tip about an edge?



2. A rectangular block 'A', having height h and width b has been placed on another block B as shown in the figure. Both blocks have equal mass and there is no friction between the block 'B' and ground. A horizontal time dependent force; $F = kt$ is applied on the block 'B'. At what time will block 'A' topple? Assume that friction between the two blocks is large enough to prevent the block 'A' from slipping.

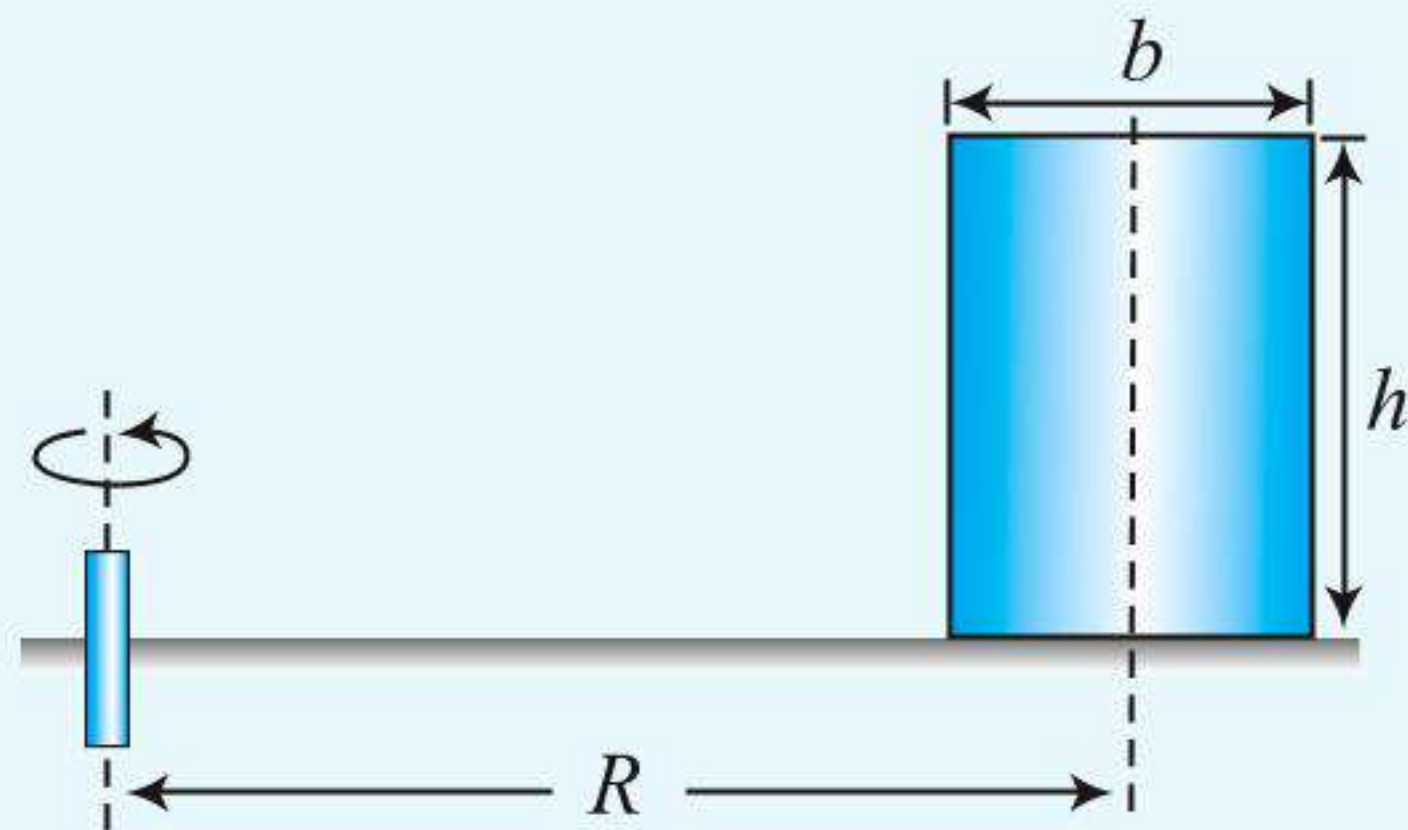


3. A tall block of mass $M = 50$ kg and base width $b = 1$ m and height $h = 3$ m is kept on a rough inclined surface with coefficient of friction $\mu = 0.8$ as shown in figure. The

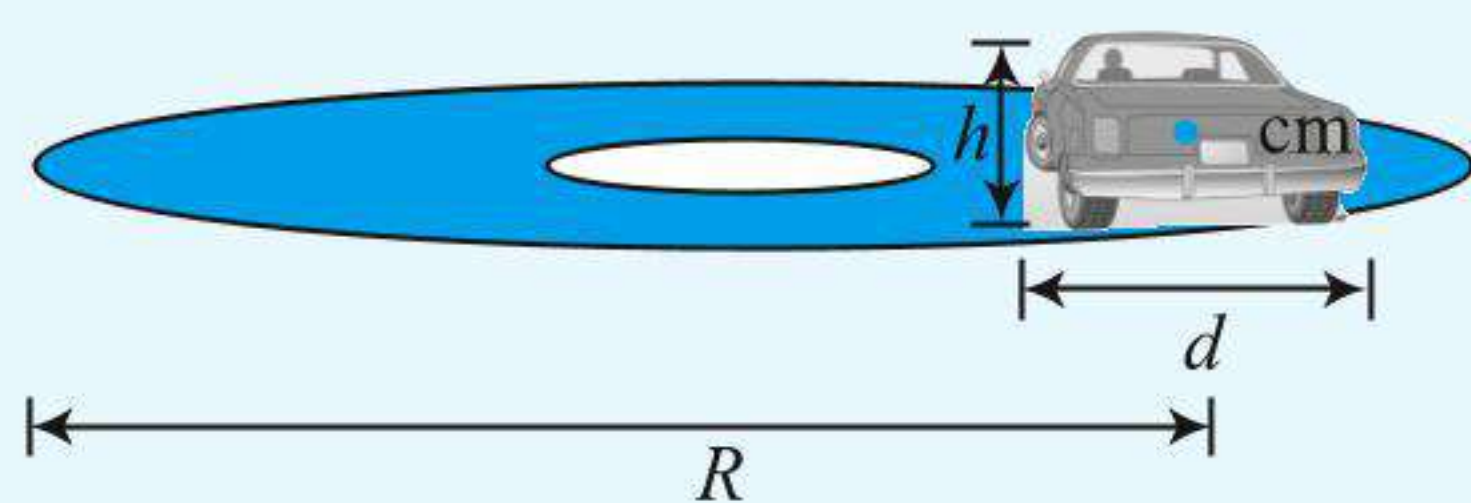


angle of inclination with the horizontal is 37° . Determine whether the block slides down or topples over.

4. A rectangular block rests on a horizontal rotating disc, as shown in the figure. Find at what angular velocity ω , the block falls off the disc, if the distance between the axis of the disc and block is R and the coefficient of friction $\mu > \frac{b}{h}$, where b is the width of the block and h is its height.



5. A car moves with speed v on a horizontal circular track of radius R . A head-on view of the car is shown in figure. The height of the car's centre of mass above the ground is h , and the separation between its inner and outer wheels is d . The road is dry, and the car does not skid. Show that the maximum speed the car can have without overturning is given by $v_{\max} = \sqrt{\frac{gRd}{2h}}$. To reduce the risk of rollover, should one increase or decrease h ? Should one increase or decrease the width d of the wheel base?



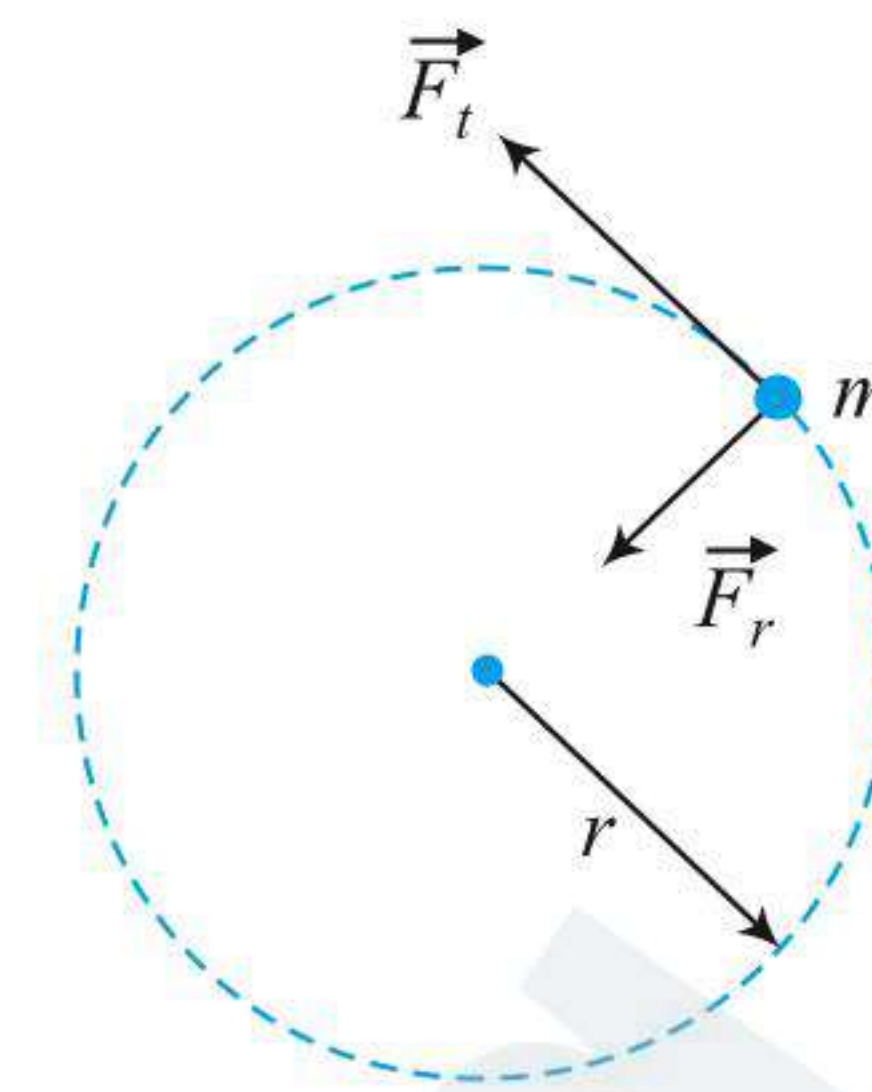
ANSWERS

1. $\frac{2mg}{3}$ 2. $\frac{2dgm}{kh}$

NEWTON'S SECOND LAW FOR ROTATIONAL MOTION

We know a net force on an object causes an acceleration of the object and that the acceleration is proportional to the net force which can be verified by Newton's second law. Now we will learn the rotational analog of Newton's second law: the angular acceleration of a rigid object rotating about a fixed axis is proportional to the net torque acting about that axis. Before discussing the more complex case of rigid-object rotation, however, it is instructive first to discuss the case of a particle moving in a circular path about some fixed point under the influence of an external force.

Consider a particle of mass m rotating in a circle of radius r about some fixed point under the influence of an external force. The components of the force in tangential direction and radial direction are $\vec{F}_t$ and $\vec{F}_r$ respectively are shown in figure.



The radial net force causes the particle to move in the circular path with a centripetal acceleration. The tangential force provides a tangential acceleration $\vec{a}_t$, and we can write

$$F_t = ma_t \quad \dots(i)$$

The magnitude of the net torque due to $\vec{F}_t$ on the particle about an axis perpendicular to the page through the center of the circle is

$$\tau = F_t r = (ma_t)r \quad \dots(ii)$$

Because the tangential acceleration is related to the angular acceleration through the relationship $a_t = r\alpha$, the net torque can be expressed as

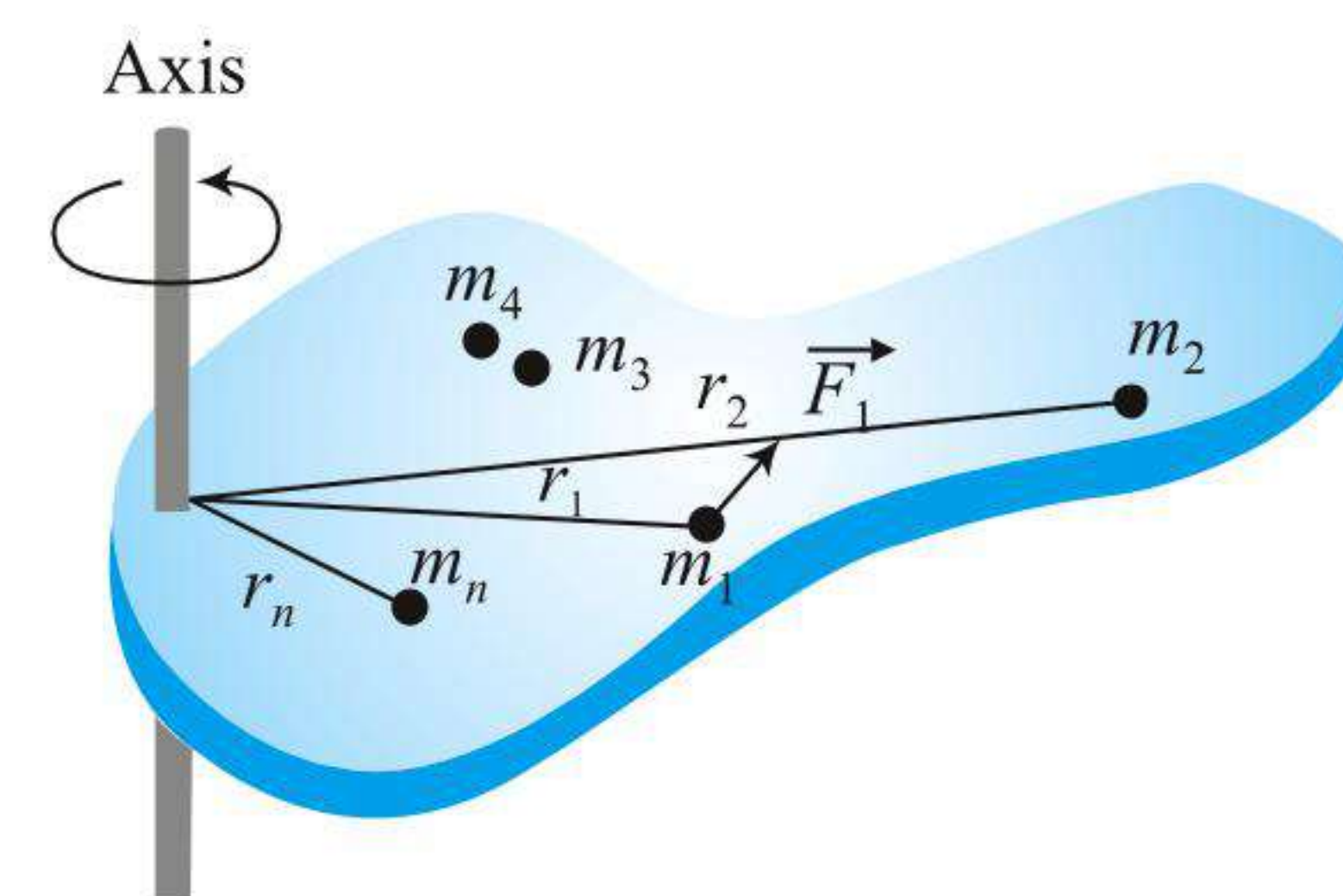
$$\tau = (mr\alpha)r = (mr^2)\alpha \quad \dots(iii)$$

Let us denote the quantity mr^2 with the symbol I for now. We will say more about this quantity below. Using this notation, Eq. (iii) can be written as

$$\tau = I\alpha \quad \dots(iv)$$

Equation (iv) is the form of Newton's second law we have been seeking. It indicates that the net external torque τ is directly proportional to the angular acceleration α . The constant of proportionality is $I = mr^2$, which is called the **moment of inertia of the particle**. The SI unit for moment of inertia is kg m^2 .

Now let us extend this discussion to a rigid object of arbitrary shape rotating about a fixed axis passing through a point O as in figure.



The object can be regarded as a collection of particles of masses $m_1, m_2, \dots, m_n$. If we impose a Cartesian coordinate system on the object, each particle rotates in a circle about the origin with some tangential acceleration produced by external tangential force. For any given particle, we know from Newton's second law that,

$$F_i = m_i a_i$$

The external torque $\vec{\tau}_i$ associated with the force $\vec{F}_i$ acts about the origin and its magnitude is given by,

$$\tau_i = r_i F_i = r_i m_i a_i$$

Because $a_i = r_i \alpha$, the expression for τ_i becomes, $\tau_i = m_i r_i^2 \alpha$

Although each particle in the rigid object may have a different translational acceleration a_i , they all have the same angular acceleration α . With that in mind, we can add the torques on all the particles making up the rigid object to obtain the net torque on the object about an axis through O due to all external forces:

$$\Sigma \tau_{\text{ext}} = \sum_i \tau_i = \sum_i m_i r_i^2 \alpha = \left(\sum_i m_i r_i^2 \right) \alpha \quad \dots(\text{v})$$

where α can be taken outside the summation because it is common to all particles and $\Sigma m r^2 = m_1 r_1^2 + m_2 r_2^2 + \dots + m_n r_n^2$ represents the sum of the individual moments of inertia. The latter quantity is the **moment of inertia I of the body**. The expression for $\Sigma \tau_{\text{ext}}$ becomes

$$\Sigma \tau_{\text{ext}} = I \alpha \quad \dots(\text{vi})$$

Rotational analog of Newton's second law for a rigid body rotating about a fixed axis

Net external torque = moment of inertia $\times$ Angular acceleration
or $\tau_{\text{ext}} = I \alpha$

The above equation is called 'torque equation'.

The torque equation is, however, more complex than the force equation. Why?

- Force is an absolute value, whereas torque depends on the point about which it is being calculated.
- Mass is an absolute value for a body, whereas MI can have infinite values for a body. MI is defined only for a given axis of the body.

Note:

Remember the following points about the torque equation:

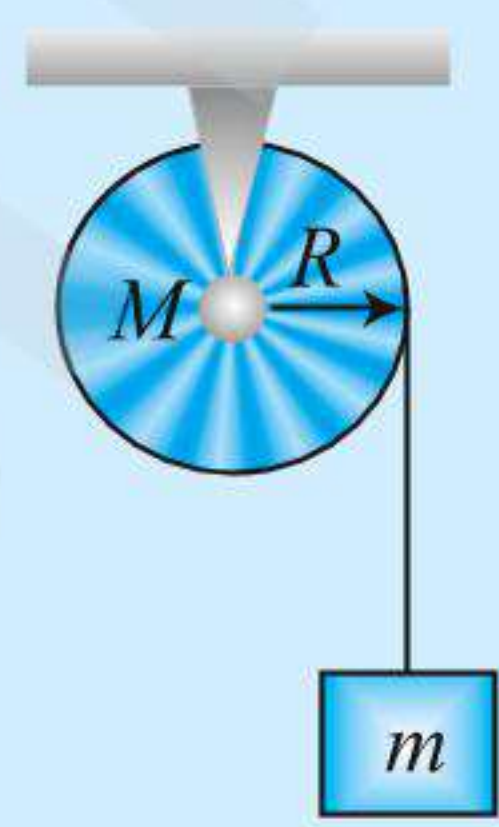
Torque equation can be applied only about two points in a body. These are as follows:

- Centre of mass.
- Point which has zero velocity/acceleration. This point is commonly referred to as the center of rotation or instantaneous center of rotation.

Now, remember that the axis of rotation passes through the point and is perpendicular to the surface (valid only for two dimensional bodies).

ILLUSTRATION 3.34

A block of mass m is attached at the end of an inextensible string which is wound over a rough pulley of mass M and radius R (see figure). Assume the string does not slide over the pulley. Find the acceleration of the block when released.



Sol. This problem is different from the problems which we have solved in laws of motion where we assumed the string is sliding over the pulley without friction. Here, the pulley is rough, and string is not sliding.

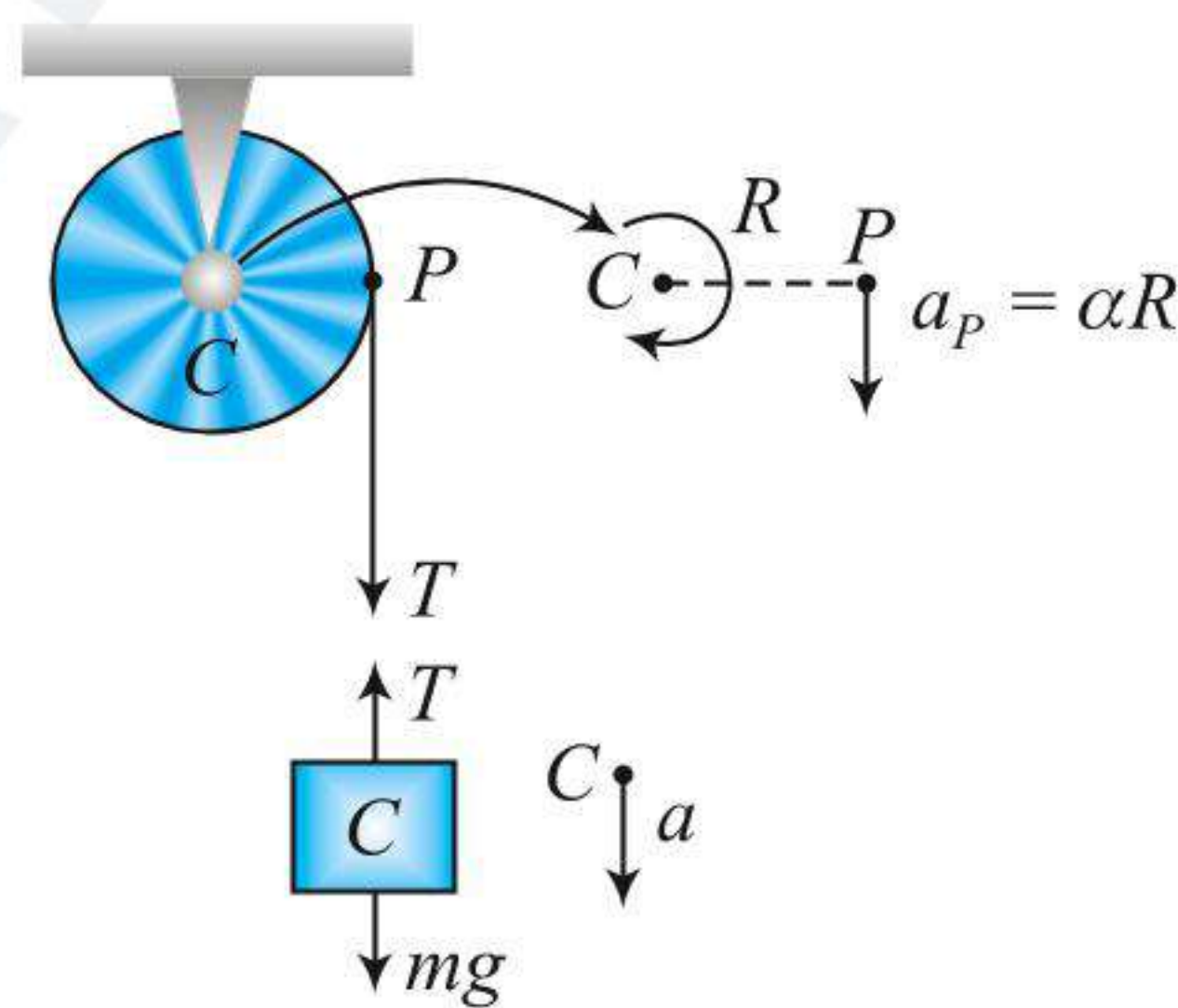
The free-body diagrams of the given situation are shown in figure.

The motion of the block is translational, here we can apply equation of motion of the block:

$$mg - T = ma \quad \dots(\text{i})$$

Motion of the pulley is pure rotational.

If we take a point P on the rim of the pulley, its tangential acceleration should be equal to αR . This acceleration should be equal to the acceleration of the block.



$$\text{Hence } a = \alpha R \quad \dots(\text{ii})$$

Torque equation for the pulley: $\tau_c = I_c \alpha$

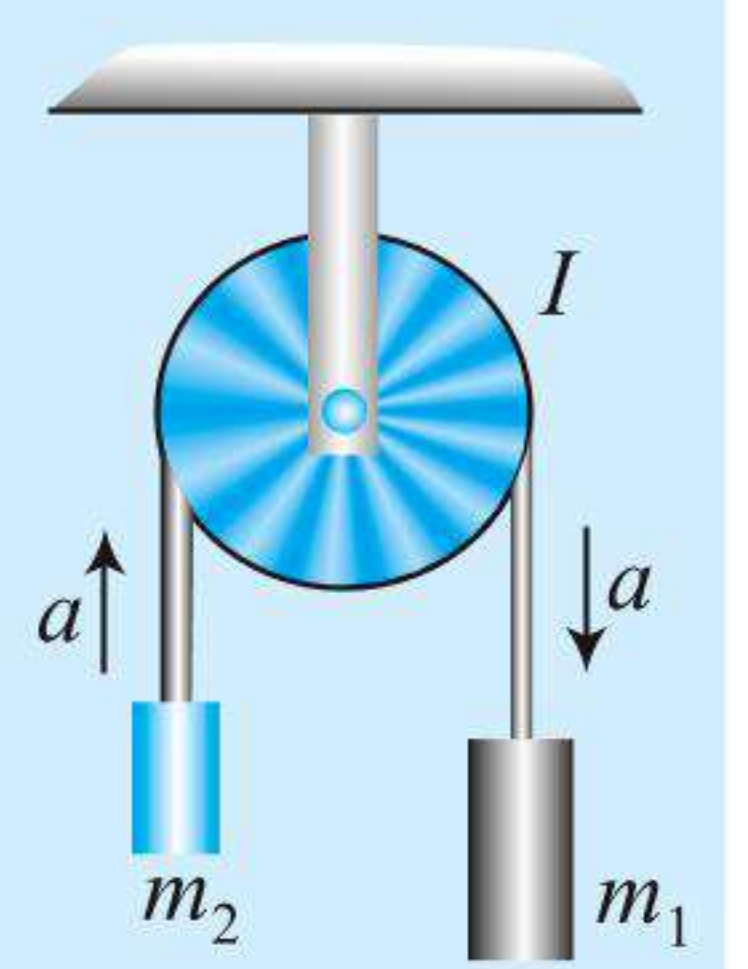
$$TR = \left(\frac{MR^2}{2} \right) \alpha \Rightarrow \alpha = \frac{2T}{MR} \quad \dots(\text{iii})$$

After solving Eqs. (i), (ii) and (iii), we get

$$a = \frac{2mg}{2m + M}$$

ILLUSTRATION 3.35

Find the acceleration of m_1 and m_2 in an Atwood's Machine if there is friction present between surface of pulley and the thread and thread does not slip over the surface of pulley. Moment of inertia of pulley is I .



Sol. Atwood's machine consists of two weights with masses m_1 and m_2 connected by a string or rope, which is guided over a pulley. The Atwood's machine we analyzed in the chapter laws of motion, was subject to the condition that the rope glides without friction over the pulley so that the pulley does not rotate (or, equivalently, that the pulley is massless). With the concepts of rotational dynamics, we can take another look at the Atwood's machine and consider the case where friction occurs between the rope and the pulley, causing the rope to turn the pulley without slipping.

In the chapter laws of motion, we saw that the magnitude of the tension, T , is the same everywhere in a string. However, now friction forces are involved to keep the string attached to the pulley, and we cannot assume that the tension is constant. Instead, the string tension is separately determined in each segment of the string from which one of the two masses hang. Thus, there are two different string tensions, T_1 and T_2 , in the free-body diagrams for the two masses (as shown in figure). As there is friction between pulley and thread, tension in thread throughout will not remain same. Let on two sides of the pulley tension in thread be T_1 and T_2 and the masses are going with acceleration a . As thread does not slip of the surface of pulley, pulley will rotate and the linear acceleration of the particles on pulley at its rim will also be a , thus its angular acceleration can be given as

$$\alpha = \frac{a}{R} \quad \dots(\text{i})$$

Now we write motion equations for masses m_1 and m_2 as

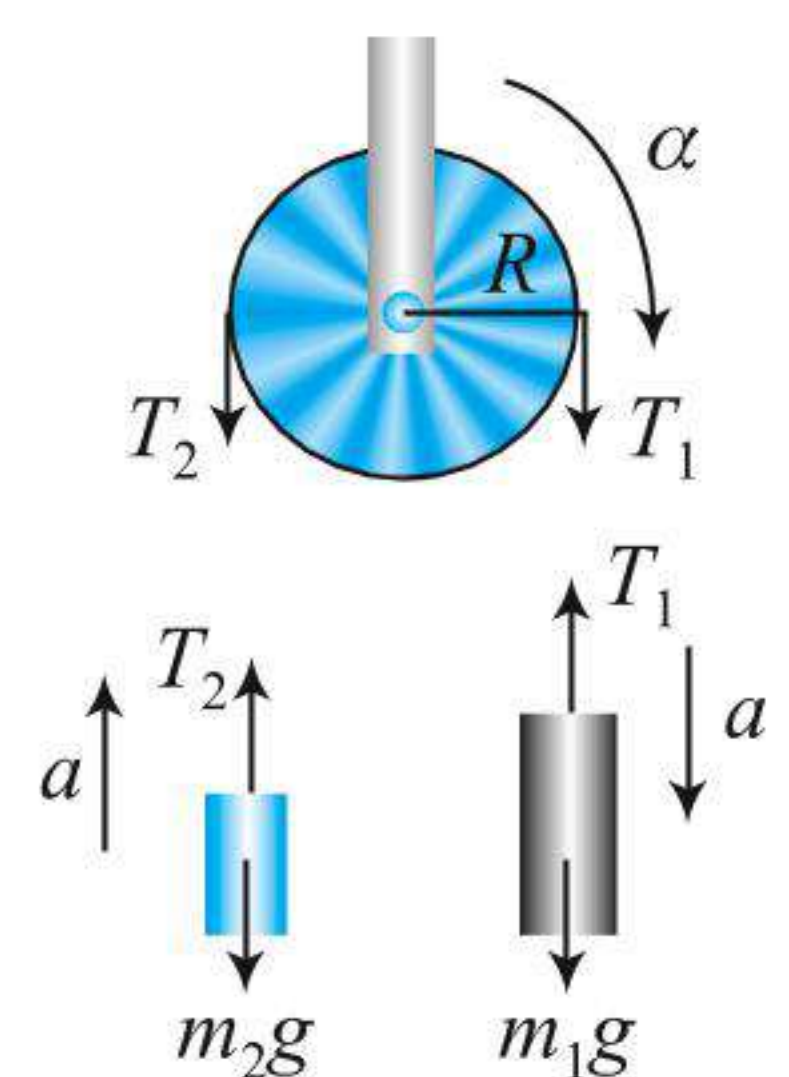
$$T_2 - m_2 g = m_2 a \quad \dots(\text{ii})$$

$$\text{or } m_1 g - T_1 = m_1 a \quad \dots(\text{iii})$$

A net torque does act on the pulley. Torques are applied by different tensions in string on both sides of pulley. The magnitude of the net torque on the pulley due to the string tensions,

$$\tau = \tau_1 - \tau_2 = T_1 R - T_2 R = R(T_1 - T_2) \quad \dots(\text{iv})$$

These two torques have opposite signs, because one is acting clockwise and the other counterclockwise. The net torque is



related to the moment of inertia of the pulley and its angular acceleration by the relation

$$\tau = I\alpha = I \frac{a}{R} \quad \dots(v)$$

Substituting this expression for the torque into Eq. (iv) we find,

$$R(T_1 - T_2) = \tau = I \frac{a}{R}$$

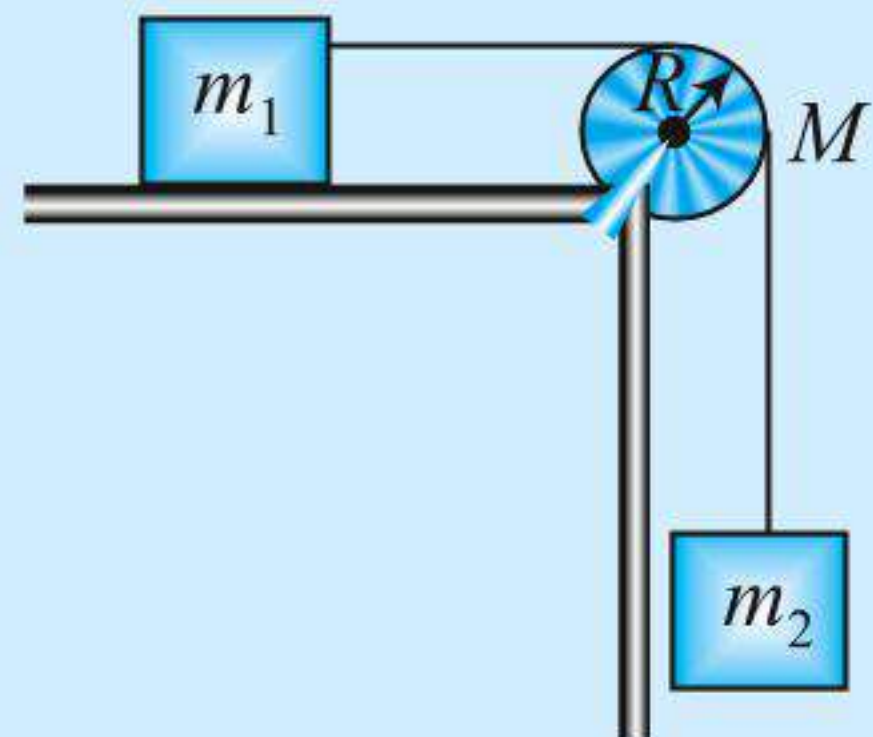
$$\text{or } T_1 - T_2 = I \frac{a}{R^2} \quad \dots(vi)$$

Adding Eqs. (ii), (iii) and (vi), we get

$$(m_1 - m_2)g = \left(m_1 + m_2 + \frac{I}{R^2}\right)a \quad \text{or } a = \frac{(m_1 - m_2)g}{\left(m_1 + m_2 + \frac{I}{R^2}\right)}$$

ILLUSTRATION 3.36

In figure, mass m_1 slides without friction on the horizontal surface, the frictionless pulley is in the form of a cylinder of mass M and radius R , and a string turns the pulley without slipping. Find the acceleration of each mass.



Sol. In this case the string rotates the pulley, i.e., the string does not slide. The pulley will have angular acceleration. Hence, the tension on the string on both the sides of the pulley will not be equal. The motion of m_1 and m_2 will be translational. Positive direction of angular acceleration α is taken along the corresponding acceleration a_1 which is taken positive in the direction of net force.

Let these be T_1 and T_2 . The equation of motion respectively, for masses m_1 and m_2 is

$$T_1 = m_1 a \quad \dots(i)$$

$$\text{and } m_2 g - T_2 = m_2 a \quad \dots(ii)$$

Application of torque equation:

The torque equation for the pulley,

$$T_2 R - T_1 R = I\alpha \quad \dots(iii)$$

(N_2 constitutes no torque about the axis of rotation)

There is no slipping of the string over the pulley.

Constraint relation:

$$a = \alpha R \Rightarrow \alpha = \frac{a}{R} \quad \dots(iv)$$

From Eqs. (iii) and (iv), we get

$$\therefore T_2 R - T_1 R = I \frac{a}{R}$$

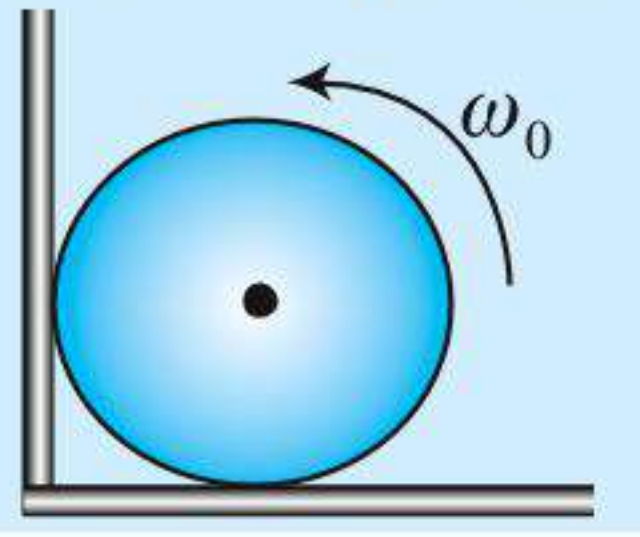
$$\text{or } T_2 - T_1 = \frac{Ia}{R^2} \quad \dots(v)$$

Solving the above equations, we get

$$a = \frac{m_2 g}{\left(m_1 + m_2 + \frac{I}{R^2}\right)}$$

ILLUSTRATION 3.37

A uniform cylinder of radius R is spun about its axis to the angular velocity ω_0 and then placed into a corner, (see figure). The coefficient of friction between the corner walls and the cylinder is μ_k . How many turns will the cylinder accomplish before it stops?

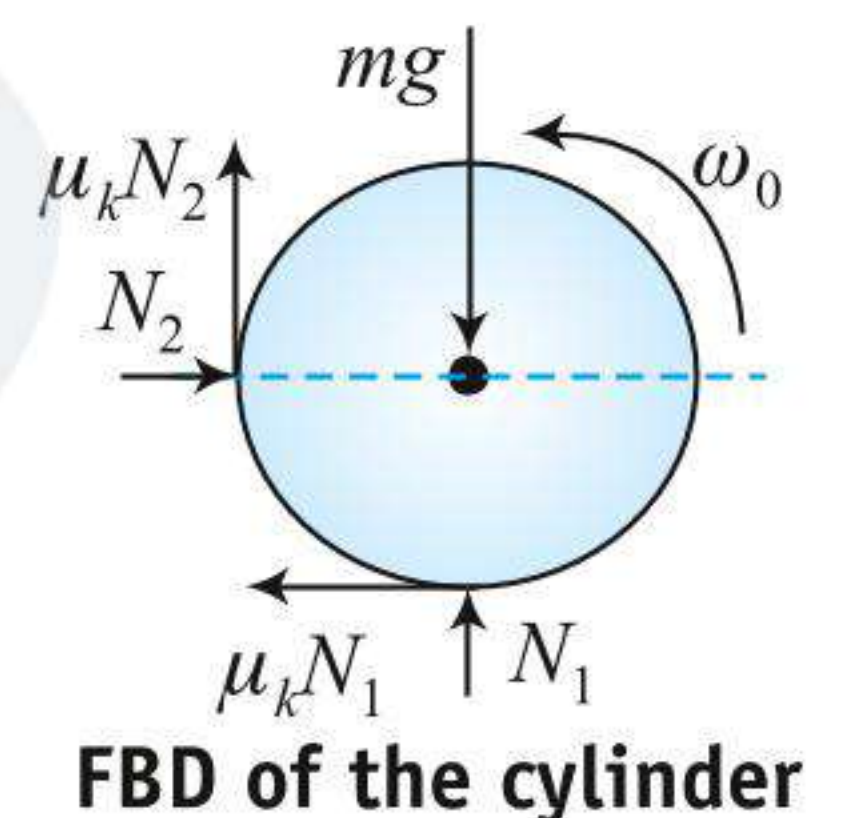


Sol. As the centre of mass of the cylinder does not accelerate, hence

$$\sum F = 0$$

$$\sum F_x = 0, N_2 - \mu_k N_1 = 0, \quad \dots(i)$$

$$\sum F_y = 0, N_1 + \mu_k N_2 - mg = 0 \quad \dots(ii)$$



FBD of the cylinder

Solving these equations:

$$N_1 = \frac{mg}{1 + \mu_k^2}, N_2 = \frac{\mu_k mg}{1 + \mu_k^2}$$

The torque on the cylinder about the axis of rotation,

$$\begin{aligned} \tau_{CM} &= mg \times 0 + N_1 \times 0 + N_2 \times 0 + \mu_k N_2 R + \mu_k N_1 R \\ &= \frac{\mu_k (1 + \mu_k)}{(1 + \mu_k^2)} mgR \end{aligned}$$

The moment of inertia about axis of rotation, $I_{CM} = \frac{1}{2} mR^2$

The torque equation, $\tau = I\alpha$

$$\frac{\mu_k (1 + \mu_k)}{1 + \mu_k^2} mgR = \left(\frac{1}{2} mR^2\right) \alpha \Rightarrow \alpha = \frac{2\mu_k (1 + \mu_k)g}{(1 + \mu_k^2)R}$$

Using equation $\omega^2 = \omega_0^2 + 2\alpha\theta$, calculate the angular displacement ω ,

$$0^2 = \omega_0^2 - 2 \left[\frac{2\mu_k (1 + \mu_k)g}{(1 + \mu_k^2)R} \right] \theta$$

(ω and α are in the opposite sense)

$$\Rightarrow \theta = \frac{(1 + \mu_k^2) \omega_0^2 R}{4\mu_k (1 + \mu_k)g}$$

Revolution accomplished,

$$n = \frac{\theta}{2\pi} = \frac{(1 + \mu_k^2) \omega_0^2 R}{2\pi \times 4\mu_k (1 + \mu_k)g} = \frac{(1 + \mu_k^2) \omega_0^2 R}{8\pi\mu_k (1 + \mu_k)g}$$

ILLUSTRATION 3.38

A uniform rod of length L and mass M is pivoted freely at one end and placed in vertical position.

(a) What is the angular acceleration of the rod when it is at an angle θ with the vertical?

(b) What is the tangential linear acceleration of the free end when the rod is horizontal?

Sol. Figure shows the rod at an angle θ to the vertical. If we take torques about the pivot we need not be concerned with the force due to the pivot.

The torque due to the weight about O ,

$$\tau_w = \frac{mgL}{2} \sin \theta$$

Using torque equation about O . Here we apply torque equation about fixed axis passing through O . We can apply torque equation about fixed axis or about centre of mass only.

$$(a) \quad \tau = I \alpha \Rightarrow \frac{mgL}{2} \sin \theta = \frac{ML^2}{3} \alpha$$

$$\text{Thus, } \alpha = \frac{3g \sin \theta}{2L}$$

$$\text{When the rod is horizontal, } \theta = \frac{\pi}{2} \text{ and } \alpha = \frac{3g}{2L}$$

(b) The tangential linear acceleration of the free end is

$$a_t = \alpha L = \frac{3g}{2}$$

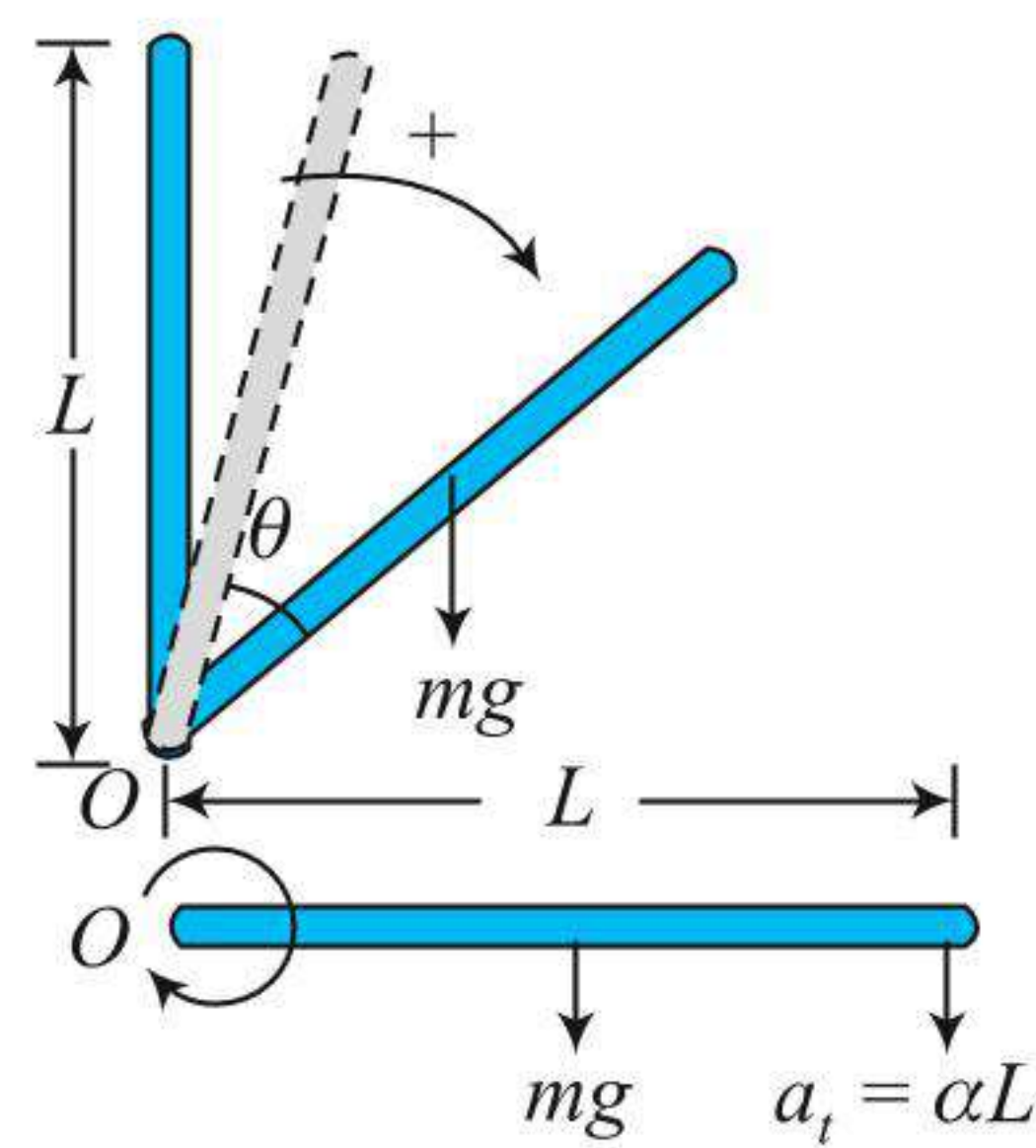
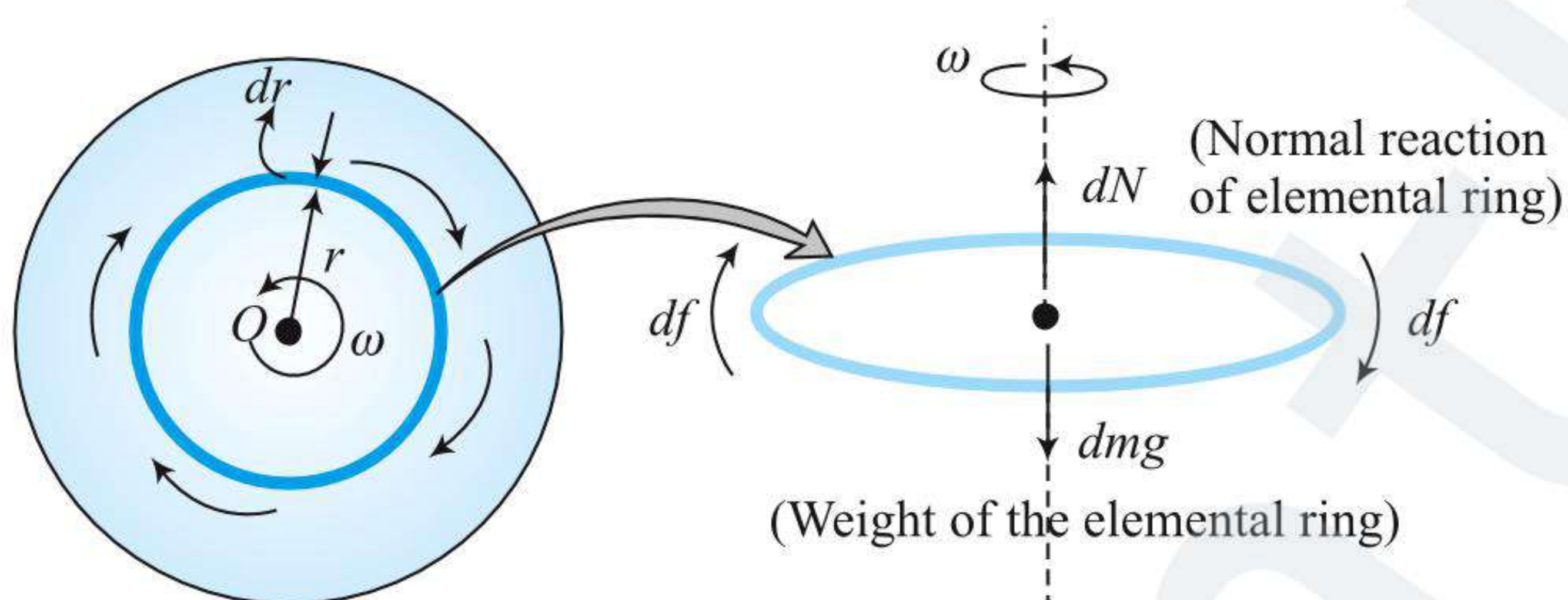


ILLUSTRATION 3.39

A uniform disc of radius R and mass M , spinning with angular velocity ω_0 is carefully lowered onto the rough horizontal surface with coefficient of friction μ . Find the time when the disc comes at rest.

Sol. The angular velocity of the disc changes due to torque of the friction force between the lower surface of the disc and floor. First, we need to calculate τ , let us take an elementary ring of radius r and thickness dr on lower surface of the disc.



The mass of the elemental ring will be distributed on its length will act in downward direction which will be equal to normal reaction of the ground.

$$dN = dm g$$

Hence magnitude of friction force on the ring

$$df = \mu dN = \mu dm g = \mu \left(\frac{M}{\pi R^2} \right) \times 2\pi r dr \times g$$

$$\Rightarrow df = \frac{2\mu M g}{R^2} r dr$$

This friction force is distributed on the length of the ring.

The torque of friction force on the elemental ring

$$d\tau = df \times r = \left(\frac{2\mu M g}{R^2} \right) r^2 dr \quad \dots(i)$$

To calculate net torque on the disc we need to integrate Eq. (i) between 0 and R .

$$\tau = \int d\tau = \frac{2\mu M g}{R^2} \int_0^R r^2 dr \quad \dots(ii)$$

From Eq. (ii) we can observe that the torque acting on the disc is constant, hence angular acceleration will be constant.

Now applying torque equation

$$\tau = I \alpha$$

$$\alpha = \frac{\tau}{I} = \frac{\frac{2}{3} \mu M g R}{\frac{MR^2}{2}}$$

$$\Rightarrow \alpha = \frac{4\mu g}{3R} \quad \dots(iii)$$

The initial angular velocity of the disc is ω_0 and finally it stops hence final angular velocity is zero.

Using, $\omega = \omega_0 + \alpha t$

$$0 = \omega_0 - \left(\frac{4\mu g}{3R} \right) t \text{ or } t = \frac{3\omega_0 R}{4\mu g}$$

CONCEPT APPLICATION EXERCISE 3.6

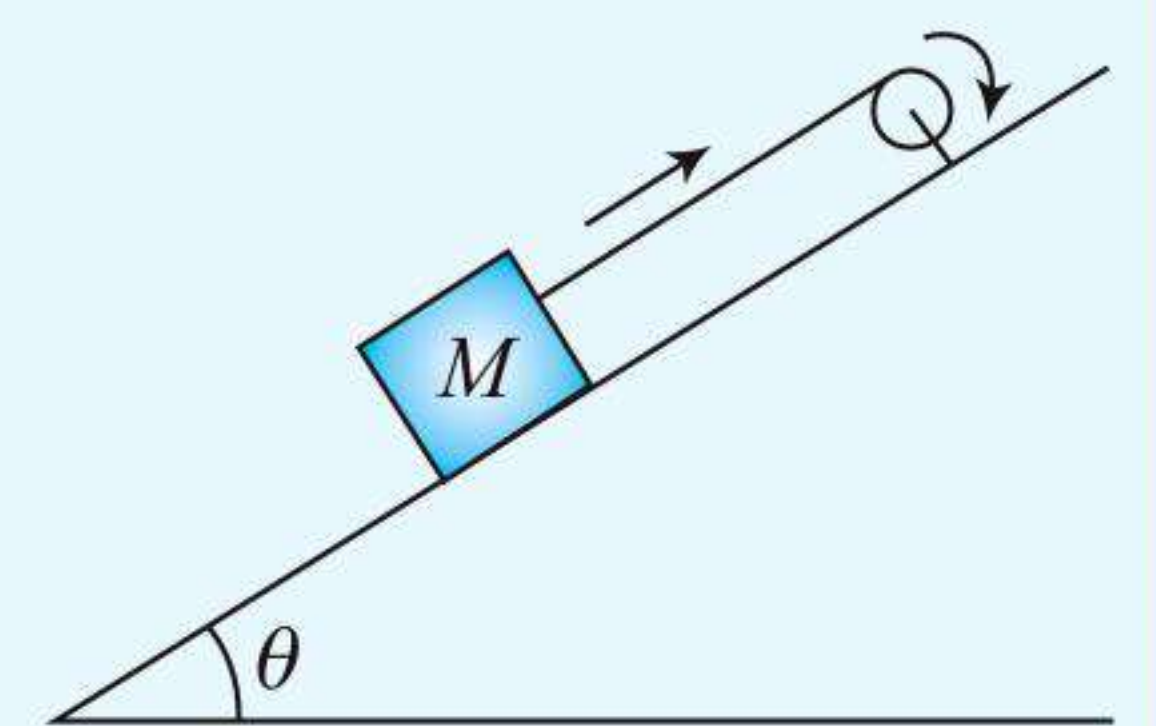
1. A uniform rod of mass m and length l can rotate in a vertical plane about a smooth horizontal axis hinged at point H .

(a) Find angular acceleration α of the rod just after it is released from initial horizontal position from rest?

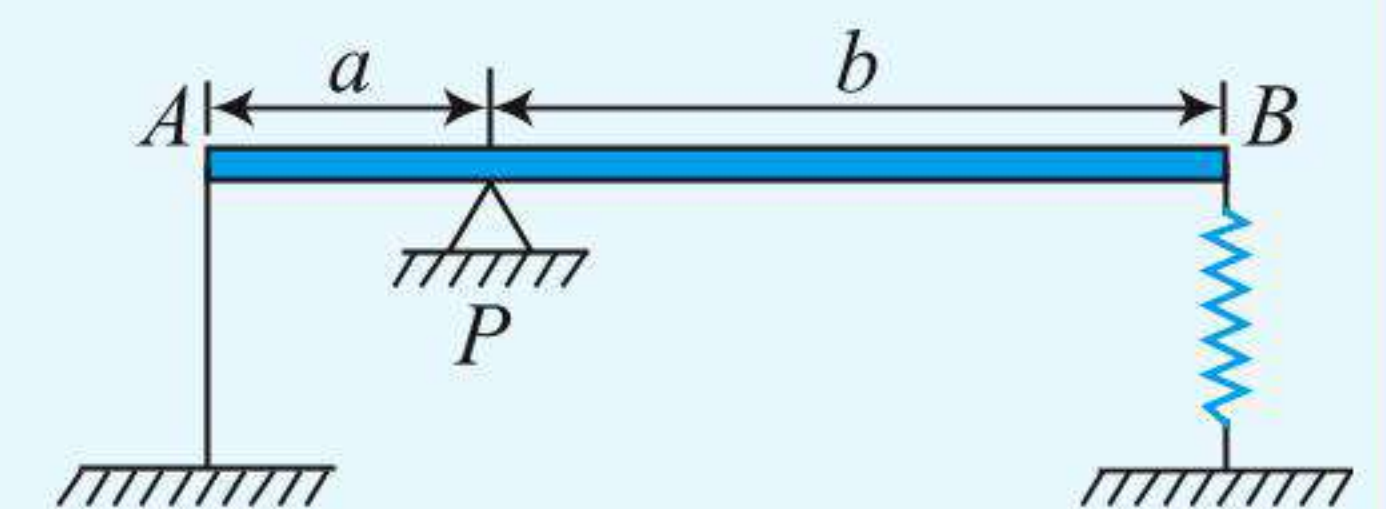
(b) Calculate the acceleration (tangential and radial) of point A at this moment.

2. A uniform rod of mass m and length l can rotate in a vertical plane about a smooth horizontal axis hinged at point H . Find the force exerted by the hinge just after the rod is released from rest, from an initial horizontal position?

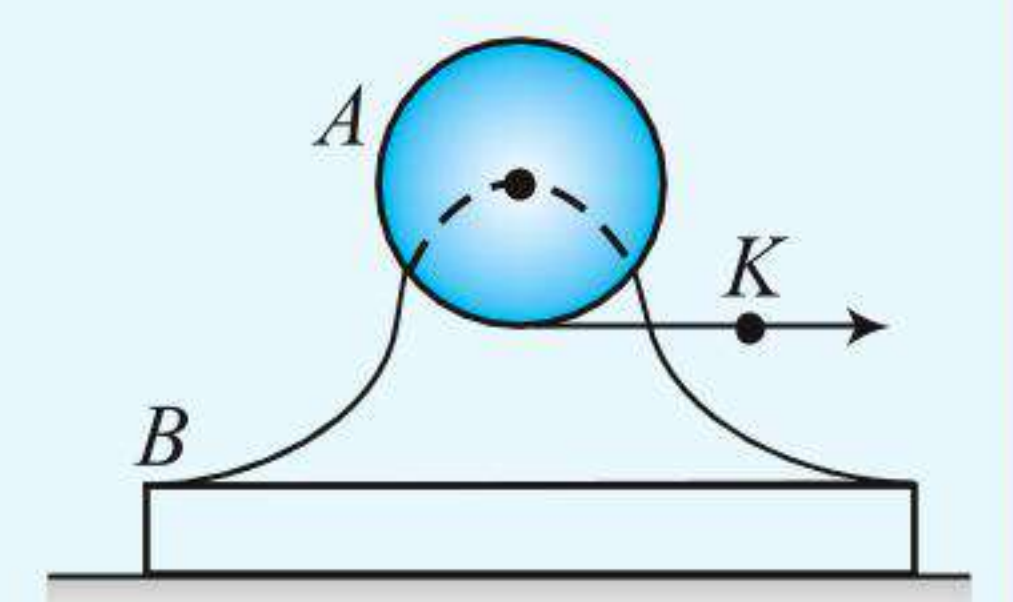
3. A wheel of radius r and moment of inertia I about its axis is fixed at the top of an inclined plane of inclination θ as shown in figure. A string is wrapped around the wheel and its free end supports a block of mass M which can slide on the plane. Initially, the wheel is rotating at a speed ω in a direction such that the block slides up the plane. How far will the block move before stopping?



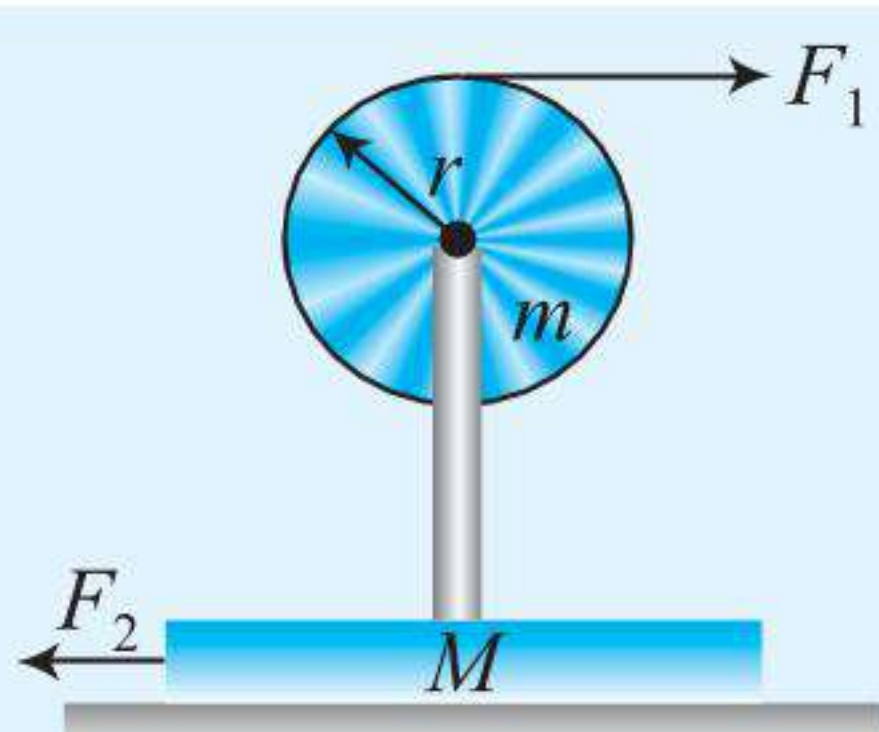
4. A uniform rod AB of mass $m = 2$ kg and length $l = 1.0$ m is placed on a sharp support P such that $a = 0.4$ m and $b = 0.6$ m. A spring of force constant $k = 600$ N/m is attached to end B as shown in figure. To keep the rod horizontal, its end A is tied with a thread such that the spring is elongated by 1 cm. Calculate reaction of support P when the thread is burnt.



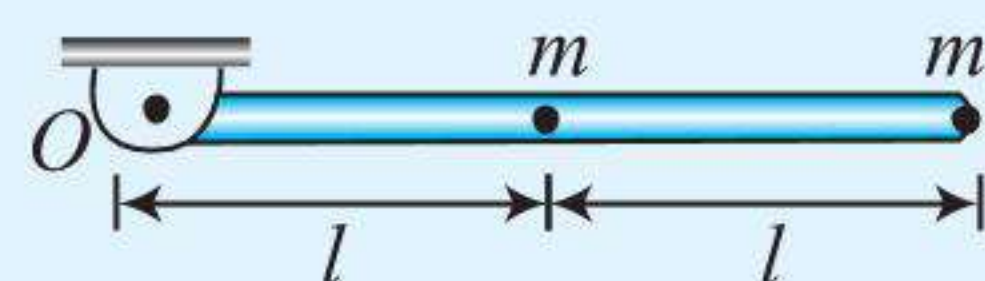
5. A uniform solid cylinder A of mass m_1 can freely rotate about a horizontal axis fixed to a mount of mass m_2 . A constant horizontal force F is applied to the end K of a light thread tightly wound on the cylinder. The friction between the mount and the supporting horizontal plane is assumed to be absent. Find the acceleration of the point K .



6. A disc of mass m and radius r is pivoted smoothly with a block of mass M . A force F_1 acts on m and another force F_2 acts on M . Assume that ground is smooth. Find the:

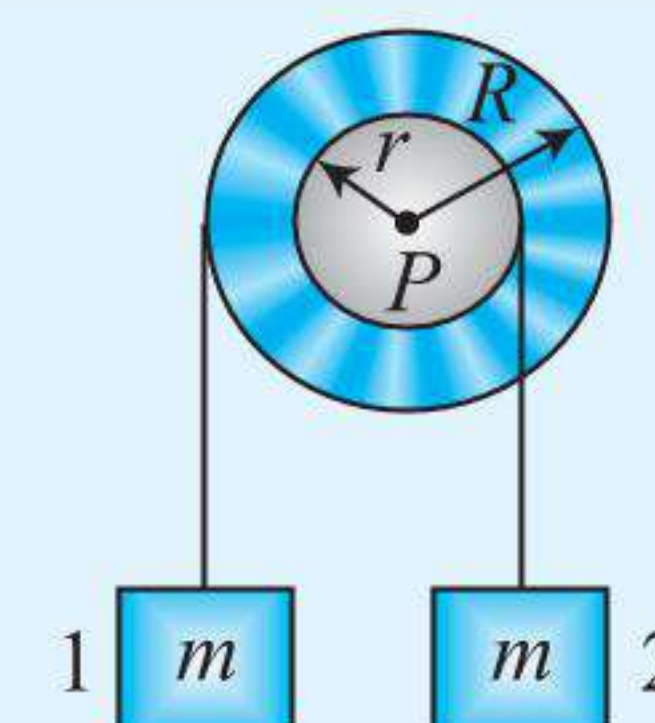


- (i) linear acceleration of the bodies.
(ii) angular acceleration of the disc.
7. A light rigid rod is connected rigidly with two identical particles each of mass m as shown. The free end of the rod is smoothly pivoted at O . The rod is released from rest from its horizontal position at $t = 0$. Find:



- (i) angular acceleration of the rod, at $t = 0$.
(ii) reaction offered by the pivot, at $t = 0$.

8. Two blocks each of mass m are hanging from a stepped pulley by an inextensible light string which passes over a stepped pulley. The mass of the pulley is m and its radius of gyration is k . If the string does not slide on the pulley, find the accelerations of the blocks.



ANSWERS

1. (a) $\frac{3g}{2l}$ (b) $\frac{3g}{2}; 0$ 2. $\frac{mg}{4}$ 3. $\frac{(I + Mr^2)\omega^2}{2Mg \sin \theta}$
4. 20 N 5. $\frac{F(3m_1 + 2m_2)}{m_1(m_1 + m_2)}$ 6. (i) $\frac{F_1 - F_2}{m + M}$ (ii) $\frac{2F_1}{mr}$
7. (i) $\frac{3g}{5l}$ (ii) $\frac{mg}{5}$ 8. $a_1 = \frac{gR(R - r)}{k^2 + (R^2 + r^2)}$, $a_2 = \frac{gr(R - r)}{k^2 + (R^2 + r^2)}$

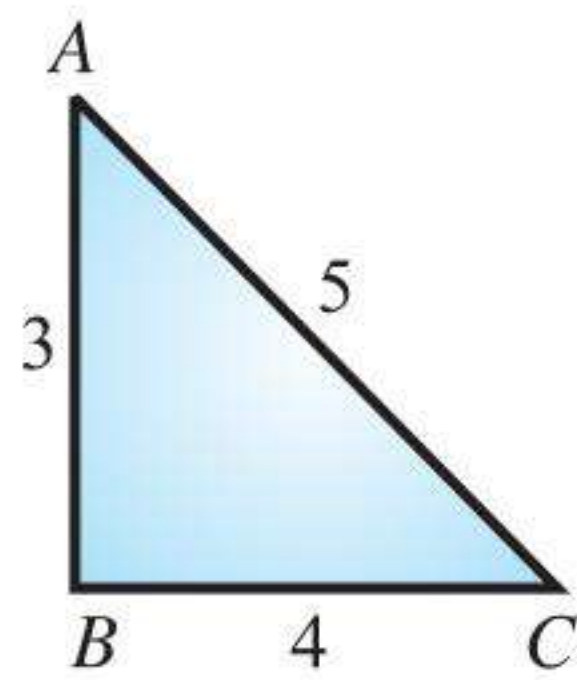
Exercises

Single Correct Answer Type

1. The wheels of an airplane are set into rotation just before landing so that the wheels do not slip on the ground. If the airplane is travelling in the east direction, what should be the direction of angular velocity vector of the wheels?

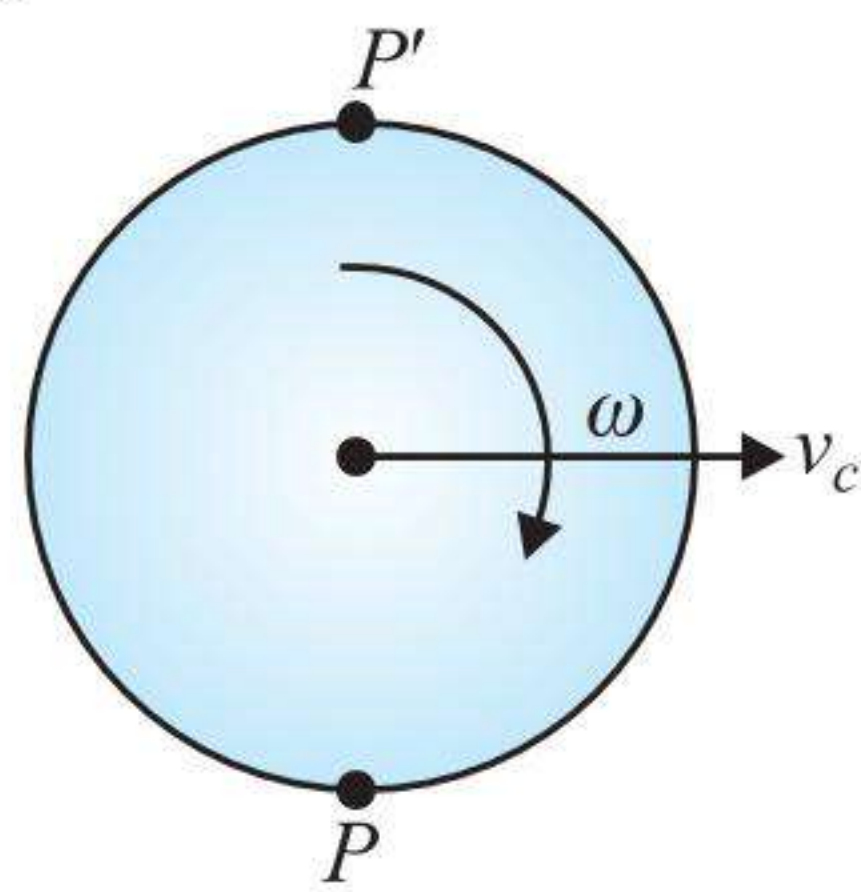
(1) East (2) West
(3) South (4) North

2. ABC is a triangular plate of uniform thickness. The sides are in the ratio shown in the figure. I_{AB} , I_{BC} , I_{CA} are the moments of inertia of the plate about AB , BC and CA respectively. Which one of the following relation is correct?



(1) I_{CA} is maximum (2) $I_{AB} > I_{BC}$
(3) $I_{BC} > I_{AB}$ (4) $I_{AB} + I_{BC} = I_{CA}$

3. A sphere is moving towards the positive x -axis with a velocity v_c and rotates clockwise with angular speed ω as shown in figure, such that $v_c > \omega R$. The instantaneous axis of rotation will be



(1) on point P (2) on point P'
(3) inside the sphere (4) outside the sphere

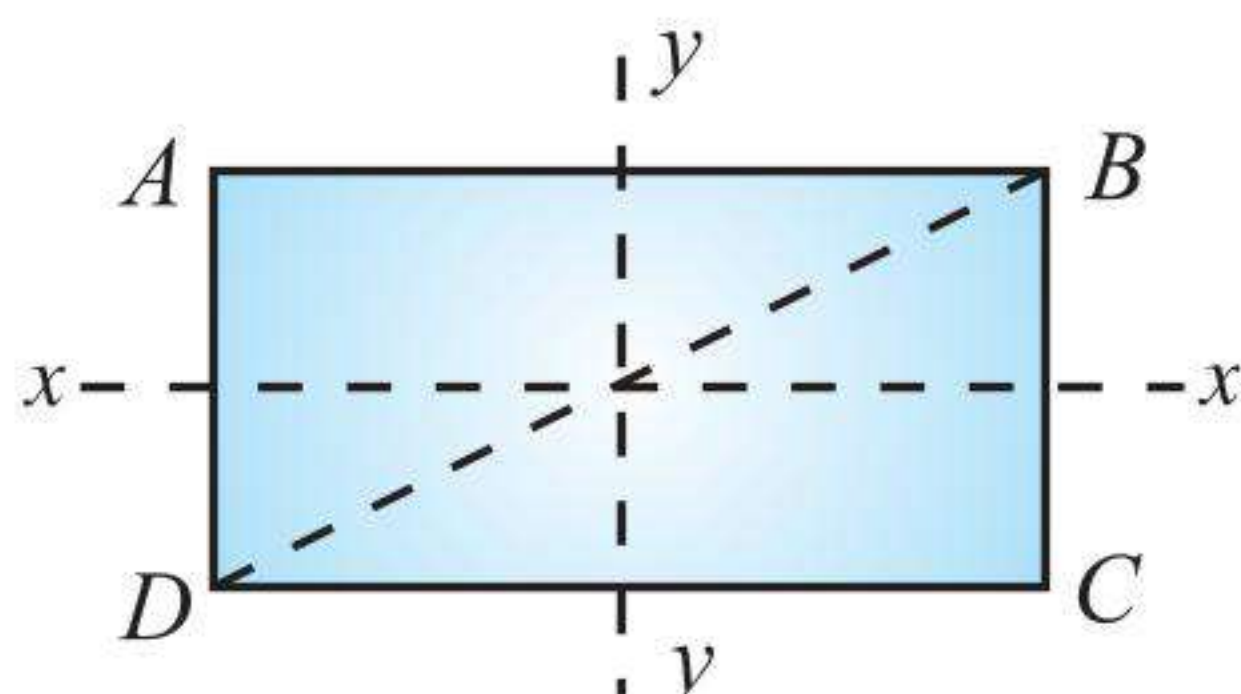
4. Let I_A and I_B be moments of inertia of a body about two axes A and B respectively. The axis A passes through the centre of mass of the body but B does not.

(1) $I_A < I_B$
(2) If $I_A < I_B$ the axes are parallel
(3) If the axes are parallel, $I_A < I_B$
(4) If the axes are not parallel, then $I_A \geq I_B$.

5. For the same total mass, which of the following will have the largest moment of inertia about an axis passing through the centre of gravity and perpendicular to the plane of body?

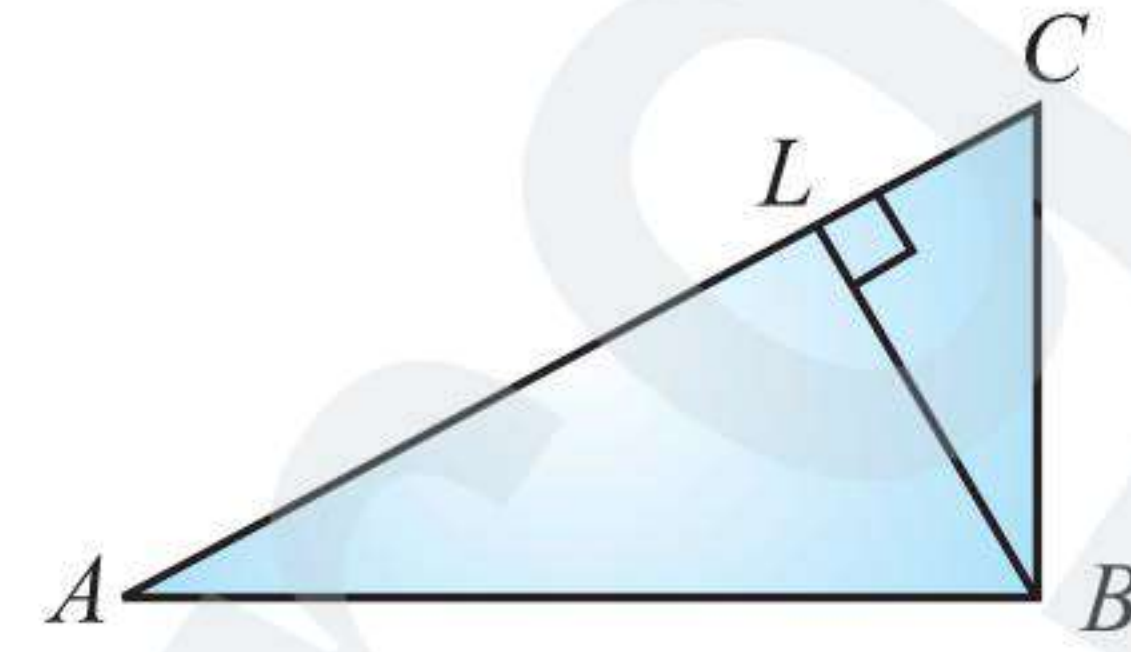
(1) A ring of radius l
(2) A disc of radius l
(3) A square lamina of side $2l$
(4) Four rods forming square of side $2l$

6. In rectangle $ABCD$, $AB = 2l$ and $BC = l$. Axes xx and yy pass through the centre of the rectangle. The moment of inertia is least about



(1) DB (2) BC
(3) xx (4) yy

7. About which axis moment of inertia in the given triangular lamina is maximum?



(1) AB (2) BC
(3) AC (4) BL

8. Three point masses m_1 , m_2 and m_3 are located at the vertices of an equilateral triangle of side a . What is the moment of inertia of the system about an axis along the altitude of the triangle passing through m_1 ?

(1) $(m_1 + m_2) \frac{a^2}{4}$ (2) $(m_2 + m_3) \frac{a^2}{4}$
(3) $(m_1 + m_3) \frac{a^2}{4}$ (4) $(m_1 + m_2 + m_3) \frac{a^2}{4}$

9. The density of a rod continuously increases from A to B . It is easier to set it into rotation by

(1) clamping the rod at A and applying a force F at B , perpendicular to the rod
(2) clamping the rod at B and applying a force F at A , perpendicular to the rod
(3) clamping the rod at mid point of AB and applying a force F at A , perpendicular to the rod
(4) clamping the rod at mid-point of AB and applying force F at B , perpendicular to the rod

10. Two rings of same radius and mass are placed such that their centres are at a common point and their planes are perpendicular to each other. The moment of inertia of the system about an axis passing through the centre and perpendicular to the plane of one of the rings is (mass of the ring = m , radius = r)

(1) $\frac{1}{2} mr^2$ (2) mr^2
(3) $\frac{3}{2} mr^2$ (4) $2mr^2$

11. The moment of inertia of a solid sphere about an axis passing through the centre of gravity is $\frac{2}{5} MR^2$; then its radius of gyration about a parallel axis at a distance $2R$ from first axis is

(1) $5R$ (2) $\sqrt{\frac{22}{5}} R$
(3) $\frac{5}{2} R$ (4) $\sqrt{\frac{12}{5}} R$

12. From a given sample of uniform wire, two circular loops P and Q are made, P of radius r and Q of radius nr . If the M.I. of Q about its axis is four times that of P about its axis (assuming the wire to be of diameter much smaller than either radius), the value of n is

(1) $(4)^{\frac{2}{3}}$ (2) $(4)^{\frac{1}{3}}$
(3) $(4)^{\frac{1}{2}}$ (4) $(4)^{\frac{1}{4}}$

13. Two circular discs A and B are of equal masses and thicknesses but made of metal with densities d_A and d_B ($d_A > d_B$). If their moments of inertia about an axis passing through their centres and perpendicular to circular faces are I_A and I_B , then

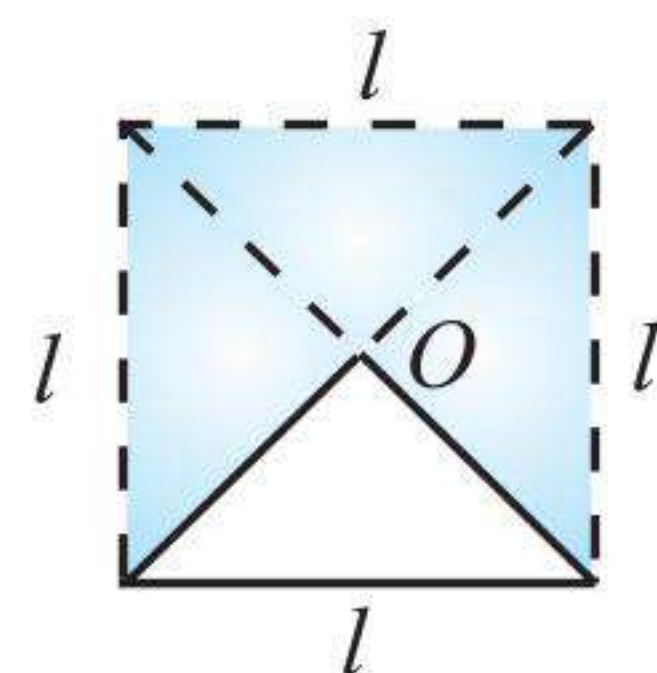
(1) $I_A = I_B$ (2) $I_A > I_B$
 (3) $I_A < I_B$ (4) $I_A \geq I_B$

14. Four identical rods are joined end to end to form a square. The mass of each rod is M . The moment of inertia of the square about the median line is

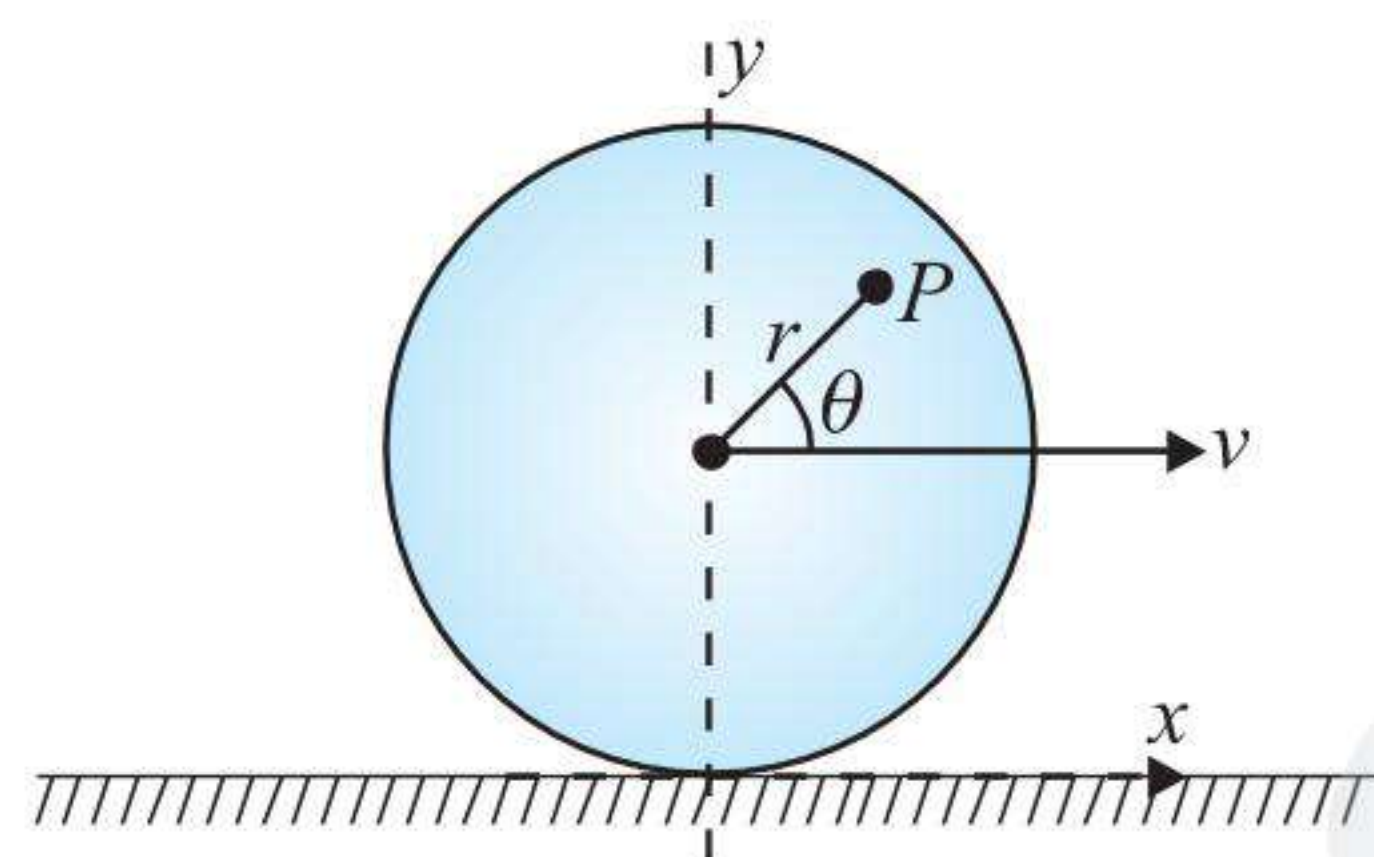
(1) $\frac{Ml^2}{3}$ (2) $\frac{Ml^2}{4}$
 (3) $\frac{Ml^2}{6}$ (4) none of these

15. An isosceles triangular piece is cut from a square plate of side l . The piece is one-fourth of the square and mass of the remaining plate is M . The moment of inertia of the plate about an axis passing through O and perpendicular to its plane is

(1) $\frac{Ml^2}{6}$ (2) $\frac{Ml^2}{12}$
 (3) $\frac{Ml^2}{24}$ (4) $\frac{Ml^2}{3}$

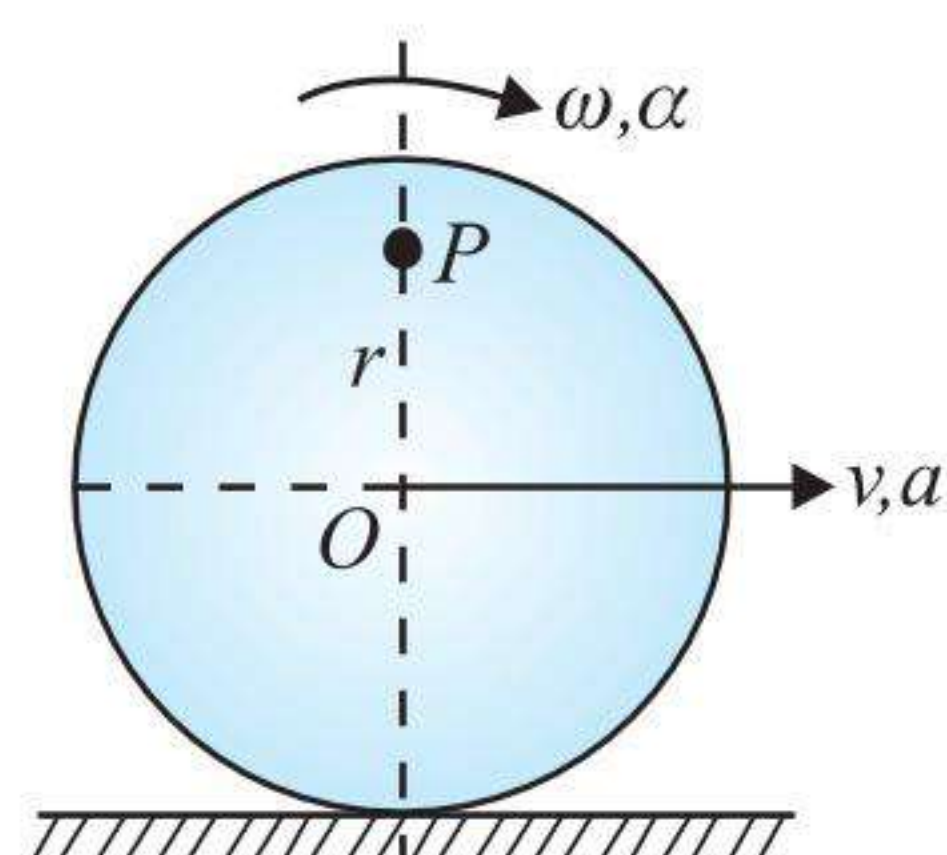


16. A disc of radius R rolls without slipping at speed v along positive x -axis. Velocity of point P at the instant shown in figure is



(1) $\vec{v}_p = \left(v + \frac{vr \sin \theta}{R} \right) \hat{i} + \frac{vr \cos \theta}{R} \hat{j}$
 (2) $\vec{v}_p = \left(v + \frac{vr \sin \theta}{R} \right) \hat{i} - \frac{vr \cos \theta}{R} \hat{j}$
 (3) $\vec{v}_p = \frac{vr \sin \theta}{R} \hat{i} + \frac{vr \cos \theta}{R} \hat{j}$
 (4) $\vec{v}_p = \frac{vr \sin \theta}{R} \hat{i} - \frac{vr \cos \theta}{R} \hat{j}$

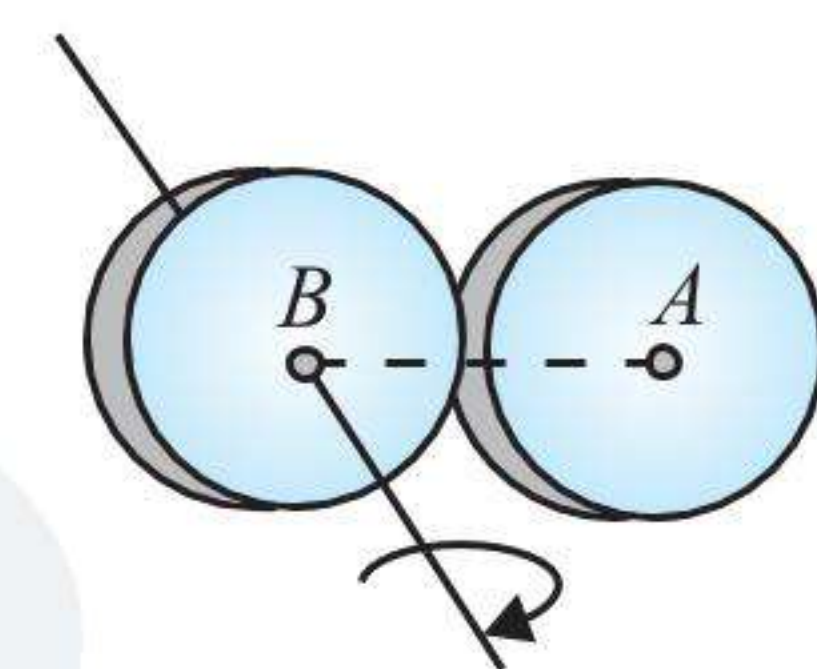
17. A disc of radius R rolls on a horizontal ground with linear acceleration a and angular acceleration α as shown in figure. The magnitude of acceleration of point P as shown in the figure at an instant when its linear velocity is v and angular velocity is ω will be



(1) $\sqrt{(a + r\alpha)^2 + (r\omega^2)^2}$ (2) $\frac{ar}{R}$
 (3) $\sqrt{r^2\alpha^2 + r^2\omega^4}$ (4) $r\alpha$

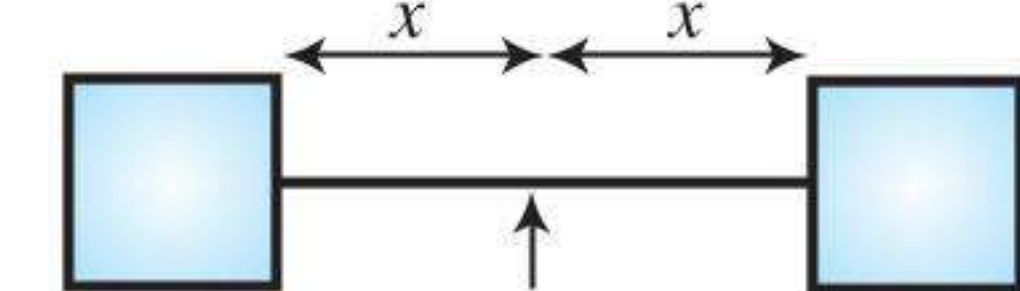
18. Two thin discs, each of mass M and radius r metre, are attached as shown in figure, to form a rigid body. The rotational inertia of this body about an axis perpendicular to the plane of disc B passing through its centre is

(1) $2Mr^2$ (2) $3Mr^2$
 (3) $4Mr^2$ (4) $5Mr^2$



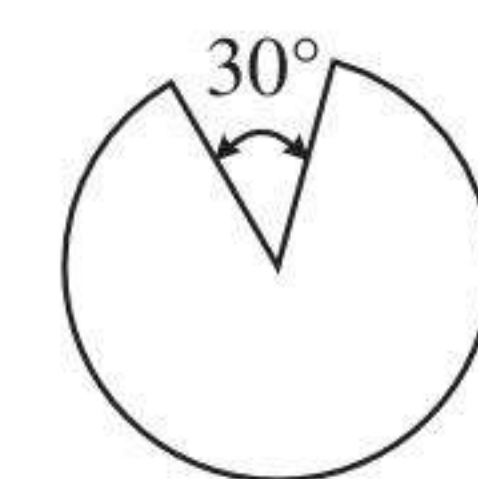
19. Two identical masses are connected to a horizontal thin massless rod as shown in the figure. When their distance from the pivot is x , a torque produces an angular acceleration α_1 . If the masses are now repositioned so that they are at distance $2x$ each from the pivot, the same torque will produce an angular acceleration α_2 such that

(1) $\alpha_2 = 4\alpha_1$ (2) $\alpha_2 = \alpha_1$
 (3) $\alpha_2 = \frac{\alpha_1}{2}$ (4) $\alpha_2 = \frac{\alpha_1}{4}$



20. From a complete ring of mass M and radius R , a 30° sector is removed. The moment of inertia of the incomplete ring about an axis passing through the centre of the ring and perpendicular to the plane of the ring is

(1) $\frac{9}{12}MR^2$ (2) $\frac{11}{12}MR^2$
 (3) $\frac{11.3}{12}MR^2$ (4) MR^2



21. A uniform disc of mass M and radius R is mounted on an axle supported in frictionless bearings. A light cord is wrapped around the rim of the disc and a steady downward pull T is exerted on the cord. The angular acceleration of the disc is

(1) $\frac{T}{MR}$ (2) $\frac{MR}{T}$
 (3) $\frac{2T}{MR}$ (4) $\frac{MR}{2T}$

22. In problem 21, the tangential acceleration of a point on the rim is

(1) $\frac{T}{M}$ (2) $\frac{MR}{T}$
 (3) $\frac{2T}{M}$ (4) $\frac{MR}{2T}$

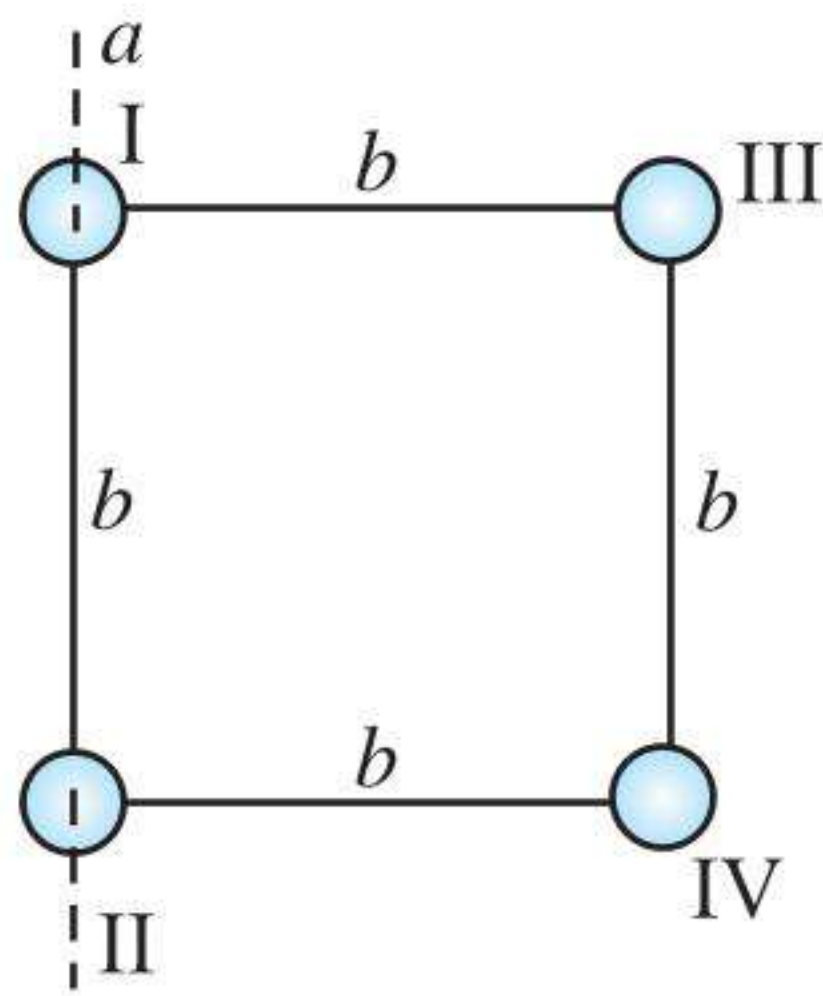
23. If we hang a body of mass m with the cord in problem 21, the tangential acceleration of the disc will be

(1) $\frac{mg}{M+m}$ (2) $\frac{mg}{M+2m}$
 (3) $\frac{2mg}{M+2m}$ (4) $\frac{M+2m}{2mg}$

24. The tension in the cord in problem 21 is

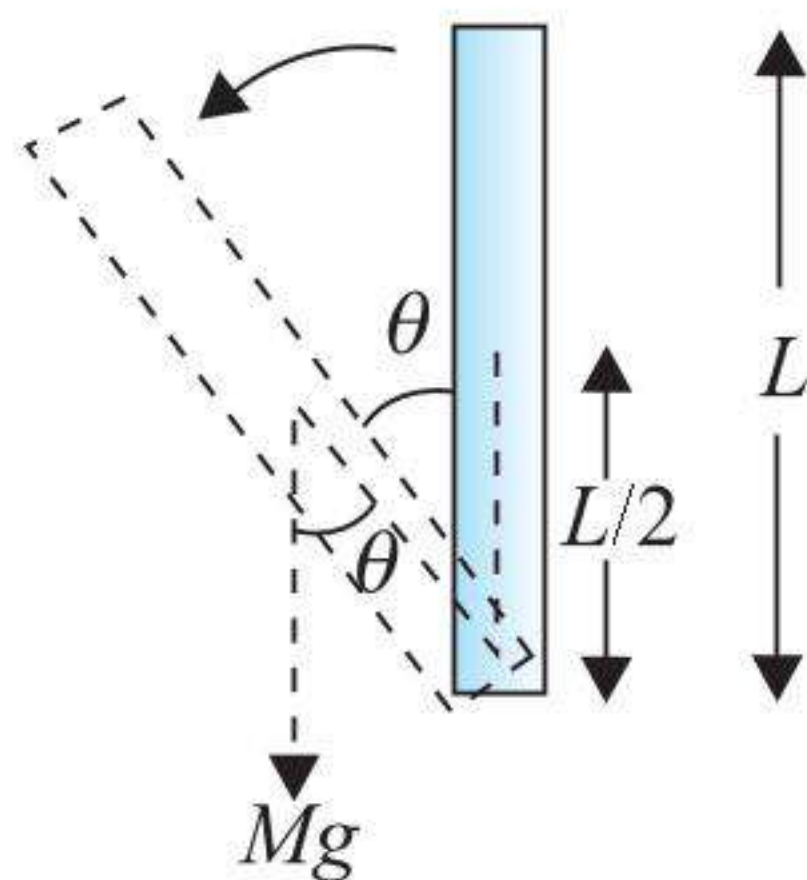
(1) $\frac{mg}{M+m}$ (2) $\frac{Mmg}{M+2m}$
 (3) $\frac{2mg}{Mmg}$ (4) none of these

25. Four spheres, each of mass m and radius r , are placed with their centres on the four corners of a square of side b . The moment of inertia of the system about any side of the square is



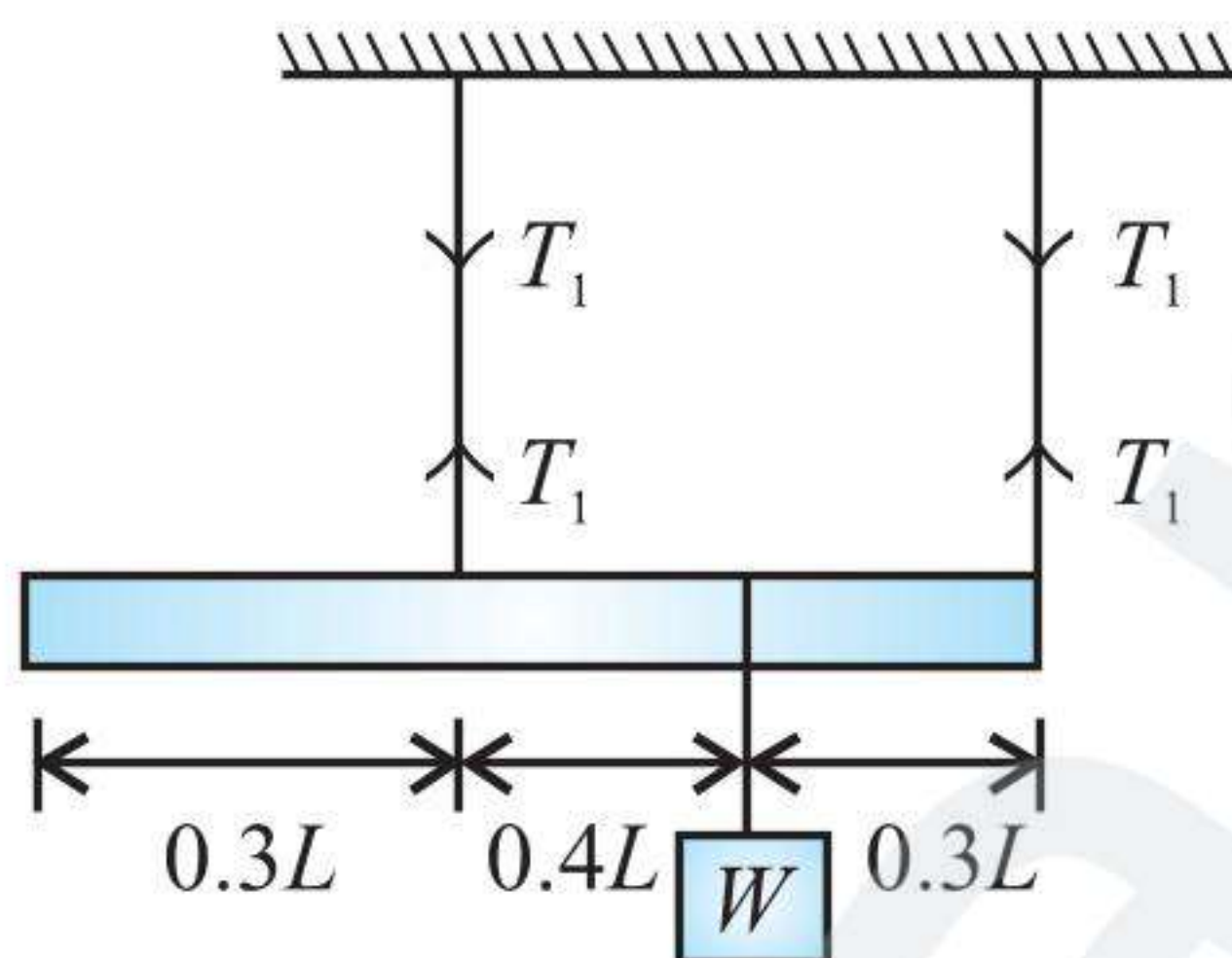
- (1) $\frac{8}{5}mr^2 + mb^2$ (2) $\frac{8}{5}mr^2 + 2mb^2$
 (3) $\frac{8}{5}mr^2 + 4mb^2$ (4) none of these

26. A uniform rod of mass M and length L is pivoted at one end such that it can rotate in a vertical plane. There is negligible friction at the pivot. The free end of the rod is held vertically above the pivot and then released. The angular acceleration of the rod when it makes an angle θ with the vertical is

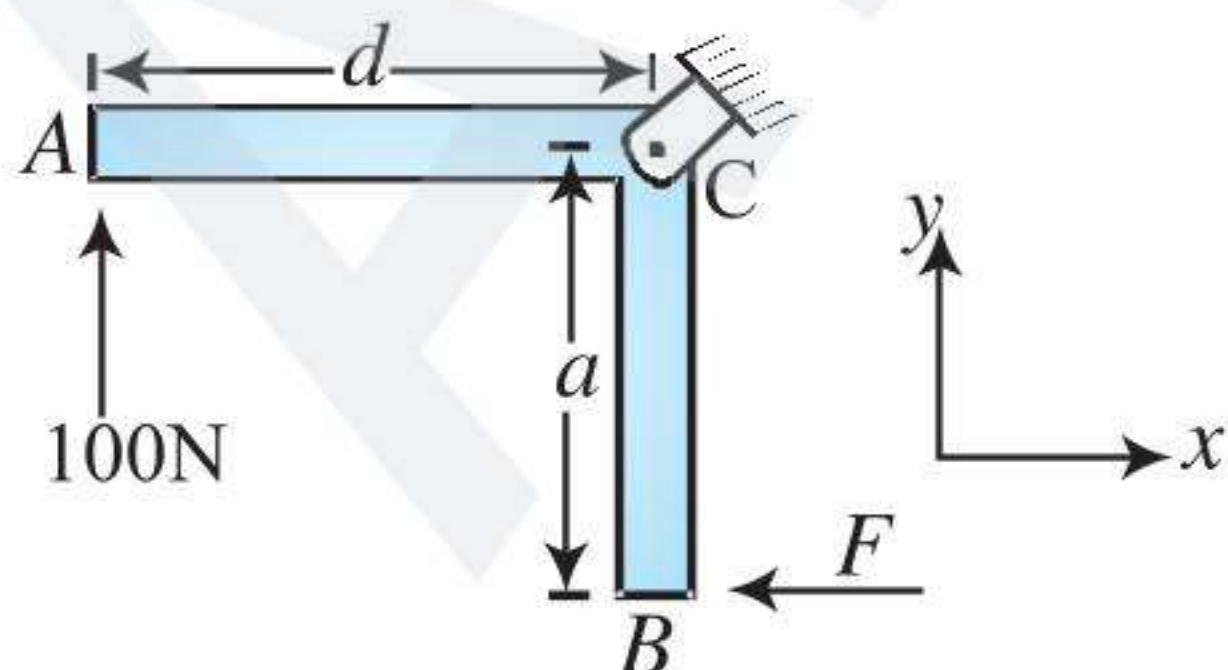


- (1) $g \sin \theta$ (2) $\frac{g}{L} \sin \theta$
 (3) $\frac{3g}{2L} \sin \theta$ (4) $6gL \sin \theta$

27. In figure, the bar is uniform and weighing 500 N. How large must W be if T_1 and T_2 are to be equal?

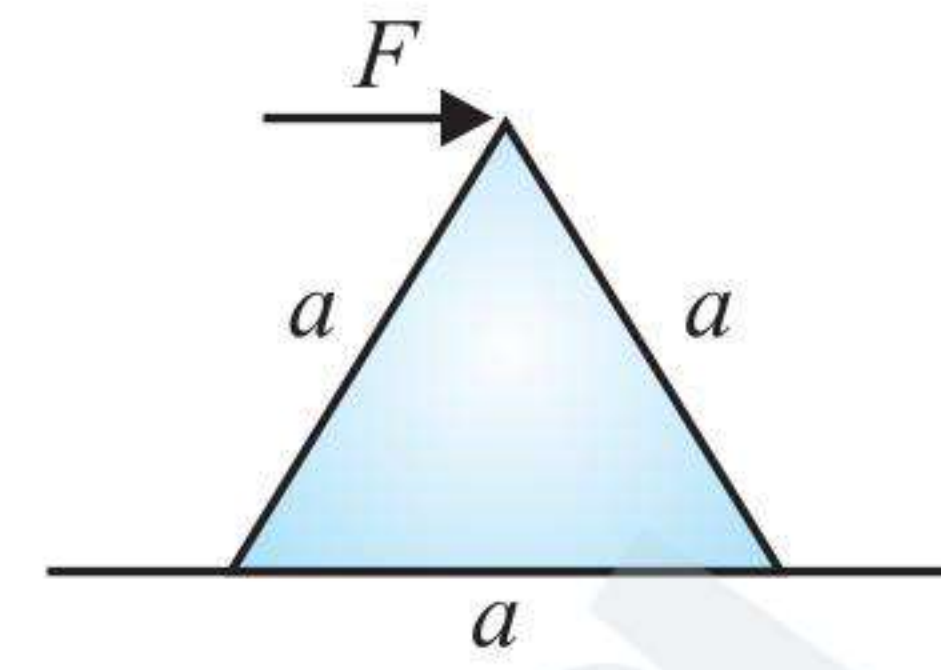


- (1) 500 N (2) 300 N
 (3) 750 N (4) 1500 N
28. Find force F required to keep the system in equilibrium. The dimensions of the system are $d = 0.3$ m and $a = 0.2$ m. Assume the rods to be massless.



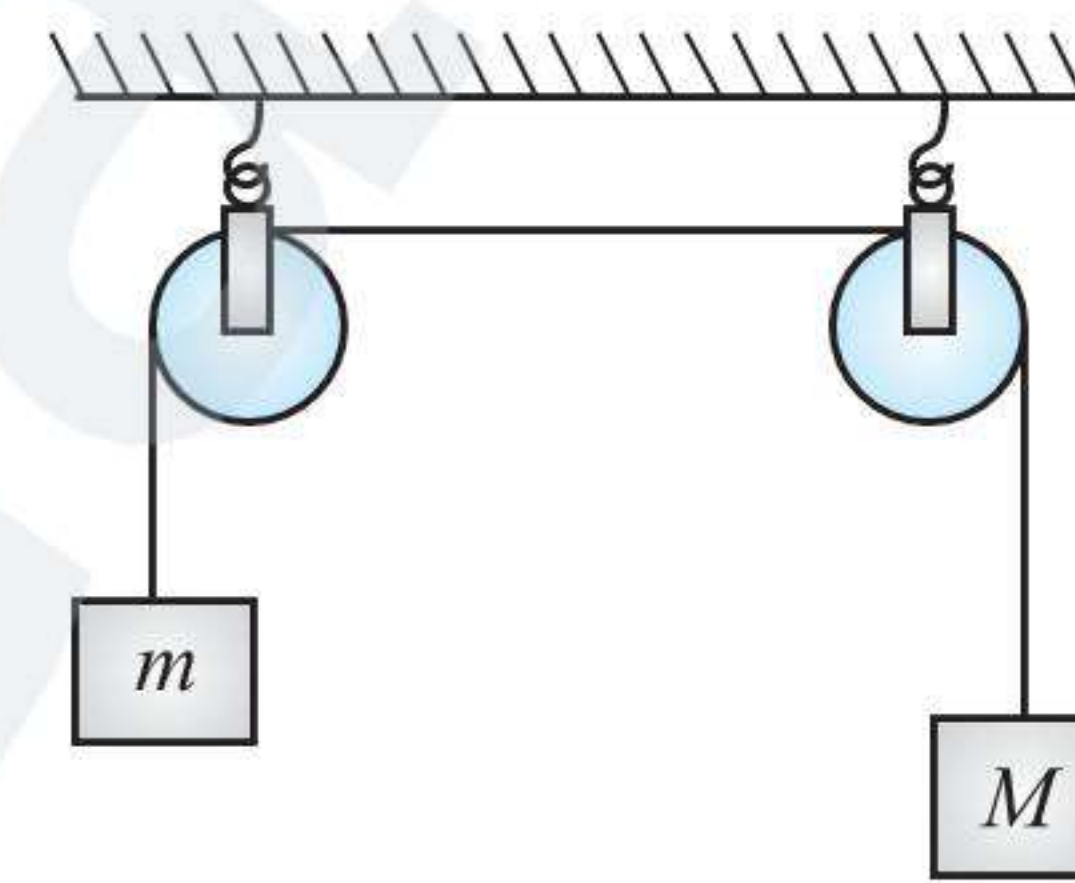
- (1) $150(\hat{i})$ (2) $150(-\hat{k})$
 (3) $150(-\hat{i})$
 (4) It cannot be in equilibrium
29. An equilateral prism of mass m rests on a rough horizontal surface with coefficient of friction μ . A horizontal force F is applied on the prism as shown in figure. If the coefficient of

friction is sufficiently high so that the prism does not slide before toppling, the minimum force required to topple the prism is



- (1) $\frac{mg}{\sqrt{3}}$ (2) $\frac{mg}{4}$
 (3) $\frac{\mu mg}{\sqrt{3}}$ (4) $\frac{\mu mg}{4}$

30. Each pulley in figure has radius r and moment of inertia I . The acceleration of the block is

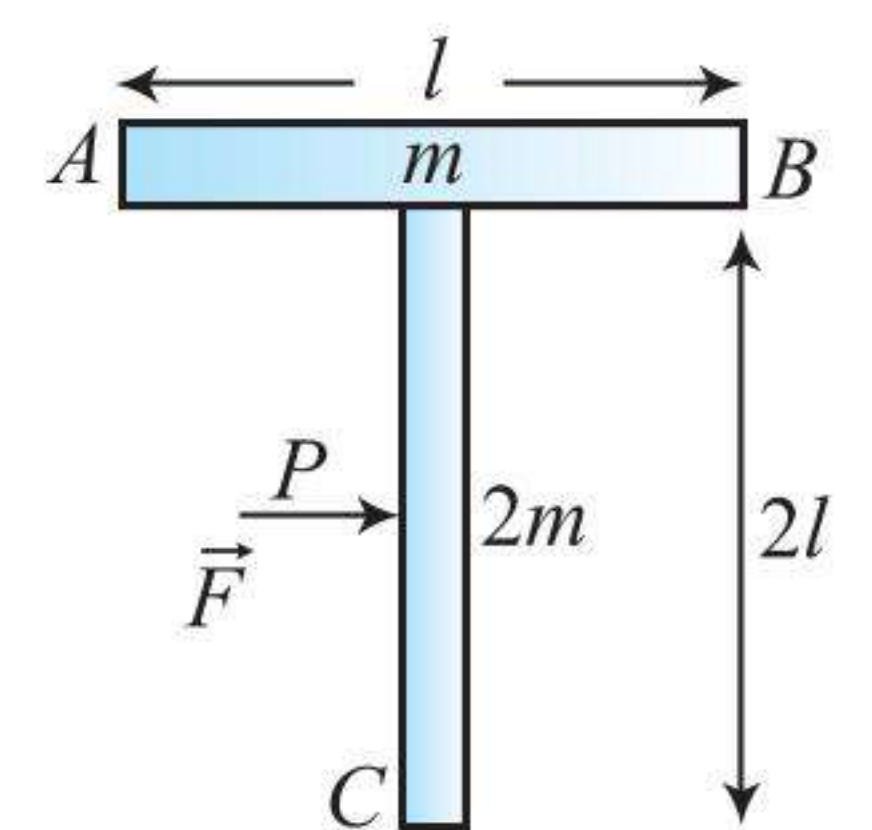


- (1) $\frac{(M-m)g}{\left(M+m+\frac{2I}{r^2}\right)}$ (2) $\frac{(M-m)g}{\left(M+m-\frac{2I}{r^2}\right)}$
 (3) $\frac{(M-m)g}{\left(M+m+\frac{I}{r^2}\right)}$ (4) $\frac{(M-m)g}{\left(M+m-\frac{I}{r^2}\right)}$

31. The line of action of the resultant of two like parallel forces shifts by one-fourth of the distance between the forces when the two forces are interchanged. The ratio of the two forces is

- (1) 1 : 2 (2) 2 : 3
 (3) 3 : 4 (4) 3 : 5

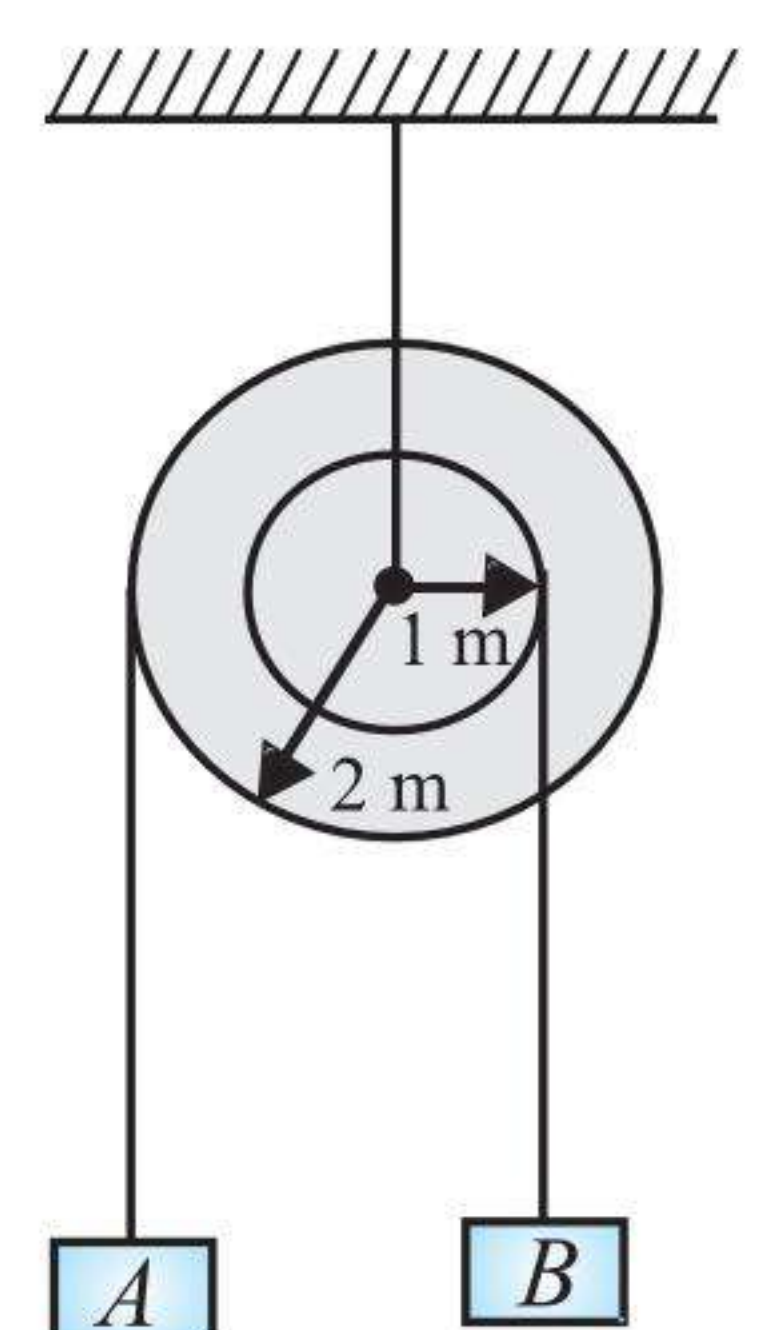
32. A T-shaped object with dimensions shown in the figure is lying on a smooth floor. A force is applied at the point P parallel to AB such that the object has only the translational motion without rotation. Find the location of P with respect to C .



- (1) $(3/4)l$ (2) l
 (3) $(4/3)l$ (4) $(3/2)l$

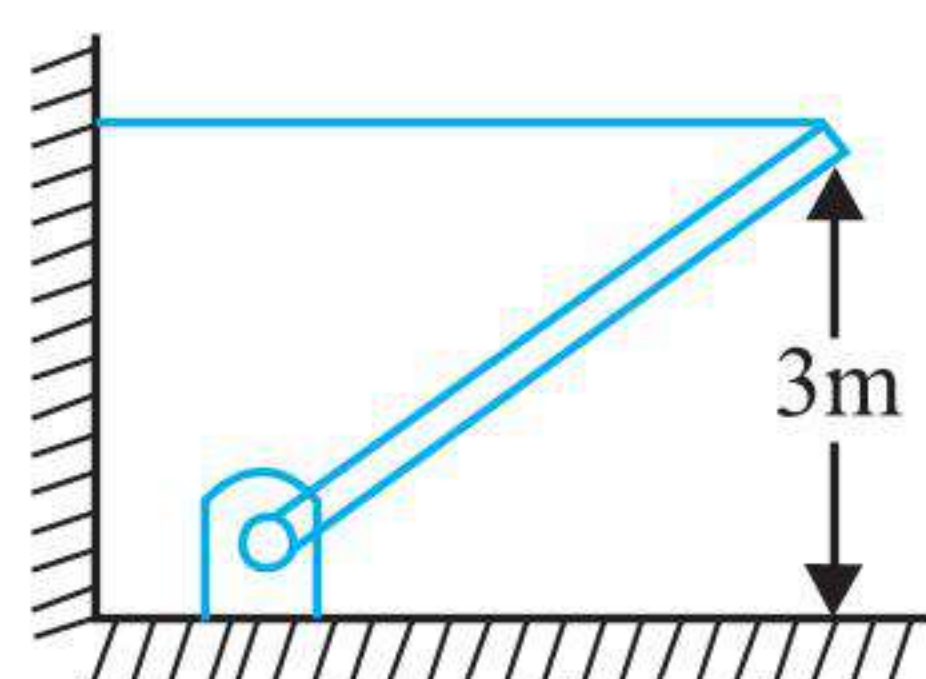
33. In the pulley system shown, if radii of the bigger and smaller pulley are 2 m and 1 m, respectively, and the acceleration of block A is 5 m s^{-2} in the downward direction, the acceleration of block B will be

- (1) 0 m s^{-2}
 (2) 5 m s^{-2}
 (3) 10 m s^{-2}
 (4) $\frac{5}{2} \text{ m s}^{-2}$



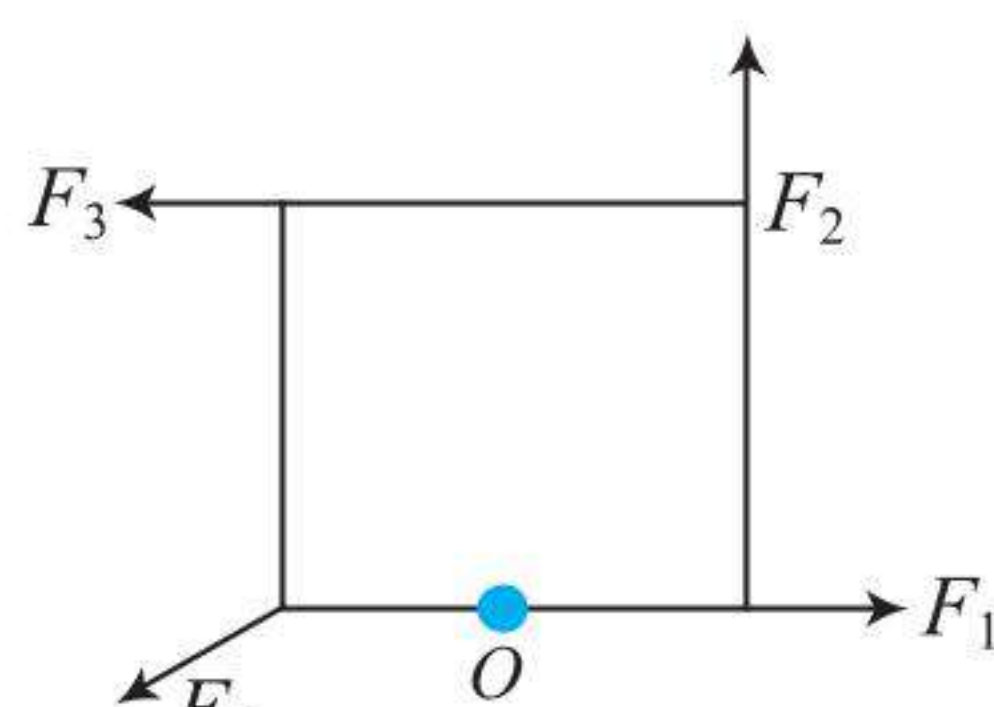
34. A uniform rod of mass 15 kg is held stationary with the help of a light string as shown in figure. The tension in the string is

(1) 150 N
(2) 225 N
(3) 100 N
(4) none of the above



35. Four forces of the same magnitude act on a square as shown in figure. The square can rotate about point O ; mid point of one of the edges. The force which can produce greatest torque is

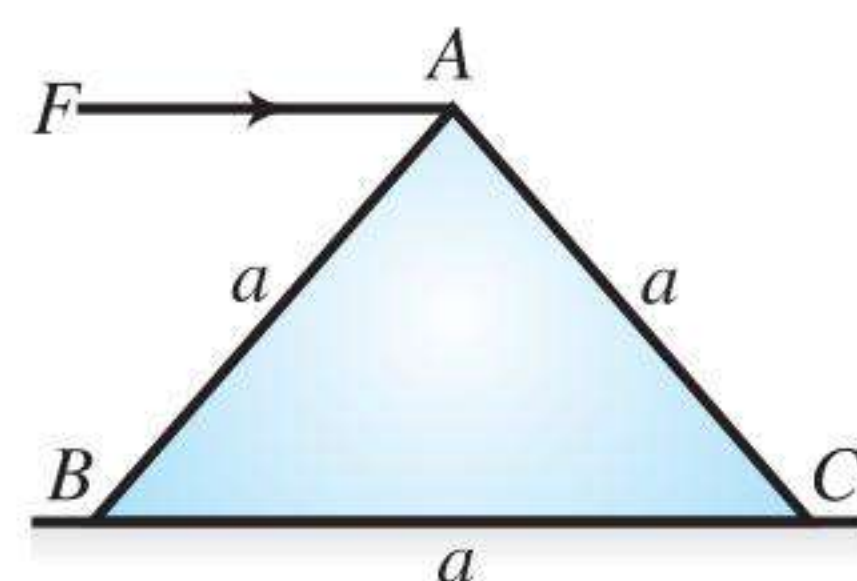
(1) F_1 (2) F_2
(3) F_3 (4) F_4



36. Given a uniform disc of mass M and radius R . A small disc of radius $R/2$ is cut from this disc in such a way that the distance between the centres of the two discs is $R/2$. Find the moment of inertia of the remaining disc about a diameter of the original disc perpendicular to the line connecting the centres of the two discs

(1) $3MR^2/32$ (2) $5MR^2/16$
(3) $11MR^2/64$ (4) none of these

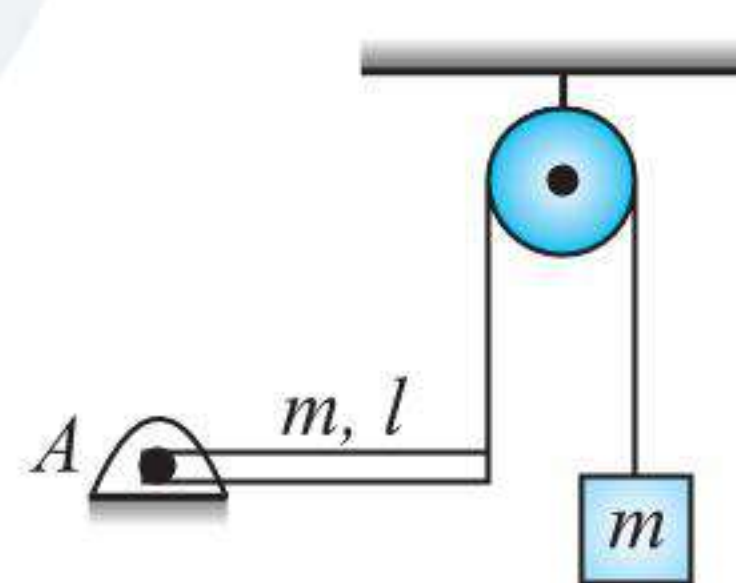
37. A horizontal force F is applied at the top of an equilateral triangular block having mass m . The minimum coefficient of friction required to topple the block before translation will be



(1) $\frac{2}{\sqrt{3}}$ (2) $\frac{1}{3}$
(3) $\frac{1}{\sqrt{3}}$ (4) $\frac{1}{2}$

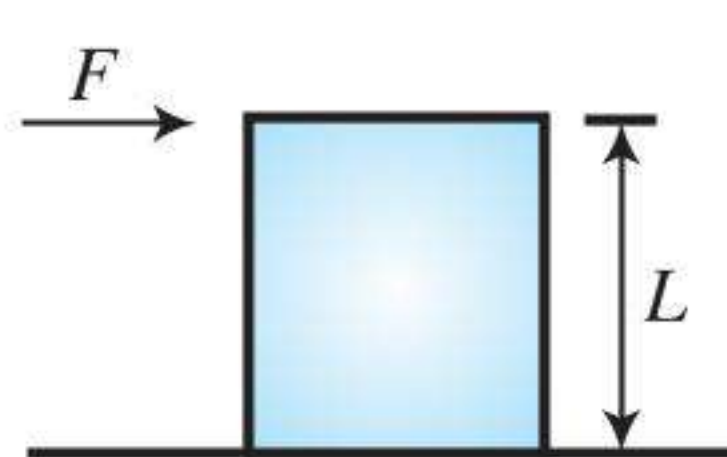
38. Uniform rod AB is hinged at end A in horizontal position as shown in the figure. The other end is connected to a block through a massless string as shown. The pulley is smooth and massless. Masses of block and rod is same and is equal to m . Then acceleration of block just after release from this position is

(1) $6g/13$ (2) $g/4$
(3) $3g/8$ (4) None



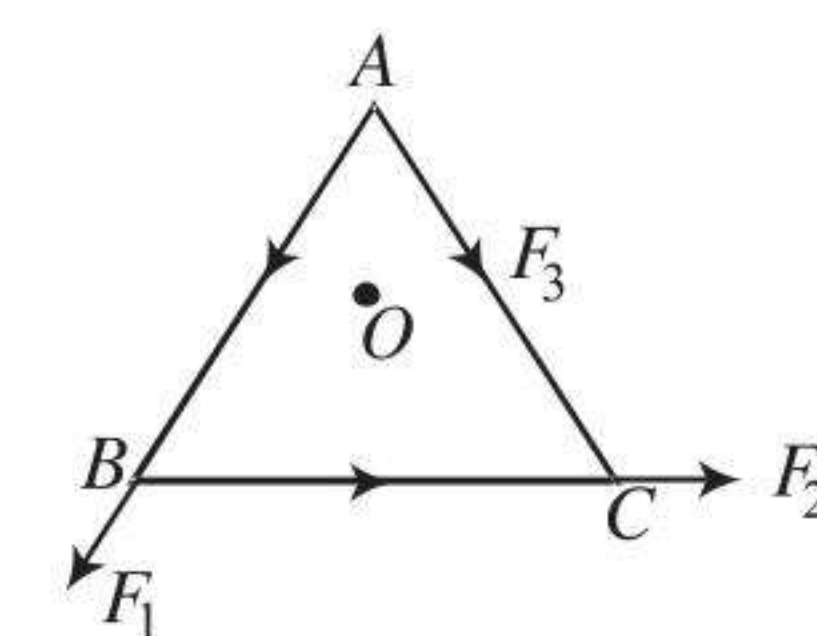
39. A cubical block of side L rests on a rough horizontal surface with coefficient of friction μ . A horizontal force F is applied on the block as shown in the figure. If the coefficient of friction is sufficiently high so that the block does not slide before toppling, the minimum force required to topple the block is

(1) infinitesimal (2) $\frac{mg}{4}$
(3) $\frac{mg}{2}$ (4) $mg(1 - \mu)$

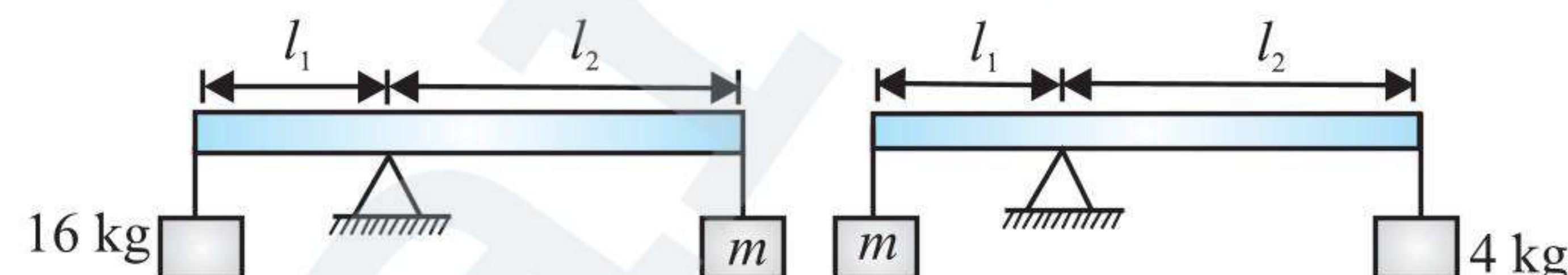


40. O is the centre of an equilateral triangle ABC . F_1 , F_2 and F_3 are the three forces acting along the sides AB , BC and AC respectively. What should be the value of F_3 so that the total torque about O is zero?

(1) $2(F_1 + F_2)$ (2) $\frac{F_1 + F_2}{2}$
(3) $F_1 - F_2$ (4) $F_1 + F_2$



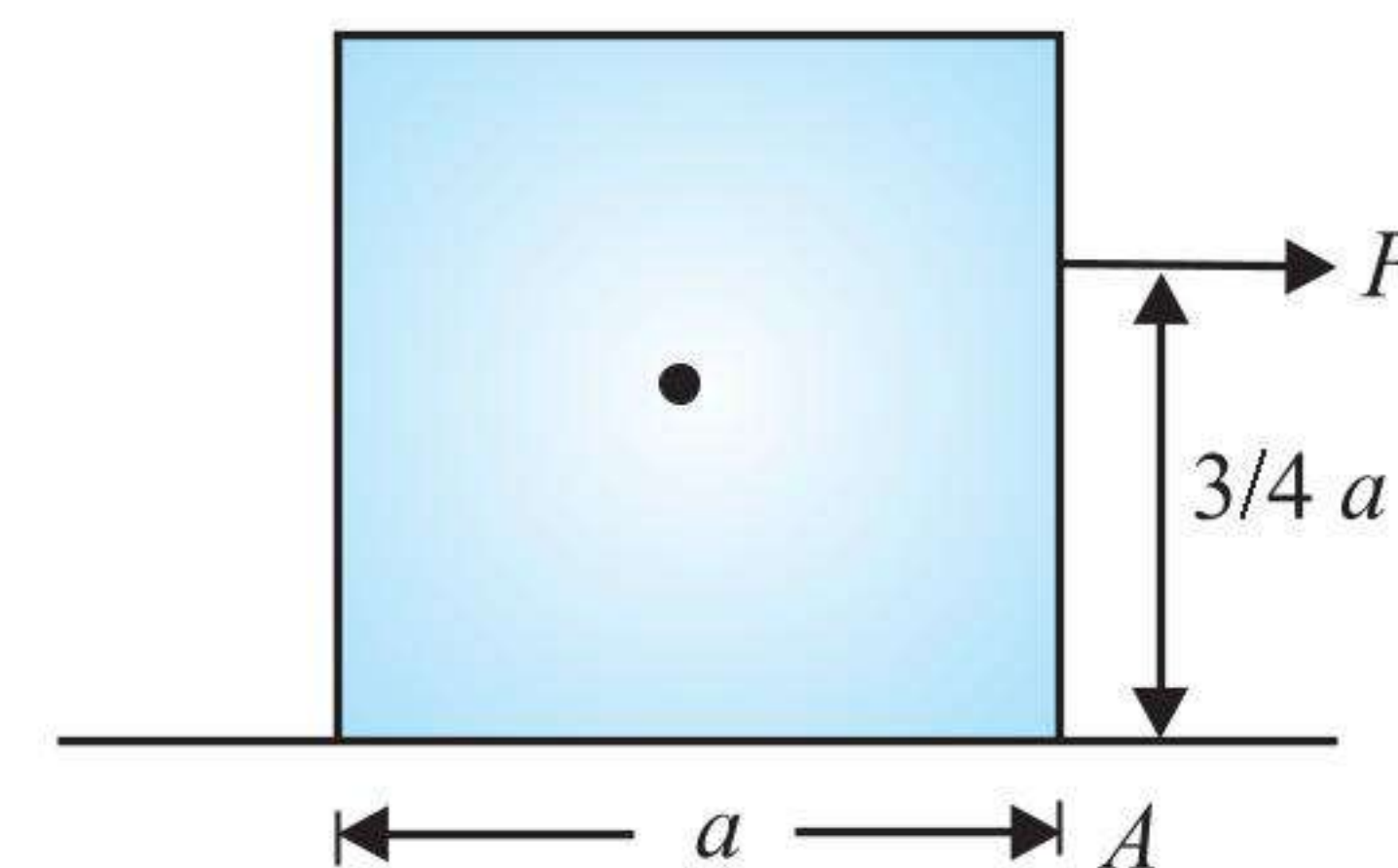
41. In an experiment with a beam balance, an unknown mass m is balanced by two known masses of 16 kg and 4 kg as shown in figure.



The value of the unknown mass m is

(1) 10 kg (2) 6 kg
(3) 8 kg (4) 12 kg

42. A cube of side a and mass m is to be tilted at point A by applying a force F as shown in figure. The minimum force required is

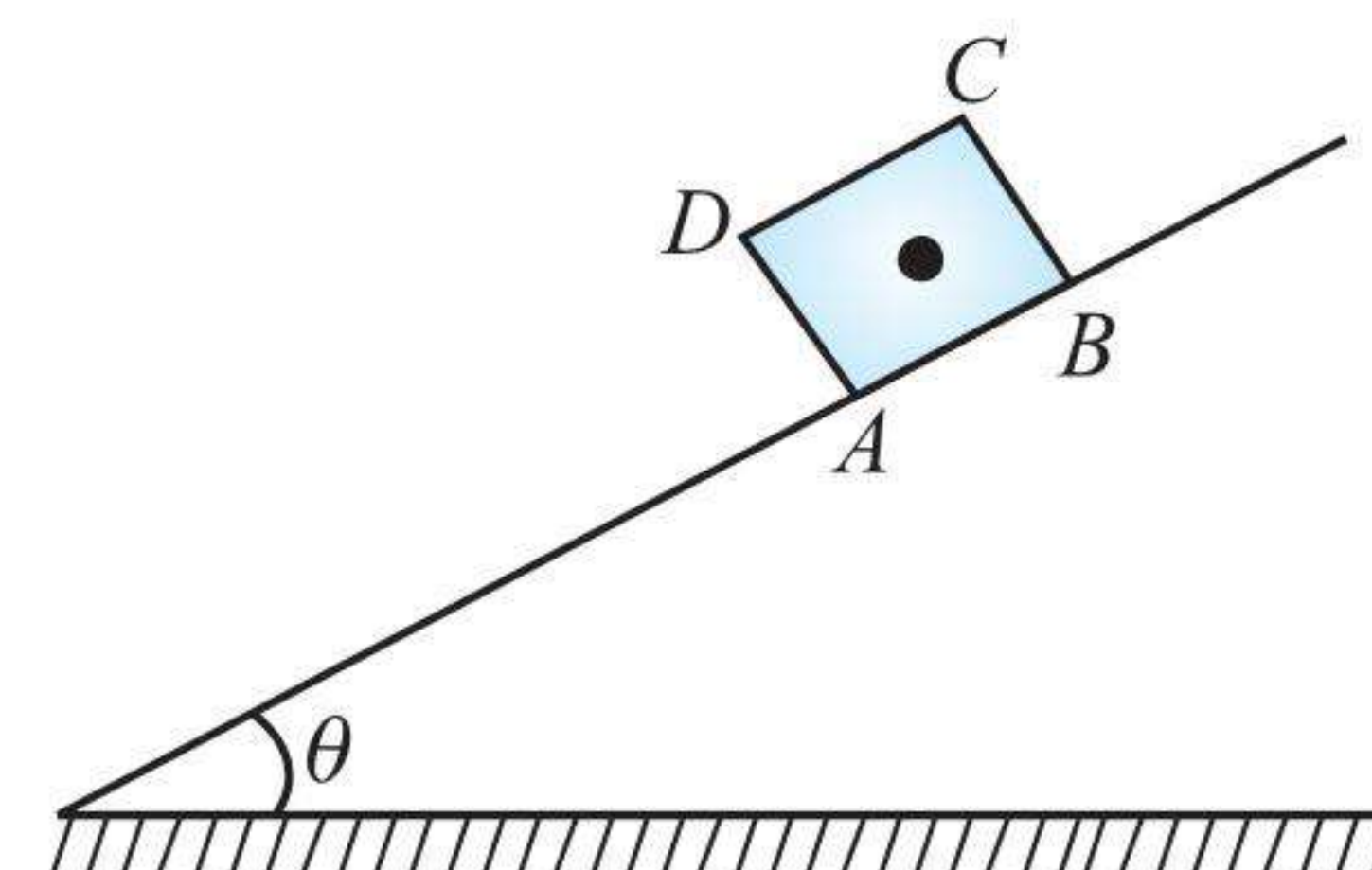


(1) mg (2) $\frac{2}{3}mg$
(3) $\frac{3}{2}mg$ (4) $\frac{3}{4}mg$

43. A ladder of length l and mass m is placed against a smooth vertical wall, but the ground is not smooth. Coefficient of friction between the ground and the ladder is μ . The angle θ at which the ladder will stay in equilibrium is

(1) $\theta = \tan^{-1}(\mu)$ (2) $\theta = \tan^{-1}(2\mu)$
(3) $\theta = \tan^{-1}(\frac{\mu}{2})$ (4) none of these

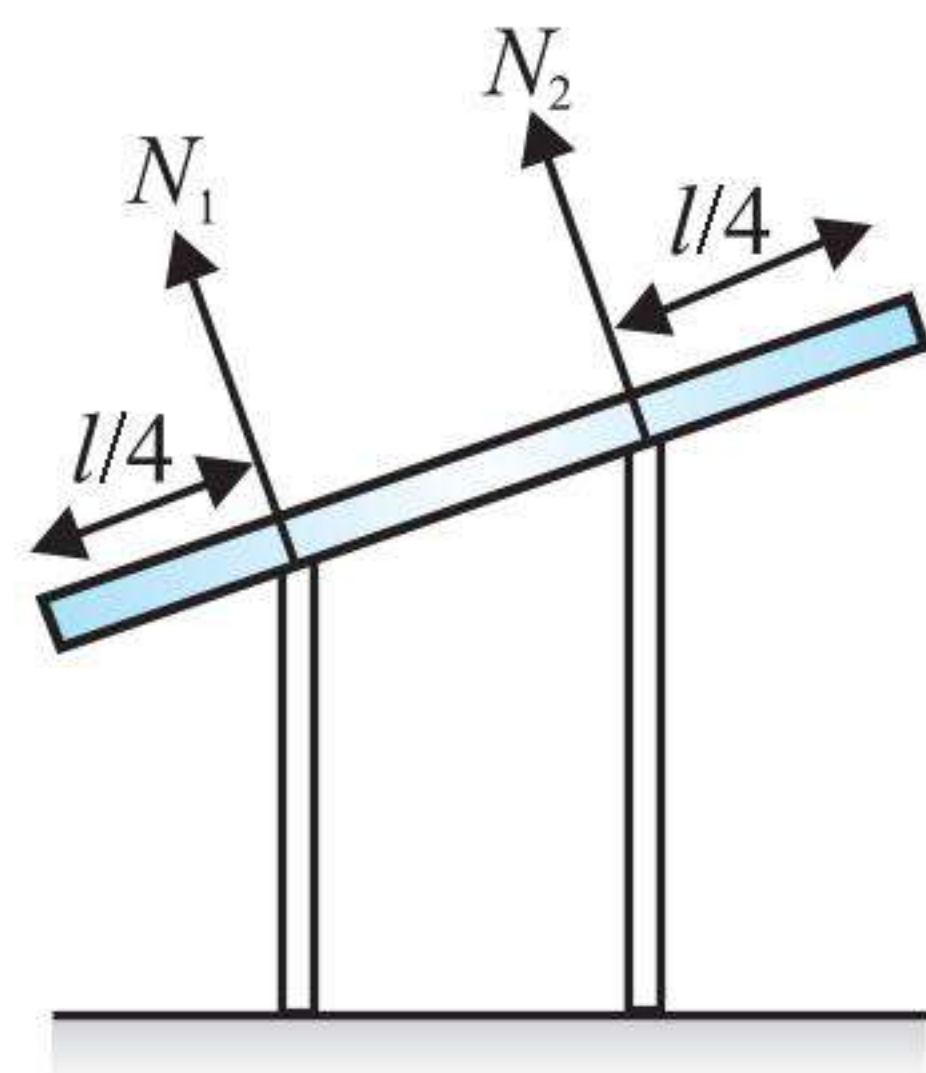
44. A cube of side a is placed on an inclined plane of inclination θ . What is the maximum value of θ for which the cube will not topple?



(1) 15° (2) 30°
(3) 45° (4) 60°

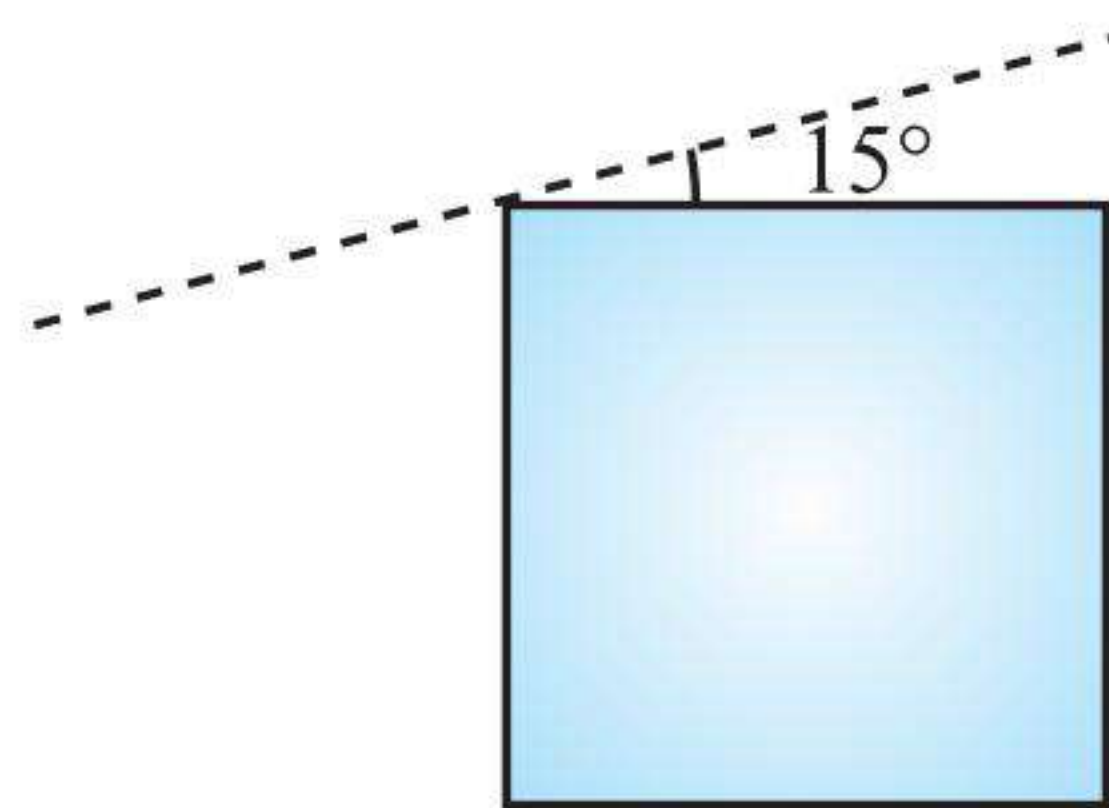
45. A uniform rod of length l is placed symmetrically on two walls as shown in figure. The rod is in equilibrium. If N_1

and N_2 are the normal forces exerted by the walls on the rod, then



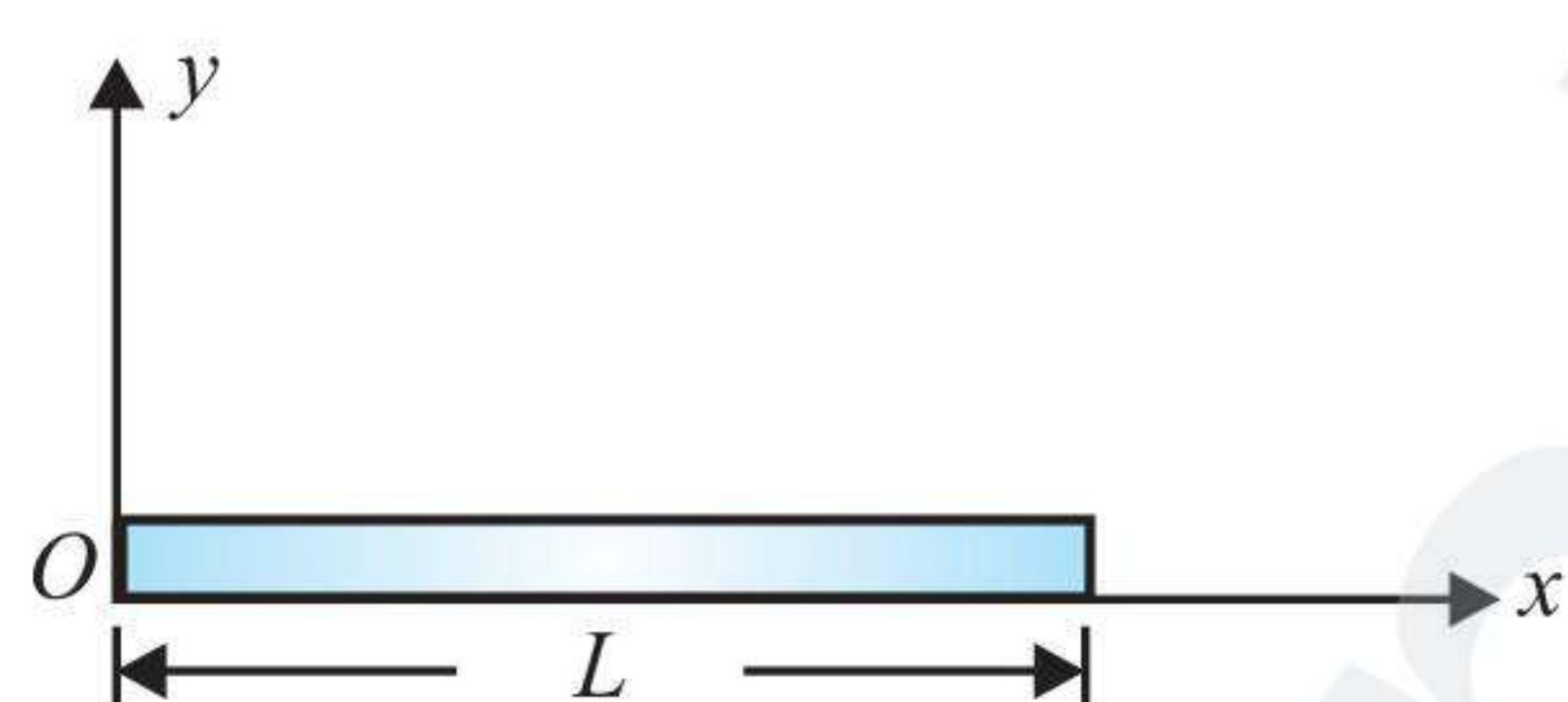
- (1) $N_1 > N_2$ (2) $N_1 < N_2$
 (3) $N_1 = N_2$
 (4) N_1 and N_2 would be in the vertical directions

46. A square plate of mass M and edge L is shown in the figure. The moment of inertia of the plate about the axis in the plane of plate and passing through one of its vertex making an angle 15° horizontal is



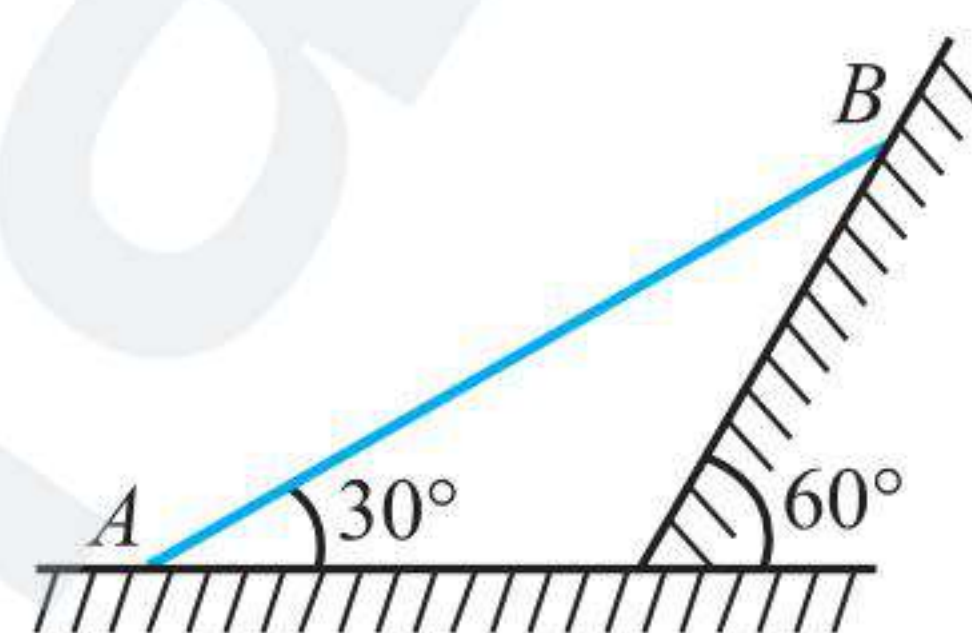
- (1) $\frac{ML^2}{12}$ (2) $\frac{11ML^2}{24}$
 (3) $\frac{7ML^2}{12}$ (4) none of these

47. The figure shows a uniform rod lying along the x -axis. The locus of all the points lying on the x - y plane, about which the moment of inertia of the rod is same as that about O , is



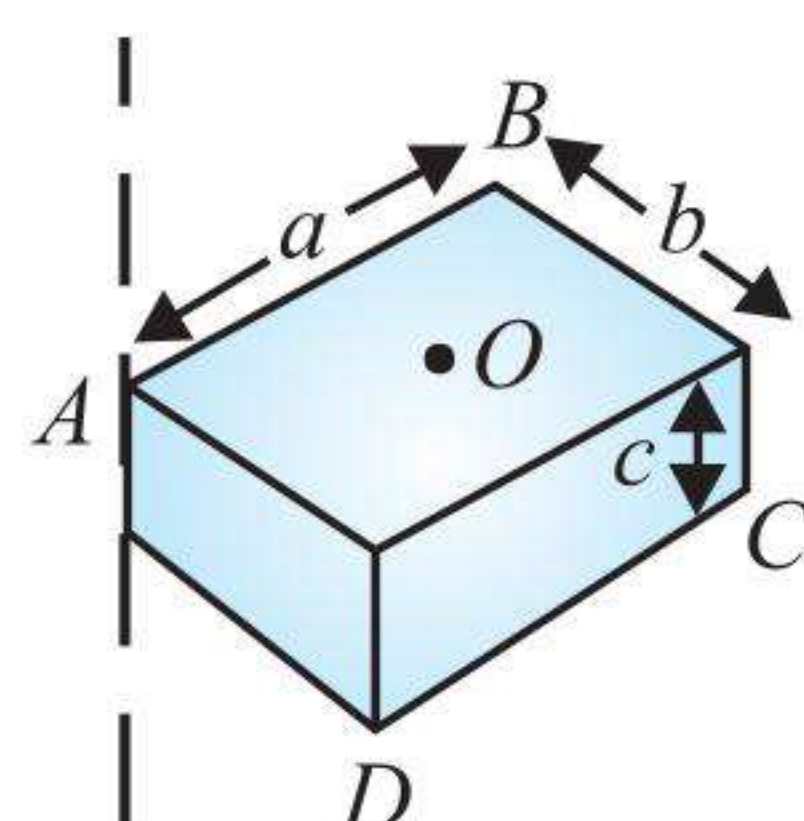
- (1) an ellipse (2) a circle
 (3) a parabola (4) a straight line

48. In the figure shown, the instantaneous speed of end A of the rod is v to the left. The angular velocity of the rod of length L must be



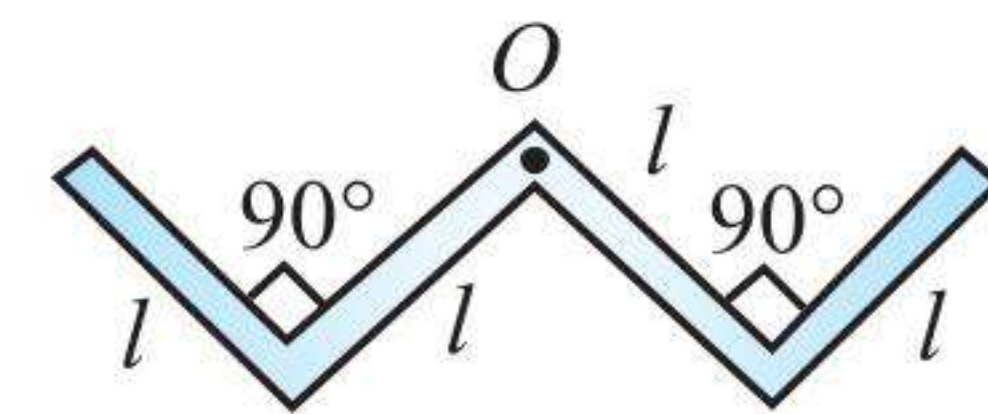
- (1) $\frac{v}{2L}$ (2) $\frac{v}{L}$
 (3) $\frac{v\sqrt{3}}{2L}$ (4) none of these

49. Figure shows a uniform solid block of mass M and edge lengths a , b and c . Its MI about an axis through one edge and perpendicular (as shown) to the large face of the block is



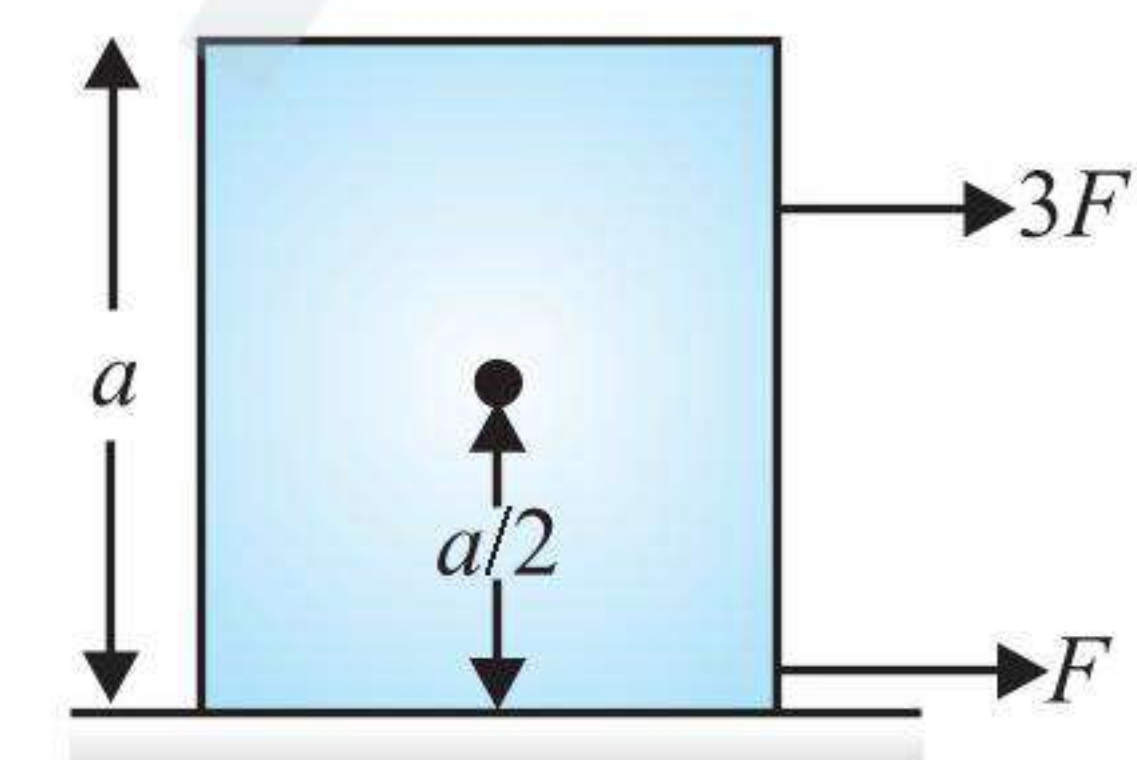
- (1) $\frac{M}{3}(a^2 + b^2)$ (2) $\frac{M}{4}(a^2 + b^2)$
 (3) $\frac{7M}{12}(a^2 + b^2)$ (4) $\frac{M}{12}(a^2 + b^2)$

50. A thin rod of length $4l$ and mass $4m$ is bent at the points as shown in figure. What is the moment of inertia of the rod about the axis passing through point O and perpendicular to the plane of the paper?



- (1) $\frac{ML^2}{3}$ (2) $\frac{10ML^2}{3}$
 (3) $\frac{ML^2}{12}$ (4) $\frac{ML^2}{24}$

51. A rectangular block of mass M and height a is resting on a smooth level surface. A force F is applied to one corner as shown in figure. At what point should a parallel force $3F$ be applied in order that the block shall undergo pure translational motion? Assume normal contact force between the block and surface passes through the centre of gravity of the block.

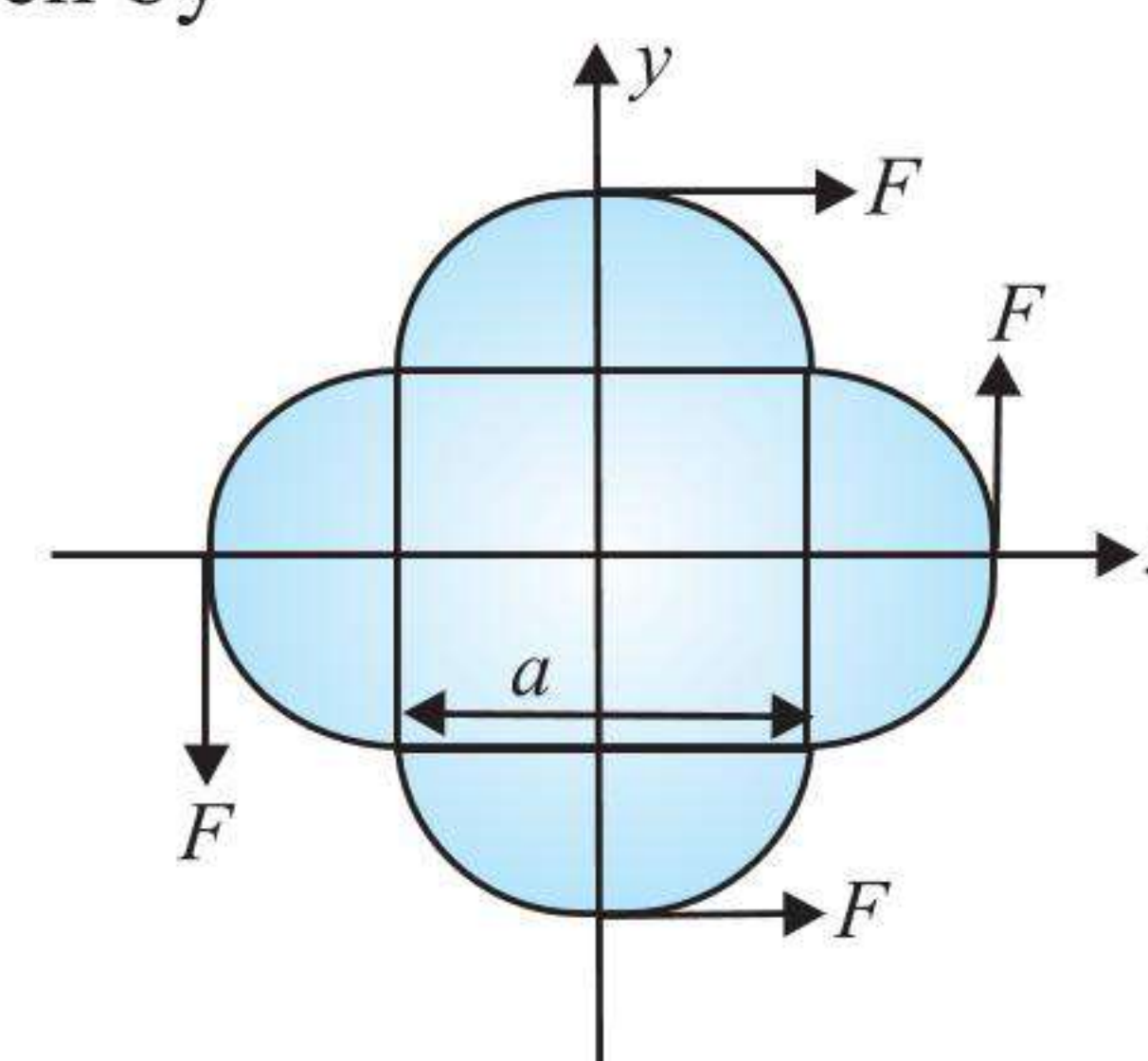


- (1) $\frac{a}{3}$ vertically above centre of gravity
 (2) $\frac{a}{6}$ vertically above centre of gravity
 (3) No such point exists
 (4) it is not possible

52. If I_1 is the moment of inertia of a thin rod about an axis perpendicular to its length and passing through its centre of mass, and I_2 is the moment of inertia (about central axis) of the ring formed by bending the rod, then

- (1) $I_1 : I_2 = 1 : 1$ (2) $I_1 : I_2 = \pi^2 : 3$
 (3) $I_1 : I_2 = \pi : 4$ (4) $I_1 : I_2 = 3 : 5$

53. A planar object made up of a uniform square plate and four semicircular discs of the same thickness and material is being acted upon by four forces of equal magnitude as shown in figure. The coordinates of point of application of forces is given by



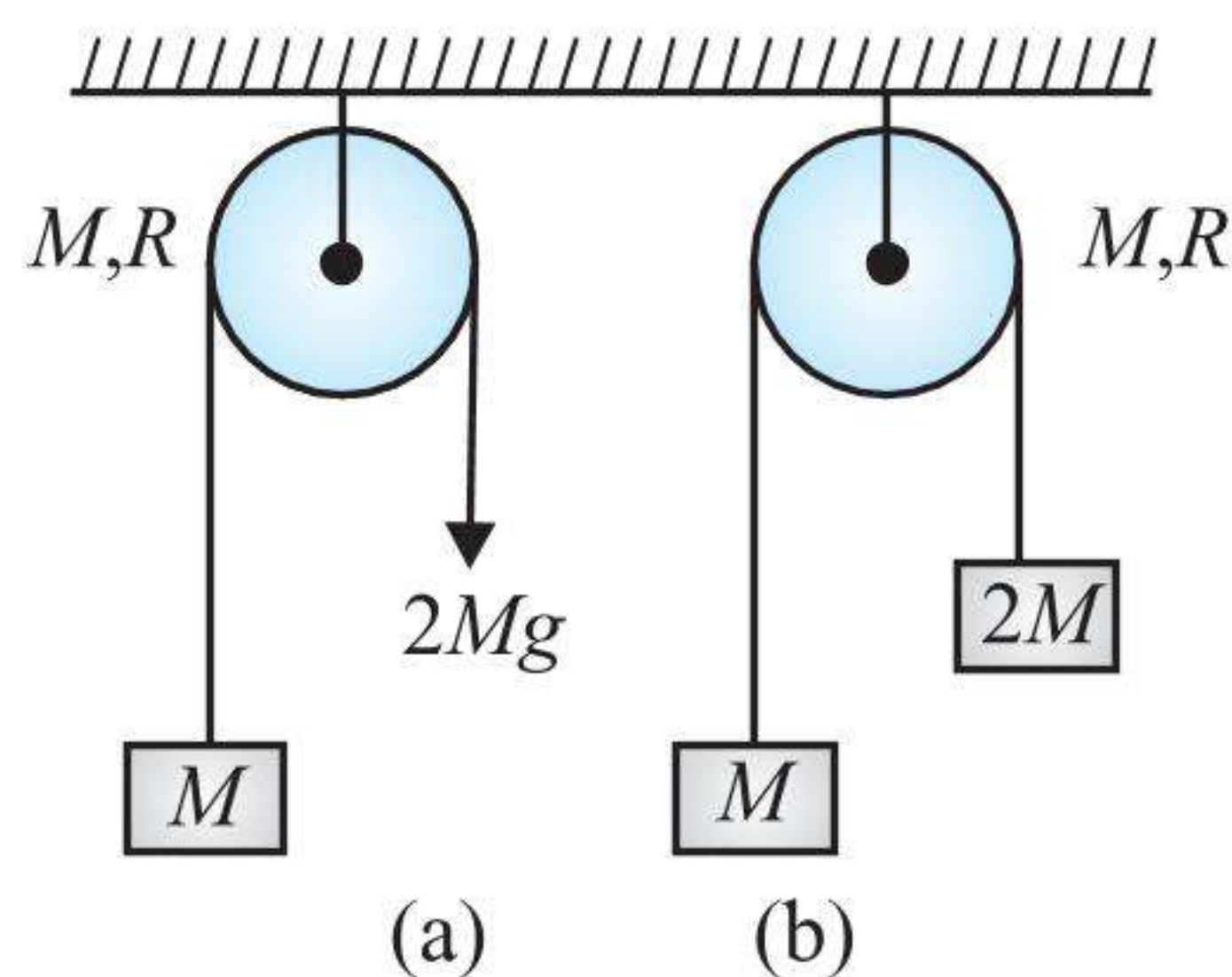
- (1) $(0, a)$ (2) $(0, -a)$
 (3) $(a, 0)$ (4) $(-a, 0)$

54. A uniform disc of radius R lies in the x - y plane, with its centre at origin. Its moment of inertia about z -axis is equal to its moment of inertia about line $y = x + c$. The value of c will be

- (1) $-\frac{R}{2}$ (2) $\pm \frac{R}{\sqrt{2}}$
 (3) $+\frac{R}{4}$ (4) $-R$

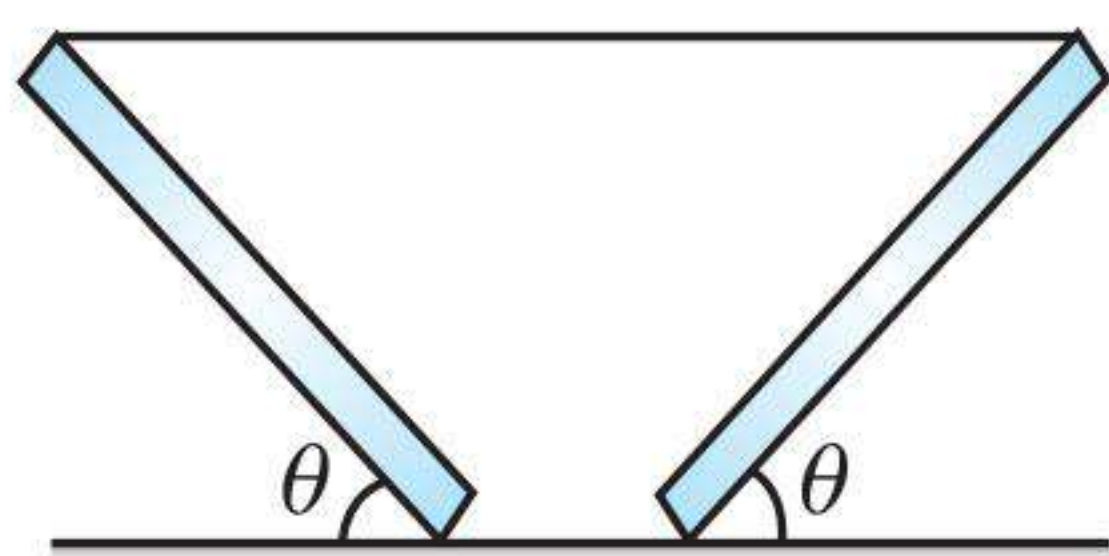
55. A cord is wrapped on a pulley (disk) of mass M and radius R as shown in figure. To one end of the cord, a block of mass

M is connected as shown and to other end in (a) a force of $2Mg$ and in (b) a block of mass $2M$. Let angular acceleration of the disk in A and B is α_A and α_B , respectively, then (cord is not slipping on the pulley).



- (1) $\alpha_A = \alpha_B$ (2) $\alpha_A > \alpha_B$
 (3) $\alpha_A < \alpha_B$ (4) None of these

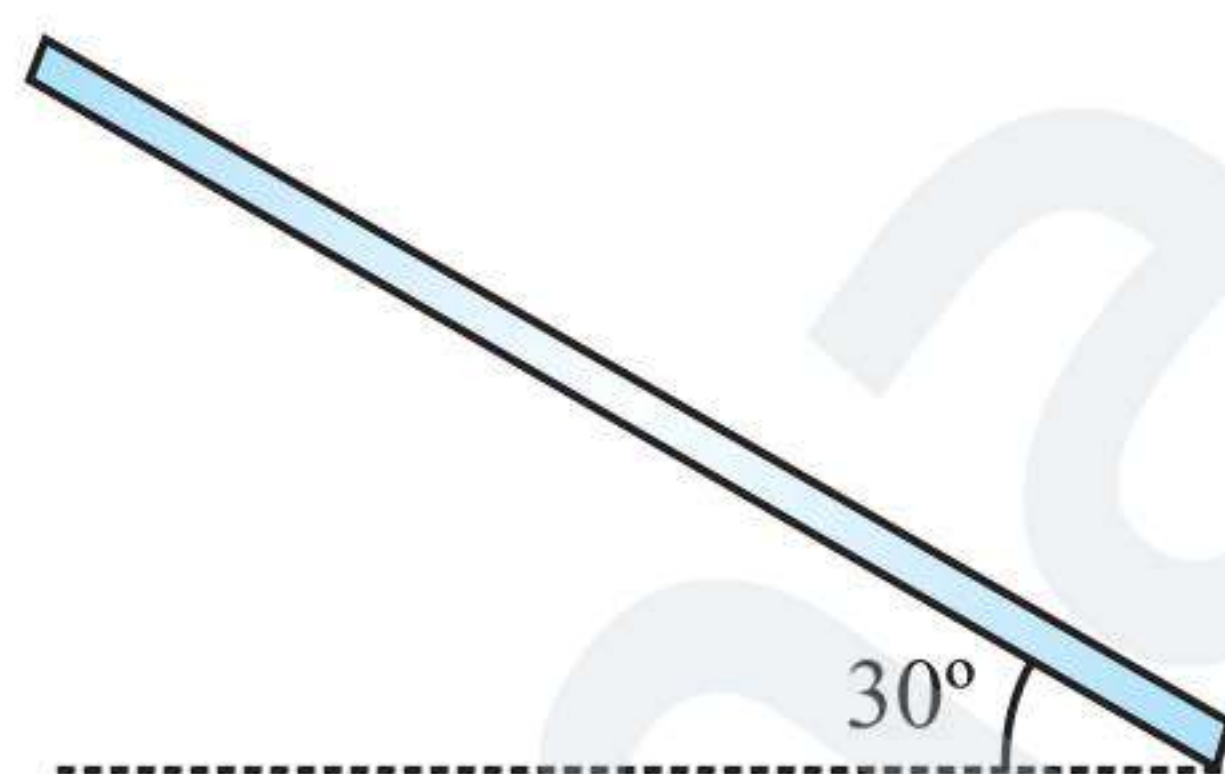
56. Two uniform boards, tied together with the help of a string, are balanced on a surface as shown in figure.



The coefficient of static friction between boards and surface is 0.5. The minimum value of θ , for which this type of arrangement is possible is

- (1) 30°
 (2) 45°
 (3) 37°
 (4) it is not possible to have this type of balanced arrangement

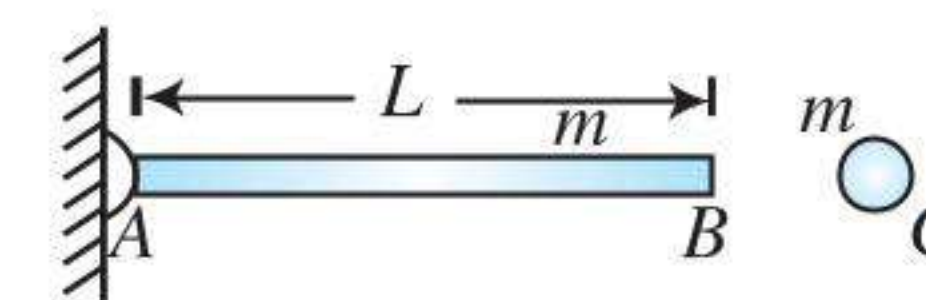
57. A slender rod of mass m and length L is pivoted about a horizontal axis through one end and released from rest at an angle of 30° above the horizontal. The force exerted by the pivot on the rod at the instant when the rod passes through a horizontal position is



- (1) $\sqrt{\frac{10}{4}} mg$ along horizontal
 (2) mg along vertical
 (3) $\frac{\sqrt{10}}{4} mg$ along a line making an angle of $\tan^{-1}\left(\frac{1}{3}\right)$ with the horizontal
 (4) $\frac{\sqrt{10}}{4} mg$ along a line making an angle of $\tan^{-1}(3)$ with the horizontal

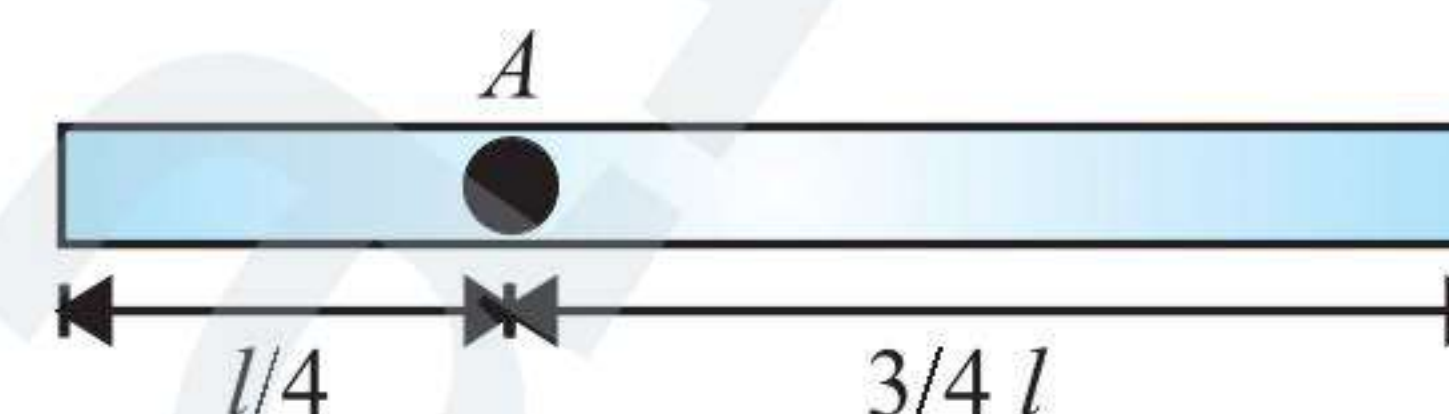
58. A uniform bar AB of mass m and a ball of the same mass are released from rest from the same horizontal position. The bar is hinged at end A . There is gravity downwards. What

is the distance of the point from point B that has the same acceleration as the ball, immediately after release?



- (1) $\frac{2L}{3}$ (2) $\frac{L}{3}$
 (3) $\frac{L}{2}$ (4) $\frac{3L}{4}$

59. A uniform rod of mass m and length l is fixed from point A , which is at a distance $l/4$ from one end as shown in the figure. The rod is free to rotate in a vertical plane. The rod is released from the horizontal position.



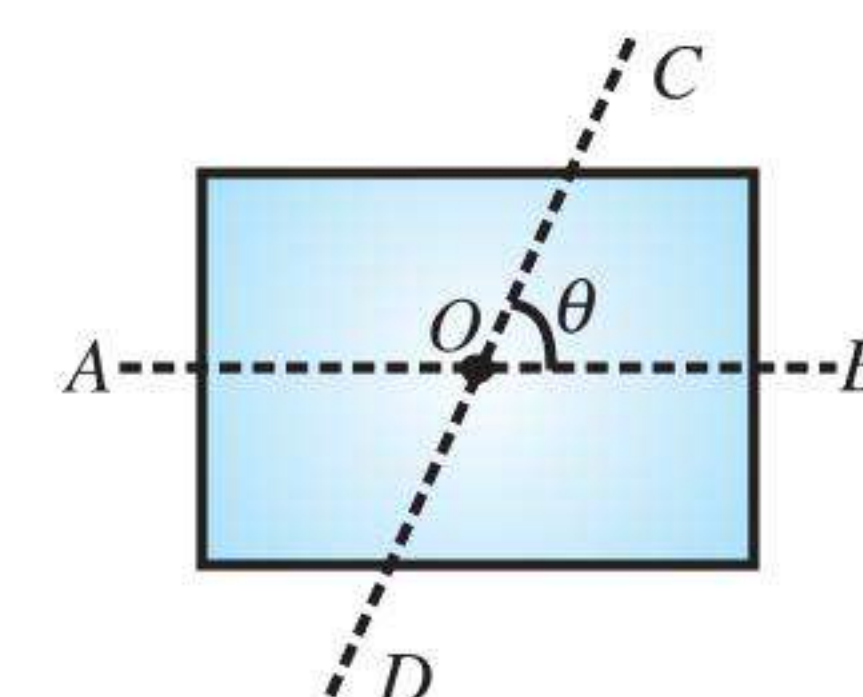
What is the reaction at the hinge, when kinetic energy of the rod is maximum?

- (1) $\frac{4}{7} mg$ (2) $\frac{5}{7} mg$
 (3) $\frac{13}{7} mg$ (4) $\frac{11}{7} mg$

60. A rod of length L is held vertically on a smooth horizontal surface. The top end of the rod is given a gentle push. At a certain instant of time, when the rod makes an angle 37° with horizontal the velocity of COM of the rod is 2 ms^{-1} . The velocity of the end of the rod in contact with the surface at that instant is

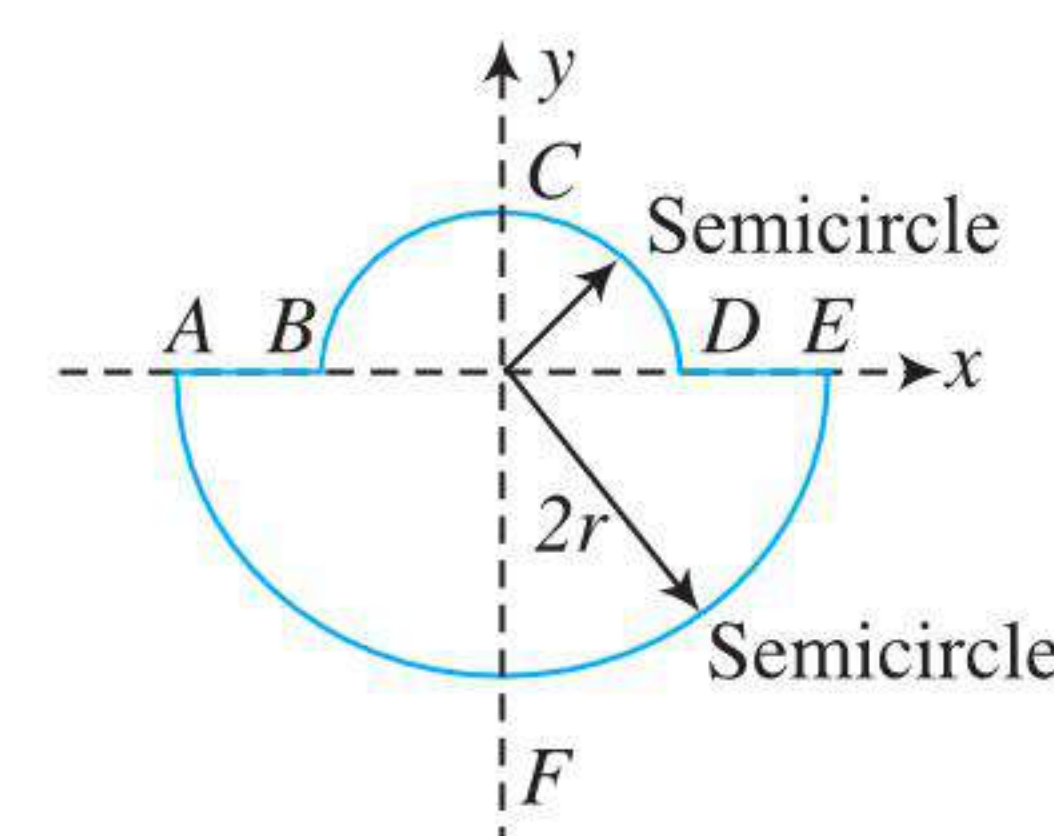
- (1) 2 ms^{-1} (2) 1 ms^{-1}
 (3) 4 ms^{-1} (4) 1.5 ms^{-1}

61. Let I be the moment of inertia of a uniform square plate about an axis AB that passes through its centre and is parallel to two of its sides. CD is a line in the plane of the plate and it passes through the centre of the plate, making an angle θ with AB . The moment of inertia of the plate about the axis CD is equal to



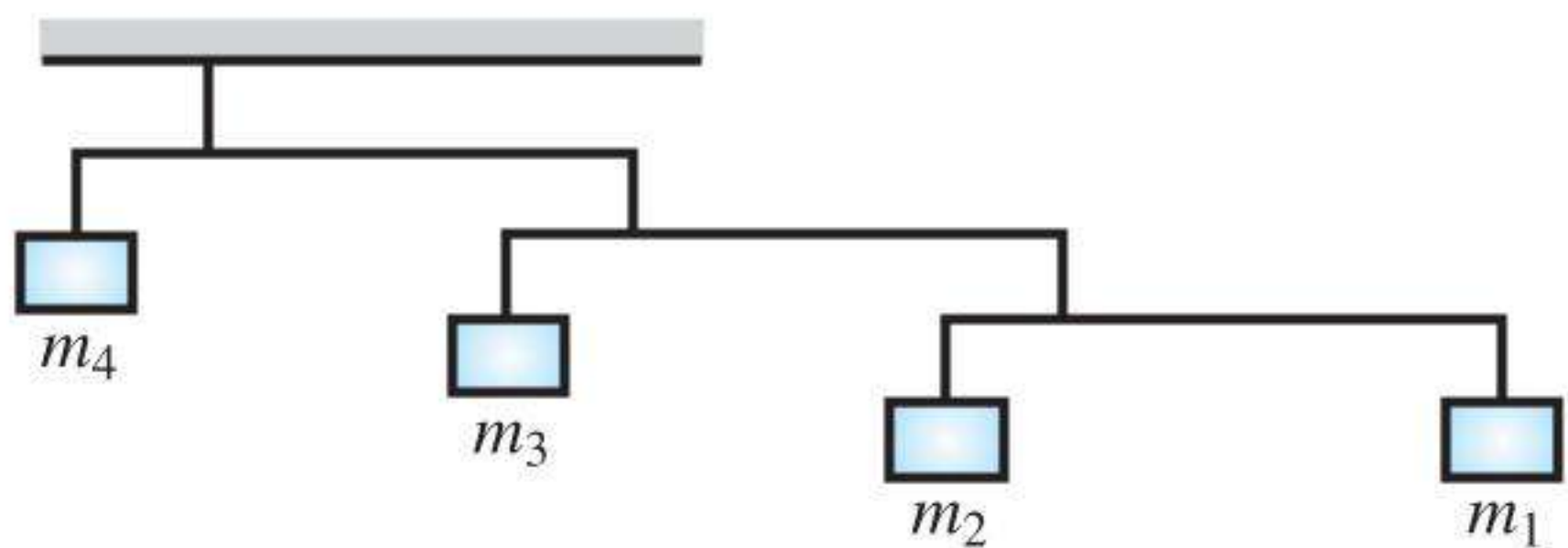
- (1) I (2) $I \sin^2 \theta$
 (3) $I \cos^2 \theta$ (4) $I \cos^2 \left(\frac{\theta}{2} \right)$

62. A uniform thin rod is bent in the form of closed loop $ABCDEF$ as shown in the figure. The ratio of moment of inertia of the loop about x -axis to that about y -axis is

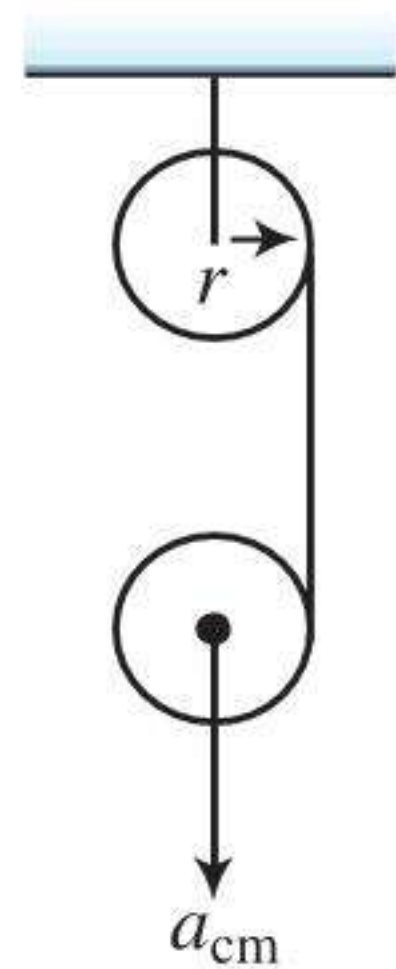


- (1) > 1 (2) < 1
 (3) $= 1$ (4) $= 1/2$

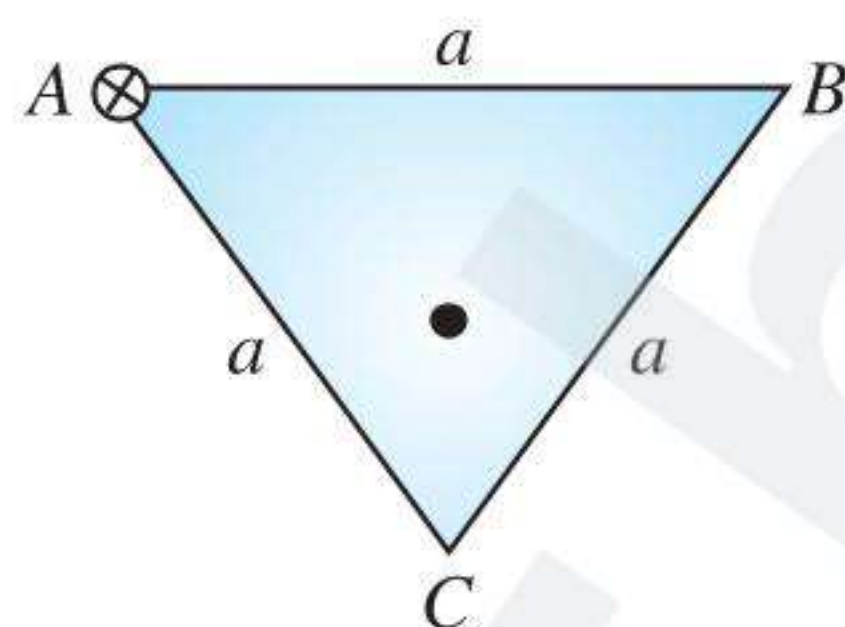
63. Figure shows an arrangement of masses hanging from a ceiling. In equilibrium, each rod is horizontal, has negligible mass and extends three times as far to the right of the wire supporting it as to the left. If mass m_4 is 48 kg then mass m_1 is equal to



- (1) 1 kg (2) 2 kg
(3) 3 kg (4) 4 kg
64. Two identical uniform discs of mass m and radius r are arranged as shown in the figure. If α is the angular acceleration of the lower disc and a_{cm} is acceleration of centre of mass of the lower disc, then relation among a_{cm} , α and r is
- (1) $a_{cm} = \frac{\alpha}{r}$ (2) $a_{cm} = 2\alpha r$
(3) $a_{cm} = \alpha r$ (4) none of these



65. A uniform triangular plate ABC of mass m and moment of inertia I (about an axis passing through A and perpendicular to plane of the plate) can rotate freely in the vertical plane about point ' A ' as shown in figure. The plate is released from the position shown in the figure. Line AB is horizontal. The acceleration of centre of mass just after the release of plate is



- (1) $\frac{mga^2}{\sqrt{3}I}$ (2) $\frac{mga^2}{4I}$
(3) $\frac{mga^2}{2\sqrt{3}I}$ (4) $\frac{mga^2}{3I}$

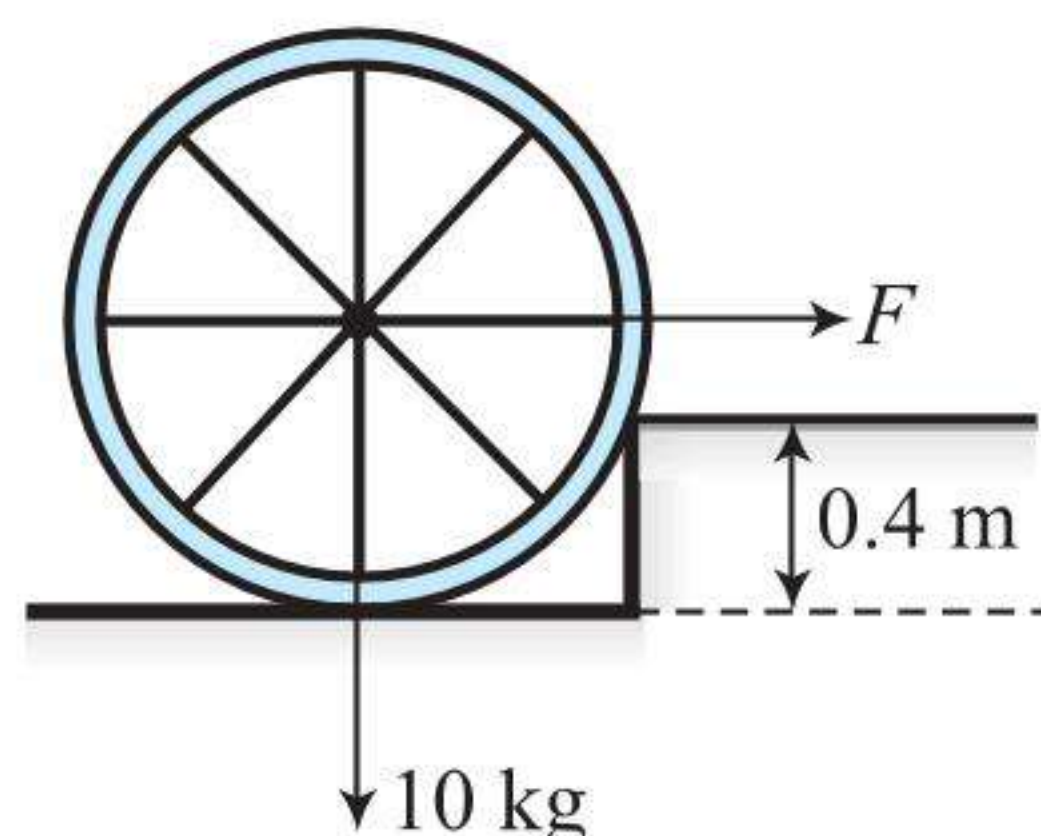
66. Three children are sitting on a see-saw in such a way that it is balanced. A 20 kg and a 30 kg boy are on opposite sides at a distance of 2m from the pivot. If the third boy jumps off, thereby destroying balance, then the initial angular acceleration of the board is (Neglect weight of board)

- (1) 0.01 rad s^{-2} (2) 1.0 rad s^{-2}
(3) 10 rad s^{-2} (4) 100 rad s^{-2}

67. A wheel of radius R has an axle of radius $R/5$. A force F is applied tangentially to the wheel. To keep the system in a state of "rotational" rest, a force F' is applied tangentially to the axle. The value of F' is

- (1) F (2) $3F$
(3) $5F$ (4) $7F$

68. Calculate the force F that is applied horizontally at the axle of the wheel which is necessary to raise the wheel over the obstacle of height 0.4 m. Radius of wheel is 1m and mass = 10 kg. F is

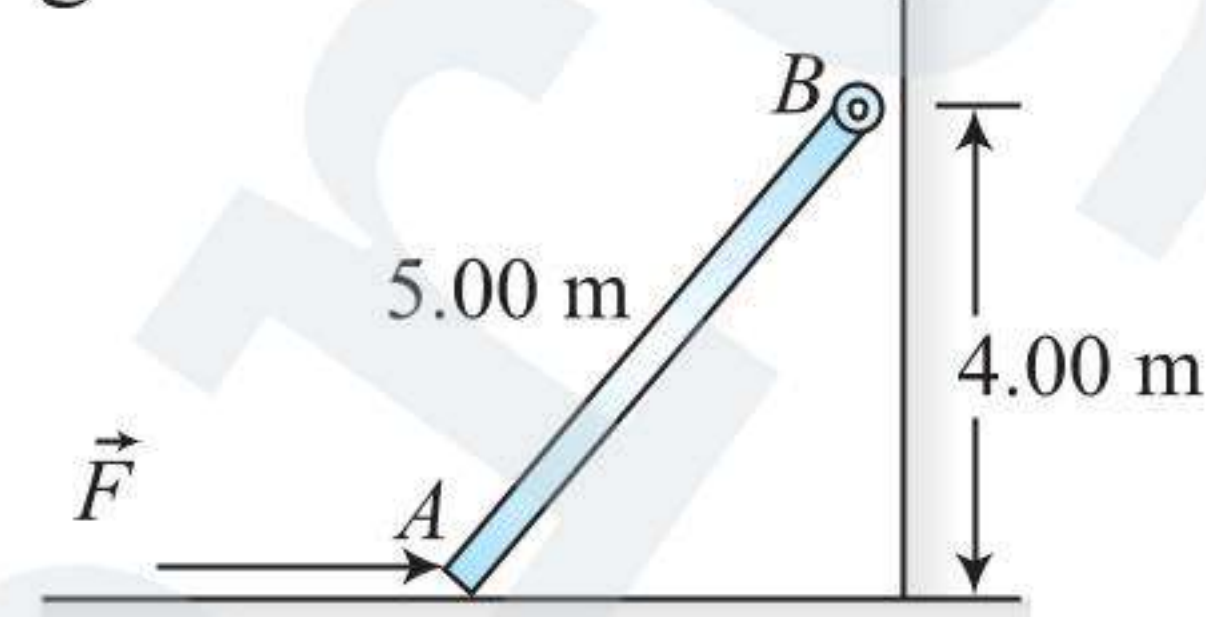


- (1) 100 N (2) 66 N
(3) 167 N (4) 133.3 N

69. A string is wrapped around a cylinder of mass M and radius R . The string is pulled vertically upwards to prevent the centre of mass from falling as the cylinder unwinds the string. The tension in the string is

- (1) $2Mg/3$ (2) $Mg/2$
(3) $Mg/3$ (4) $Mg/6$

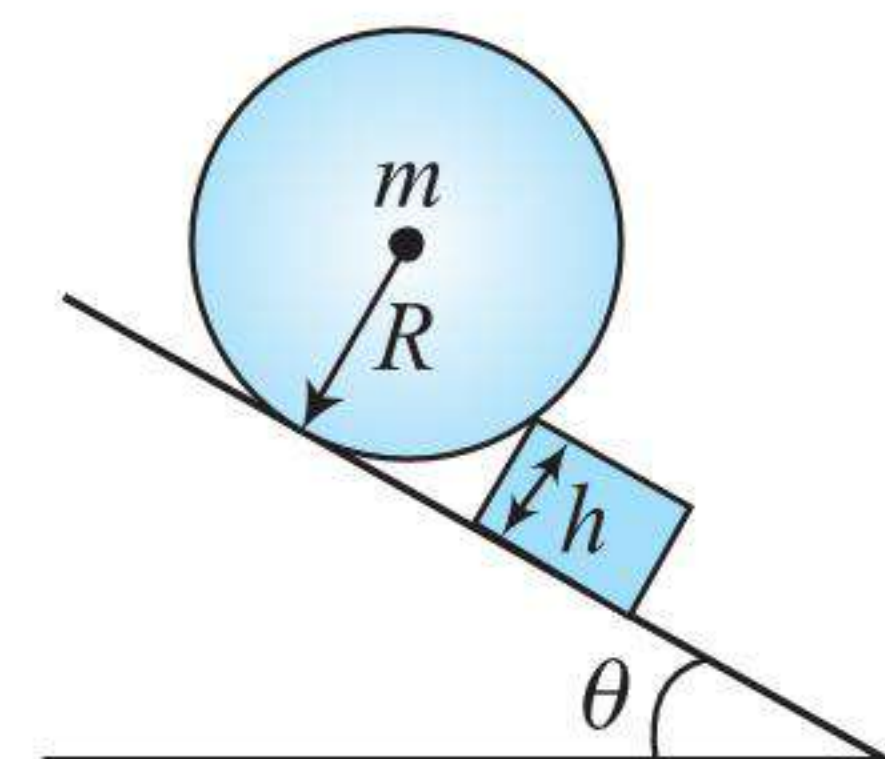
70. End A of the bar AB in figure rests on a frictionless horizontal surface and end B is hinged. A horizontal force $\vec{F}$ of magnitude 120 N is exerted on end A . You can ignore the weight of the bar. What is the net force exerted by the bar on the hinge at B ?



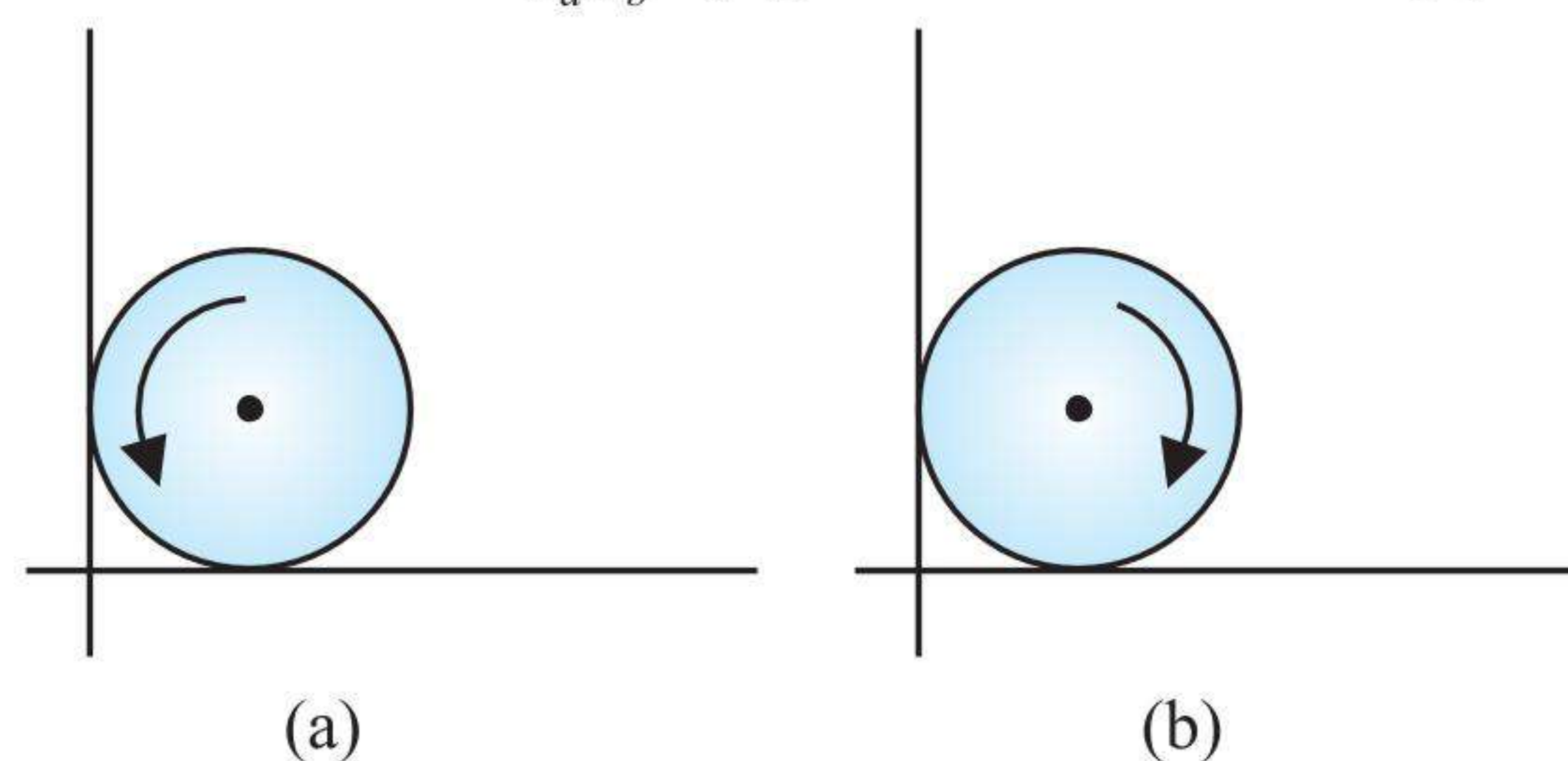
- (1) 200 N (2) 140 N
(3) 100 N (4) None of these

71. Find the minimum height of the obstacle so that the sphere can stay in equilibrium.

- (1) $\frac{R}{1 + \cos \theta}$
(2) $\frac{R}{1 + \sin \theta}$
(3) $R(1 - \sin \theta)$
(4) $R(1 - \cos \theta)$

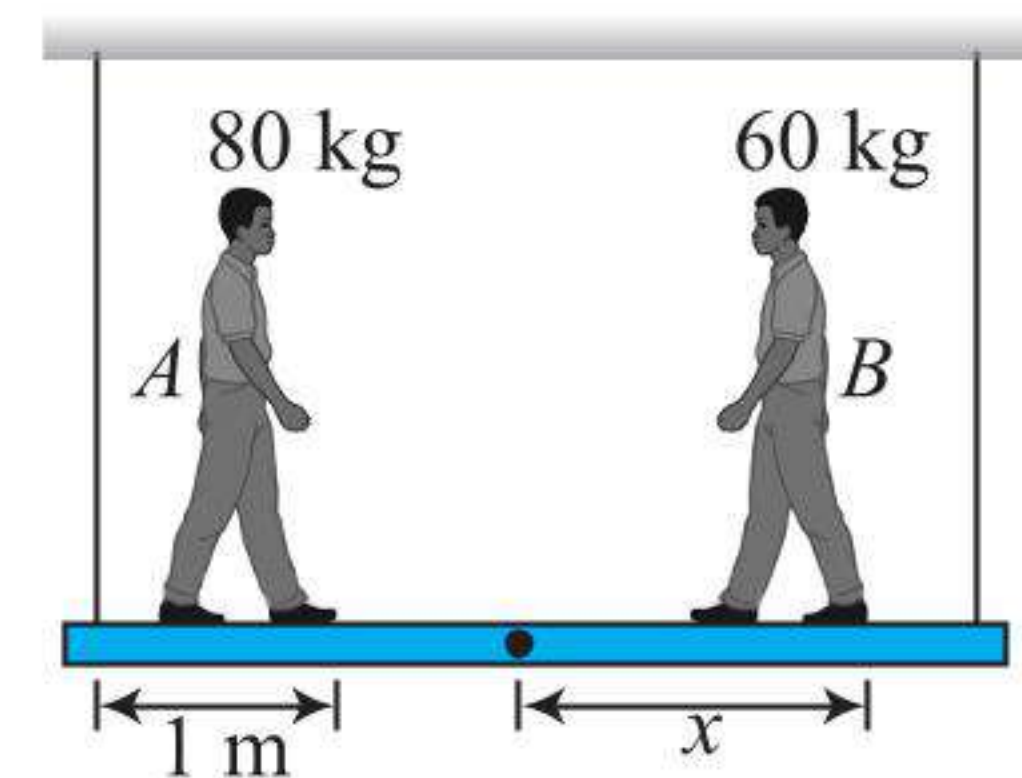


72. A sphere is placed rotating with its centre initially at rest in a corner as shown in Figs. (a) and (b). Coefficient of friction between all surfaces and the sphere is $1/3$. Find the ratio of the friction forces f_a/f_b by ground in situations (a) and (b).



- (1) 1 (2) $\frac{9}{10}$
(3) $\frac{10}{9}$ (4) None of these

73. Two painters are working from a wooden board 5 m long suspended from the top of a building by two ropes attached to the ends of the plank. Either rope can withstand a maximum tension of 1040 N. Painter A of mass 80 kg is working at a distance of 1 m from one end. Painter B of mass 60 kg is working at a distance of x m from the centre of mass of the board on the other side. Take mass of the board as 20 kg and $g = 10 \text{ m s}^{-2}$. The range of x so that both the painters can work safely is



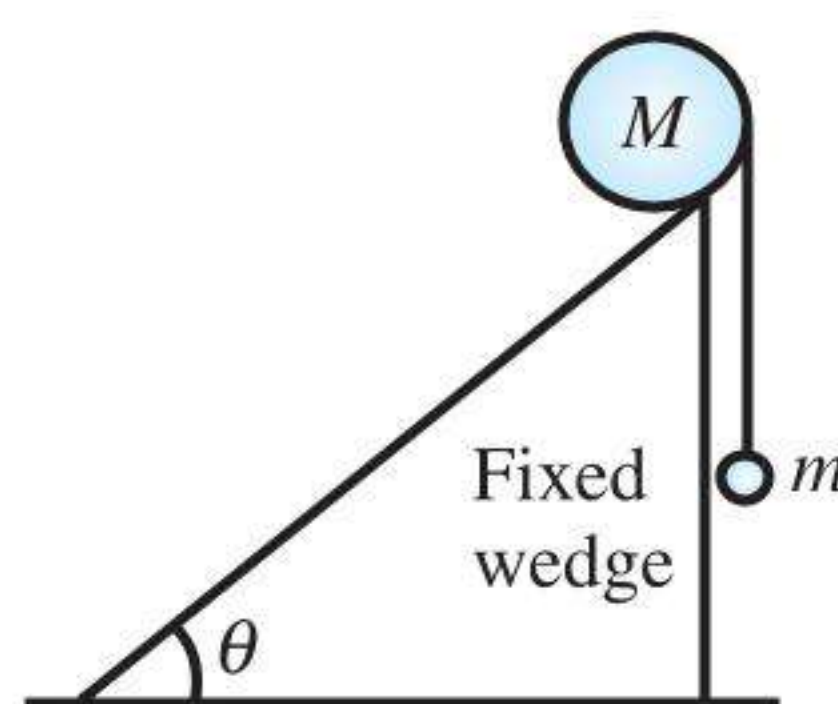
- (1) $\frac{1}{3} < x < \frac{11}{6}$ (2) $0 < x < \frac{11}{6}$
(3) $0 < x < 2$ (4) $\frac{1}{3} < x < 2$

74. In figure, a massive rod AB is held in horizontal position by two massless strings. If the string at B breaks and if the horizontal acceleration of centre of mass, vertical acceleration and angular acceleration of rod about the centre of mass are a_x , a_y and α , respectively, then



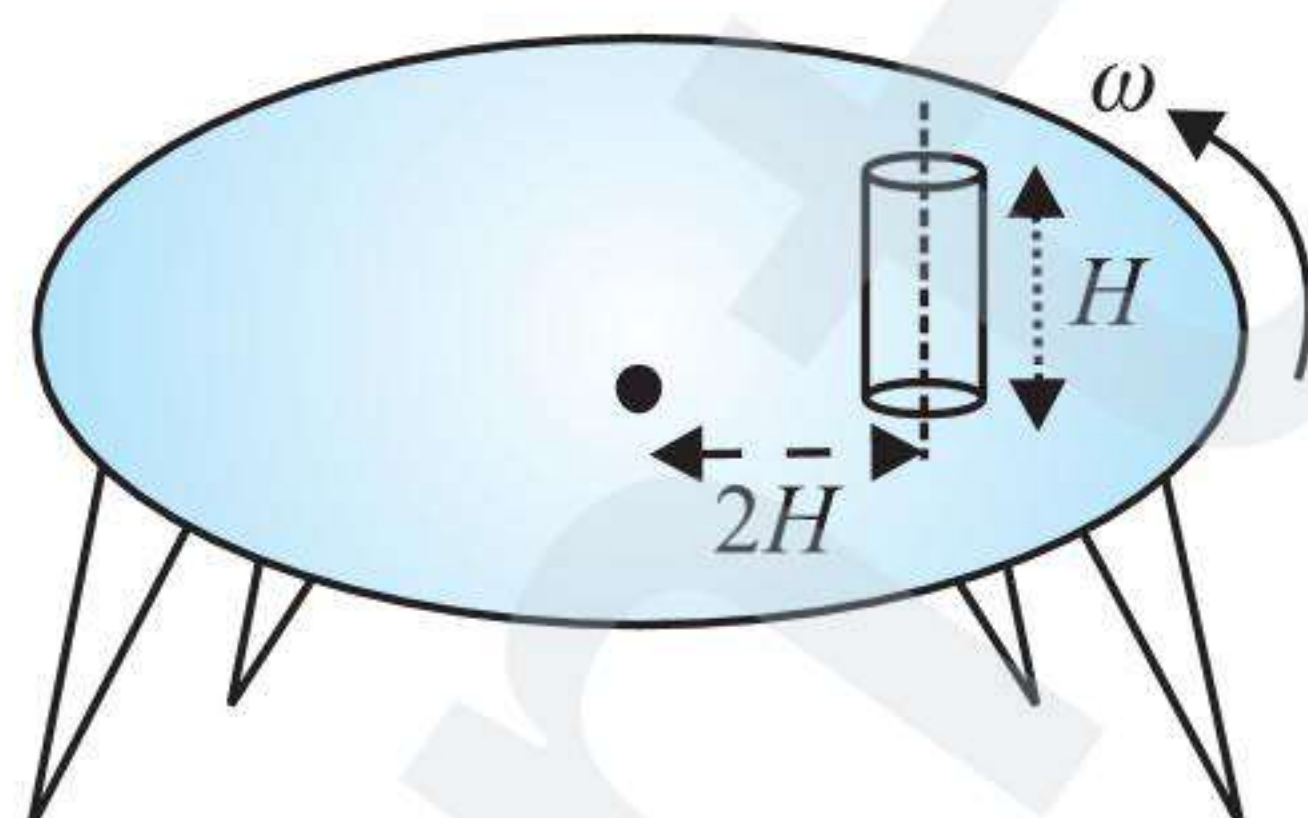
- (1) $2\sqrt{3}a_y = \sqrt{3}\alpha l + 2a_x$ (2) $\sqrt{3}a_y = \sqrt{3}\alpha l + a_x$
 (3) $a_y = \sqrt{3}\alpha l + 2a_x$ (4) $2a_y = \alpha l + 2\sqrt{3}a_x$

75. A uniform cylinder of mass M lies on a fixed plane inclined at an angle θ with horizontal. A light string is tied to the cylinder at the right most point, and a mass m hangs from the string, as shown. Assume that the coefficient of friction between the cylinder and the incline plane is sufficiently large to prevent slipping. For the cylinder the remain static, the value of m is



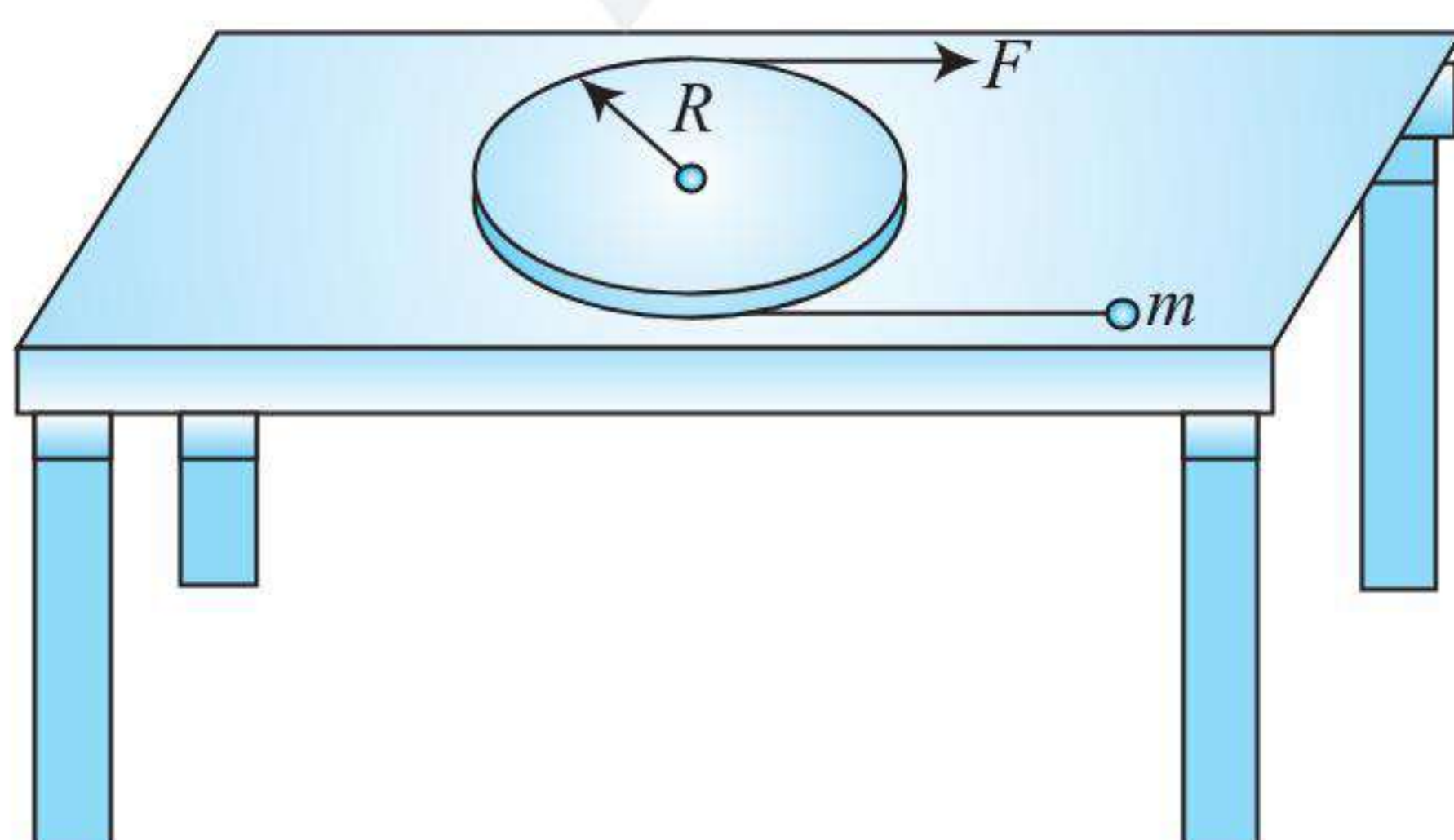
- (1) $\frac{M \sin \theta}{1 - \sin \theta}$ (2) $\frac{M \cos \theta}{1 + \sin \theta}$
 (3) $\frac{M \sin \theta}{1 + \sin \theta}$ (4) $\frac{M \cos \theta}{1 - \sin \theta}$

76. A cylinder of height H and diameter $H/4$ is kept on a frictional turntable as shown in figure. The axis of the cylinder is perpendicular to the surface of the table and the distance of axis of the cylinder is $2H$ from the centre of the table. The angular speed of the turntable at which the cylinder will start toppling (assume that friction is sufficient to prevent slipping) is



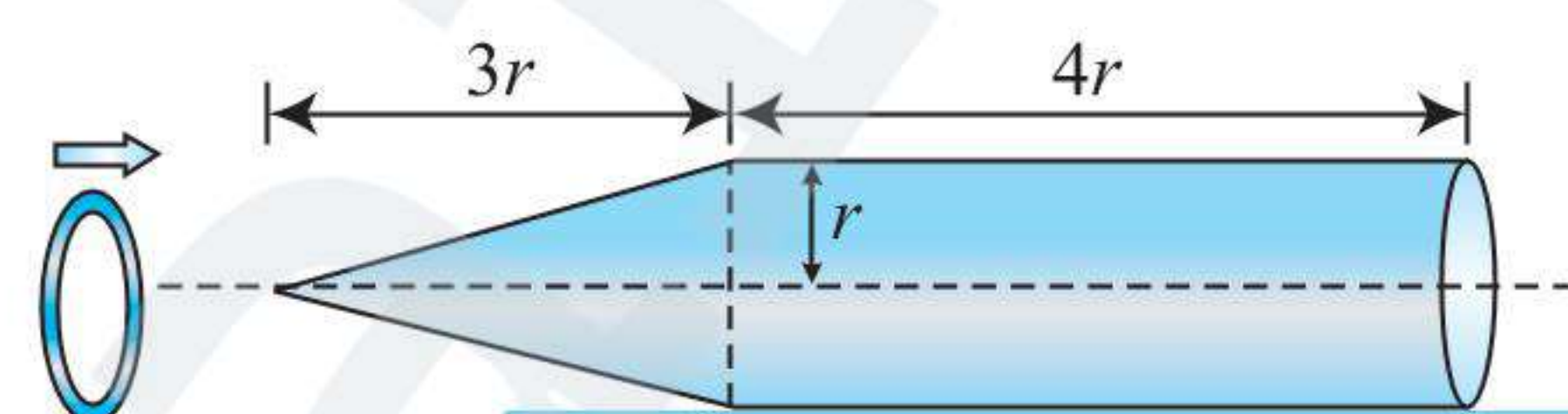
- (1) $\sqrt{\frac{g}{2} \left(\frac{1}{2} - H \right)}$ (2) $\sqrt{g \left(\frac{1}{2} - H \right)}$
 (3) $\sqrt{\frac{g}{4H}}$ (4) $\sqrt{\frac{g}{8H}}$

77. A disc of mass m and radius R placed on a smooth horizontal table as shown in figure. A light ideal string passes through the disc such that the one end of the string is attached to a small body of mass m and the other end is being pulled with a force F . The circumference of the disc is sufficiently rough so that the string does not slip over it. Find acceleration of the small body.



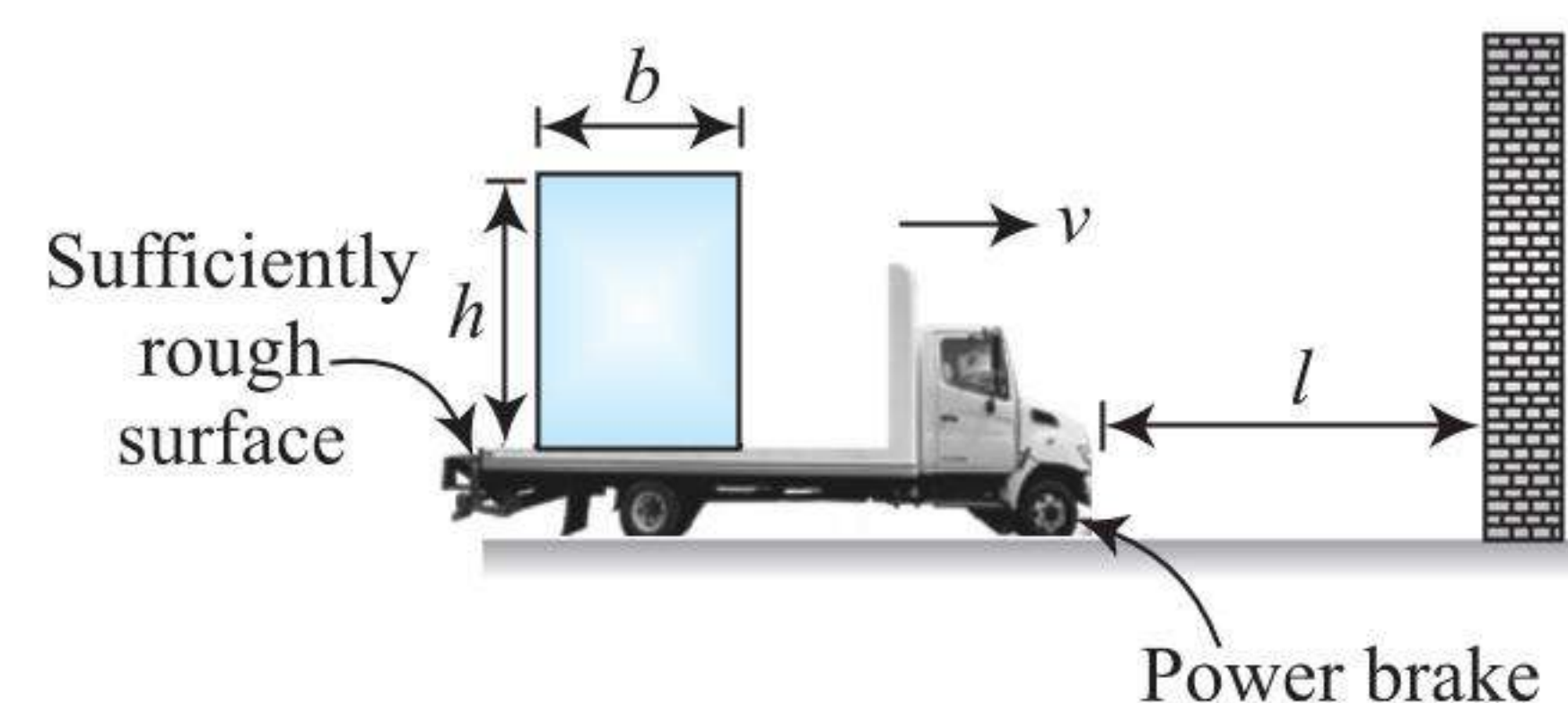
- (1) $\frac{F}{4m}$ (2) $\frac{F}{m}$
 (3) $\frac{F}{2m}$ (4) $\frac{F}{3m}$

78. A solid uniform cylindrical rod of radius r has a conical nose is placed on smooth horizontal surface as shown in figure. The length of the cylindrical and conical nose are $4r$ and $3r$ respectively. Mass of the conical part is M . A ring of radius $r/2$ is to be tightly fitted on the nose of the body. The maximum permissible mass of the ring so that the body does not topple is



- (1) $\frac{21}{5}M$ (2) $\frac{29}{6}M$
 (3) $\frac{11}{3}M$ (4) $\frac{16}{3}M$

79. Find minimum value of l so that truck can avoid the dead end, without toppling the block kept on it.



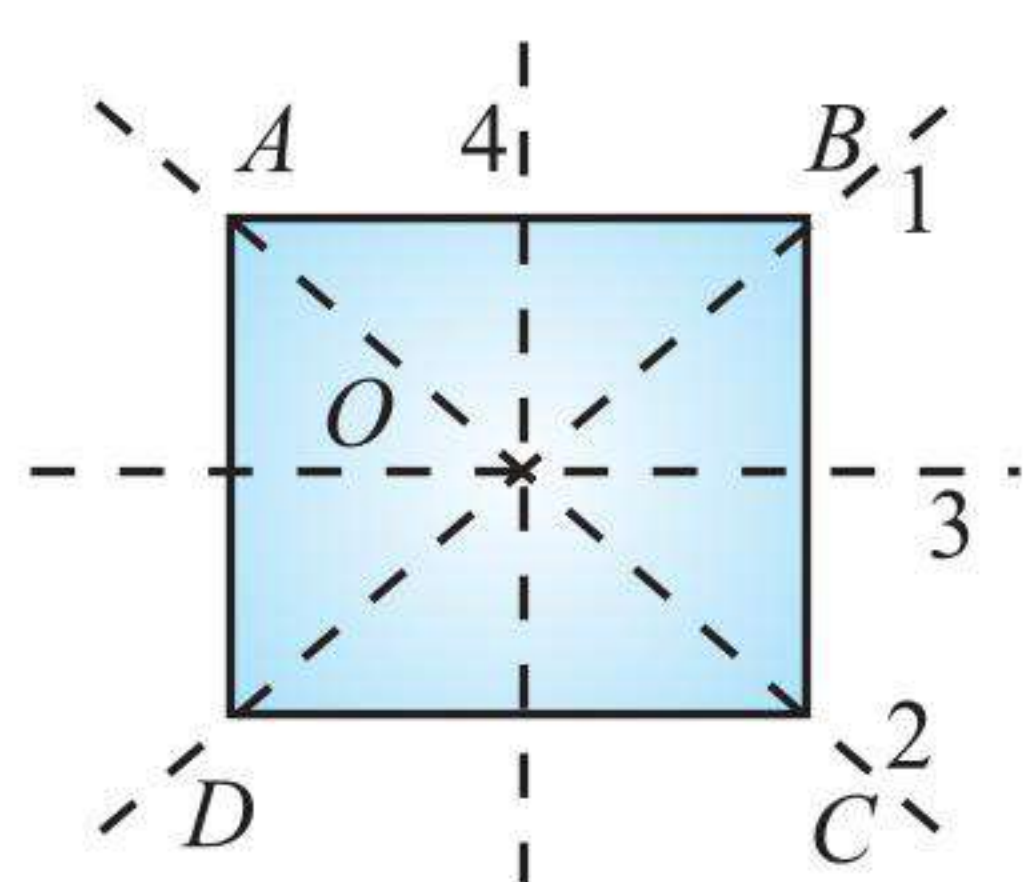
- (1) $\frac{2hv^2}{bg}$ (2) $\frac{bv^2}{2hg}$
 (3) $\frac{2bv^2}{hg}$ (4) $\frac{hv^2}{2bg}$

Multiple Correct Answers Type

- The radius of gyration of a body depends upon
 - mass of the body
 - nature of distribution of mass
 - axis of rotation
 - none of these
- A rigid body is in pure rotation, that is, undergoing fixed axis rotation. Then which of the following statement(s) are true?
 - You can find two points in the body in a plane perpendicular to the axis of rotation having the same velocity.
 - You can find two points in the body in a plane perpendicular to the axis of rotation having the same acceleration.
 - Speed of all the particles lying on the curved surface of a cylinder whose axis coincides with the axis of rotation is the same.

(4) Angular speed of the body is the same as seen from any point in the body.

3. The moments of inertia of a thin square plate $ABCD$ of uniform thickness about an axis passing through the centre O and perpendicular to the plate are



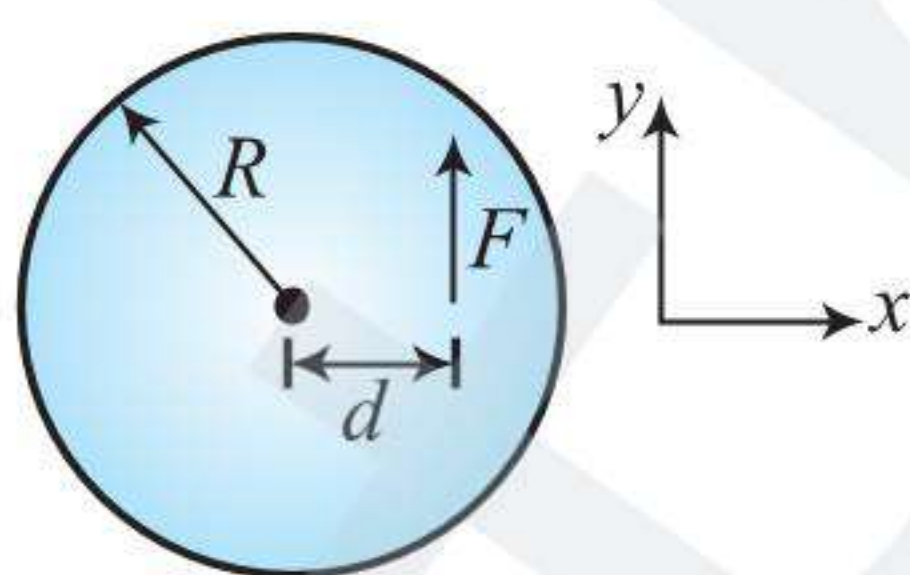
- (1) $I_1 + I_2$ (2) $I_3 + I_4$
 (3) $I_1 + I_3$ (4) $I_1 + I_2 + I_3 + I_4$

(where I_1 , I_2 , I_3 and I_4 are, respectively, the moments of inertia about axes 1, 2, 3 and 4, where axes are in the plane of the plate)

4. A bucket of water of mass 21 kg is suspended by a rope wrapped around a solid cylinder 0.2 m in diameter. The mass of the solid cylinder is 21 kg. The bucket is released from rest. Which of the following statements are correct?

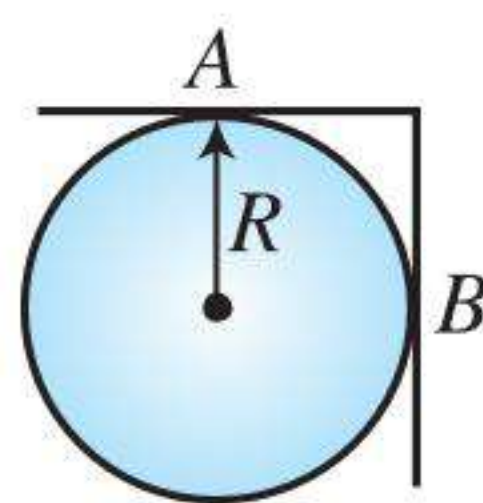
- (1) The tension in the rope is 70 N
 (2) The acceleration of the bucket is $(20/3) \text{ m/s}^2$
 (3) The acceleration of the bucket is independent of the mass of the bucket
 (4) None of these

5. A uniform thin flat isolated disc is floating in space. It has radius R and mass m . A force F is applied to it at a distance $d = (R/2)$ from the centre in the y -direction. Treat this problem as two-dimensional. Just after the force is applied



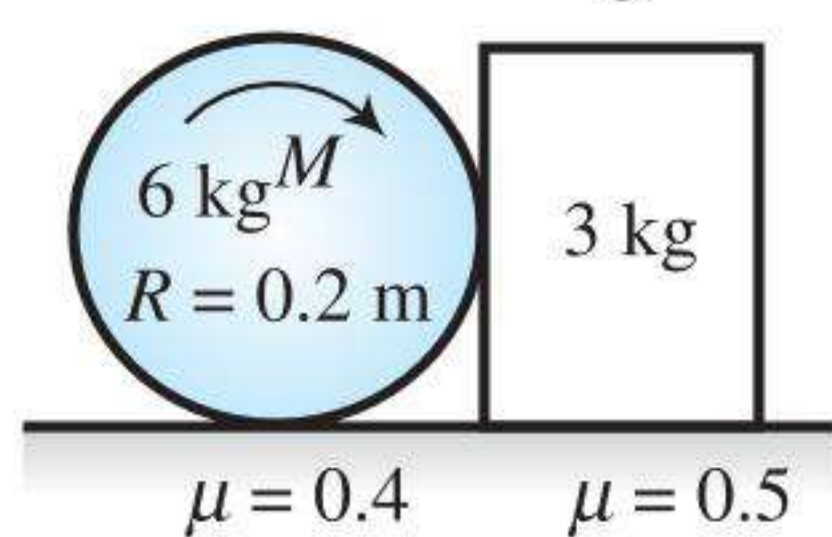
- (1) acceleration of the centre of the disc is F/m
 (2) angular acceleration of the disk is F/mR
 (3) acceleration of leftmost point on the disc is zero
 (4) point which is instantaneously unaccelerated is the rightmost point

6. A rod bent at right angle along its centre line, is placed on a rough horizontal fixed cylinder of radius R as shown in figure. Mass of rod is $2m$ and rod is in equilibrium. Assume that friction force on rod at A and B are equal in magnitude.



- (1) Normal force applied by cylinder on rod at A is $3mg/2$
 (2) Normal force applied by cylinder on rod at B must be zero
 (3) Friction force acting on rod at B is upward
 (4) Normal force applied by cylinder on rod at A is mg

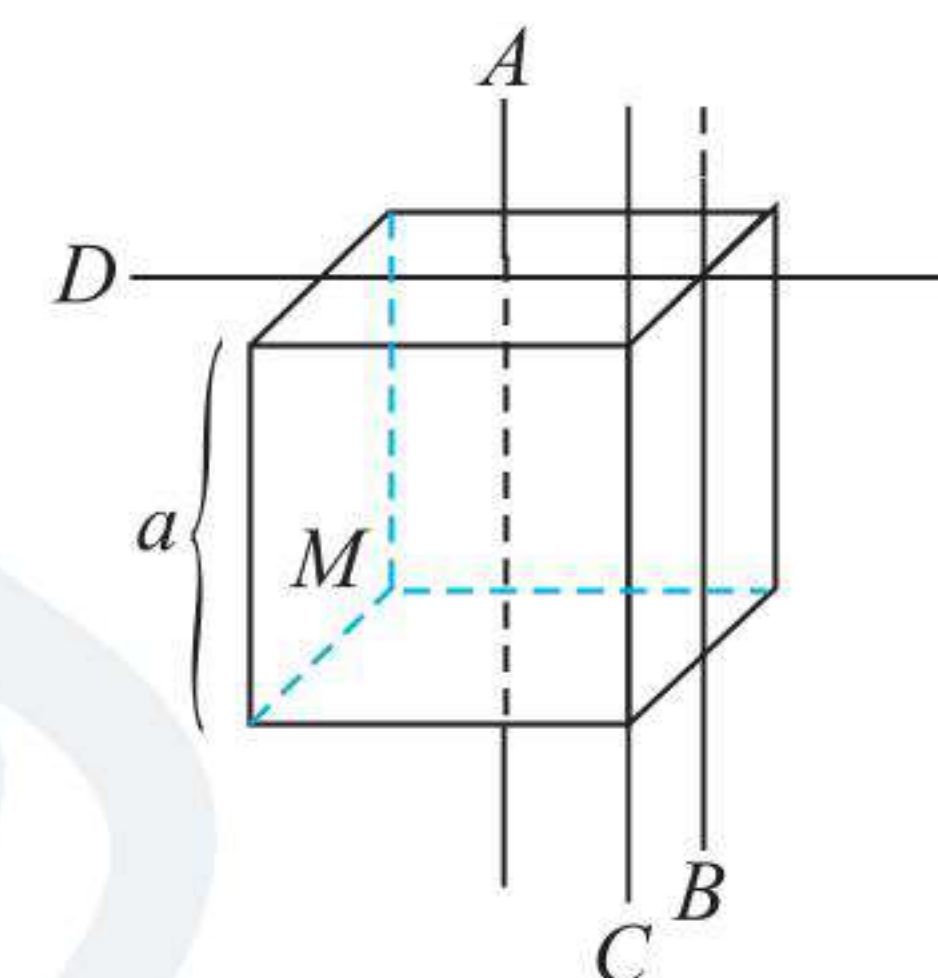
7. A clockwise torque of 6 Nm is applied to the circular cylinder as shown in the figure. There is no friction between the cylinder and the block.



- (1) The cylinder will be slipping but the system does not move forward
 (2) The system cannot move forward for any torque applied to the cylinder

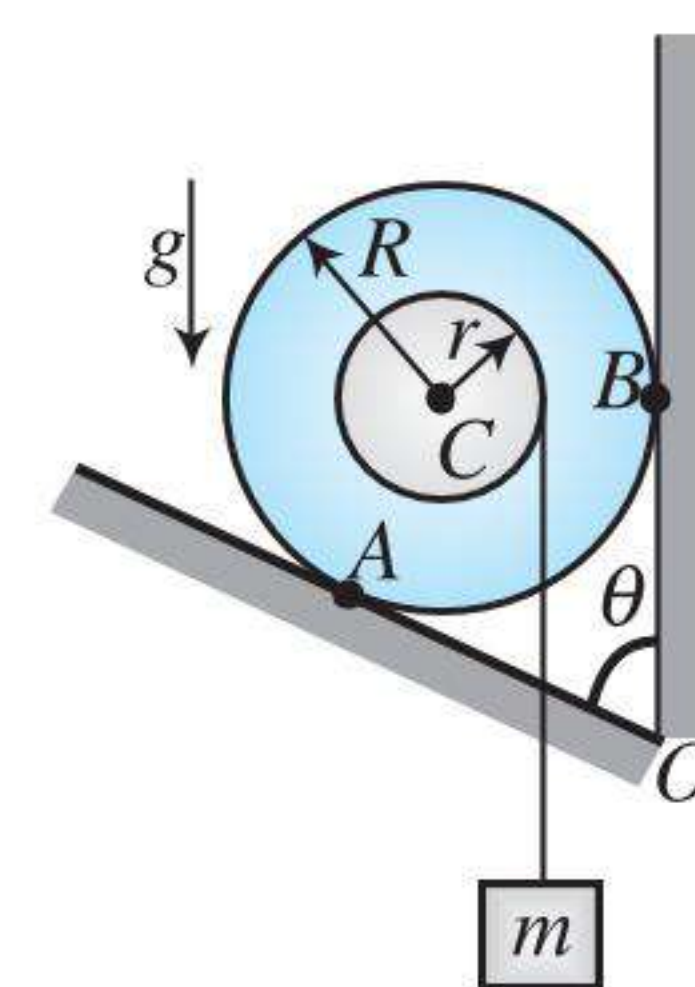
- (3) The acceleration of the system will be 1 m/s^2 forward
 (4) The angular acceleration of the cylinder is 10 rad s^{-2}

8. Illustrated below is a uniform cubical block of mass M and side a . Mark the correct statement(s).



- (1) The moment of inertia about axis A , passing through the centre of mass is $I_A = \frac{1}{6} Ma^2$
 (2) The moment of inertia about axis B , which bisects one of the cube faces is $I_B = \frac{5}{12} Ma^2$
 (3) The moment of inertia about axis C , along one of the cube edges is $I_C = \frac{2}{3} Ma^2$
 (4) The moment of inertia about axis D , which bisects one of the horizontal cube faces is $\frac{7}{12}$

9. A massless spool of inner radius r , outer radius R is placed against vertical wall and tilted split floor as shown. A light inextensible thread is tightly wound around the spool through which a mass m is hanging. There exists no friction at point A , while the coefficient of friction between spool and point B is μ . The angle between two surfaces is θ .



- (1) the magnitude of force on the spool at B in order to

maintain equilibrium is $mg \sqrt{\left(\frac{r}{R}\right)^2 + \left(1 - \frac{r}{R}\right)^2} \frac{1}{\tan^2 \theta}$

- (2) the magnitude of force on the spool at B in order to

maintain equilibrium is $mg \left(1 - \frac{r}{R}\right) \frac{1}{\tan \theta}$

- (3) the minimum value of μ for the system to remain in equilibrium is $\frac{\cot \theta}{(R/r) - 1}$

- (4) the minimum value of μ for the system to remain in equilibrium is $\frac{\tan \theta}{(R/r) - 1}$

10. For a body having pure rotation motion. The axis of rotation

- (1) may pass through the centre of mass
 (2) must pass through the centre of mass
 (3) may pass through a particle of the body
 (4) must pass through a particle of the body

11. A solid sphere is rotating about a diameter with uniform angular speed. Which of the following options is/are correct

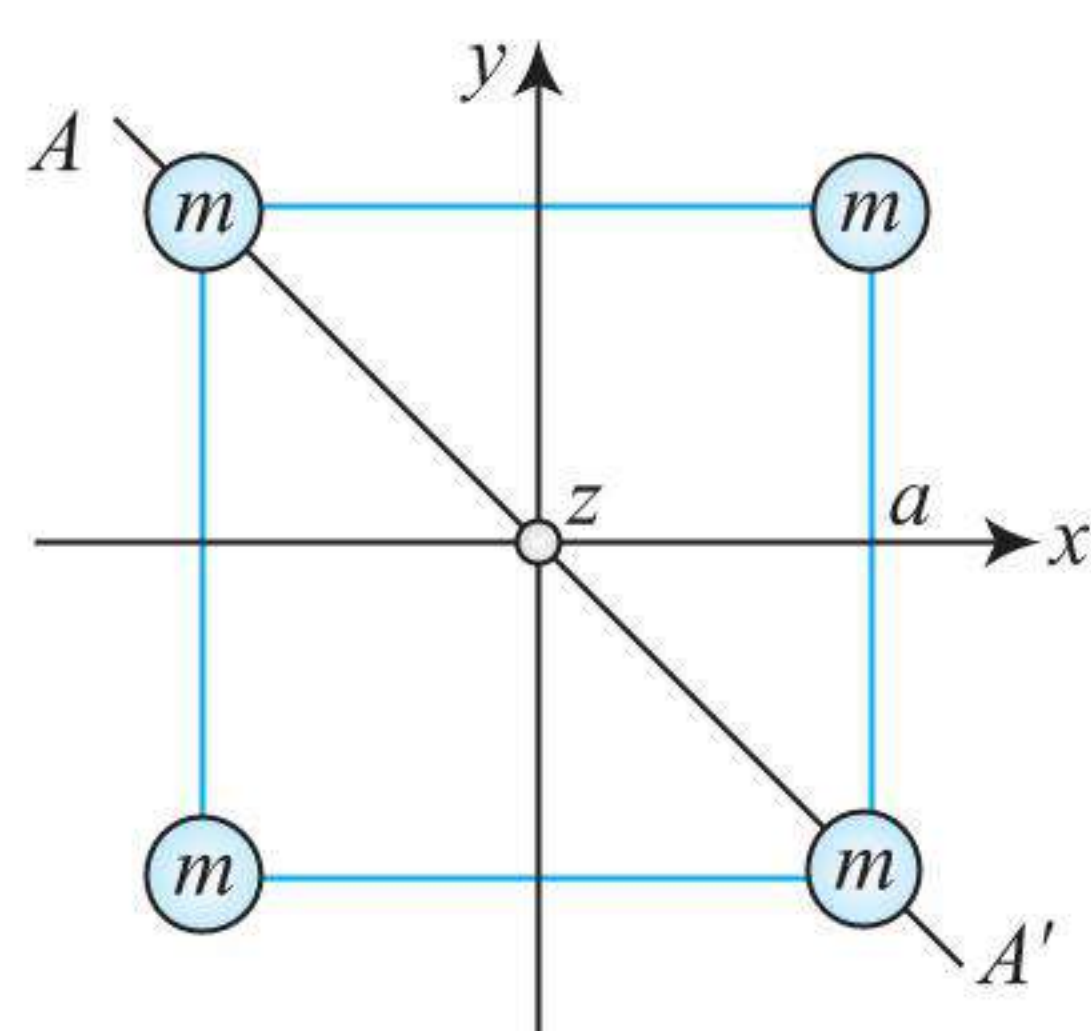
- (1) The particles on the surface of the sphere do not have any linear acceleration
 (2) The particles on the diameter mentioned above do not have any linear acceleration
 (3) Different particles on the surface have different angular speeds
 (4) All the particles on the surface have the different linear speed

12. Two identical semicircular discs of mass ' m ' each and radius ' R ' are placed in the XY (horizontal) plane and the YZ (vertical) plane, respectively. They are so placed that they have their common diameter along the Y -axis. Then, the moment of inertia of the system

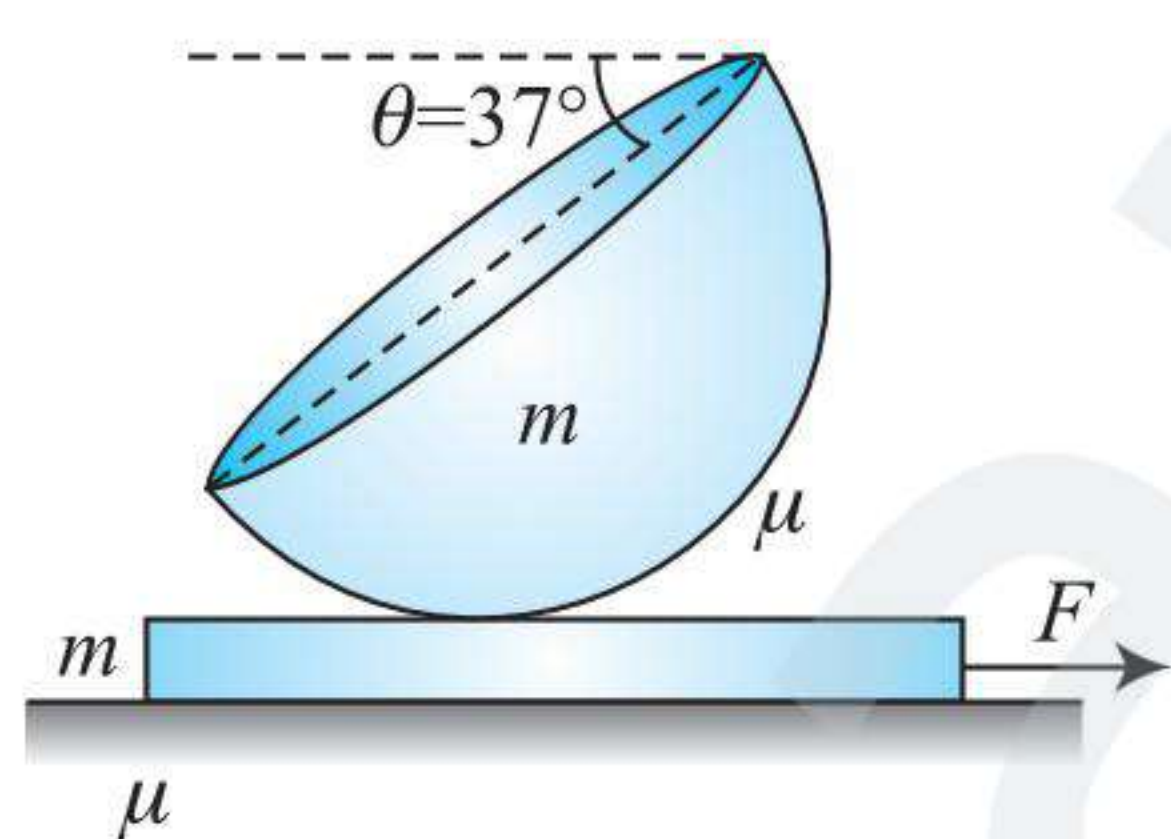
$$(1) I_x = \frac{1}{2}mR^2 \quad (2) I_y = \frac{1}{2}mR^2$$

$$(3) I_z = \frac{3}{4}mR^2 \quad (4) I_x = I_z$$

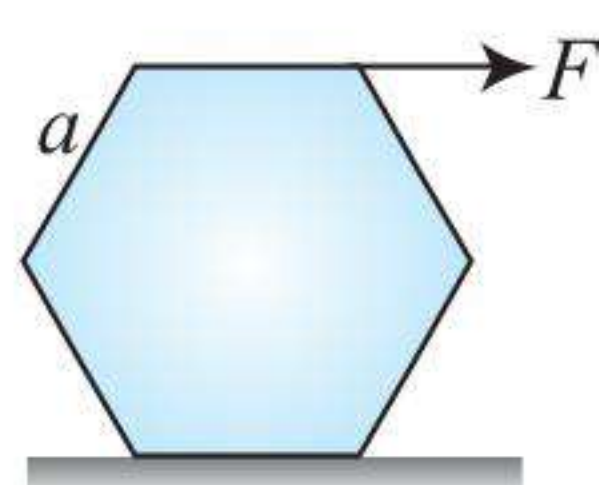
13. Four point mass, each of mass m are connected at a corner of a square of side ' a ', by massless rods as shown in the figure, x and y axis are in the plane of the system and z axis is perpendicular to the plane and passing through the centre of the square.



- (1) Moment of inertia of the system about y -axis is $I_y = ma^2$
 (2) Moment of inertia of the system about the diagonal axis AA' is $I_{AA'} = ma^2$
 (3) Moment of inertia of the system about z -axis is $I_z = ma^2$
 (4) Moment of inertia of the system about x -axis is $I_x = ma^2$
14. A force F is applied on the plank such that the hollow hemispherical shell is in equilibrium as shown in figure. The coefficient of friction μ is same between the shell with respect to plank hemispherical shell and the plank as its between the plank and the ground. Friction is just sufficient to prevent the slipping (Take $g = 10 \text{ m/s}^2$ and $m = 5 \text{ kg}$). Then



- (1) the minimum magnitude of coefficient of friction $\mu = 1/2$
 (2) the minimum magnitude of coefficient of friction $\mu = 1/4$
 (3) the magnitude of applied force is 100 N.
 (4) the acceleration of plank is 10 m/s^2 .
15. A hexagonal body lies on a horizontal rough surface. A force F acts on the side of a hexagonal body as shown in figure. Choose the correct option(s).



- (1) For the toppling, the minimum value of coefficient of friction is $\frac{1}{2\sqrt{3}}$.

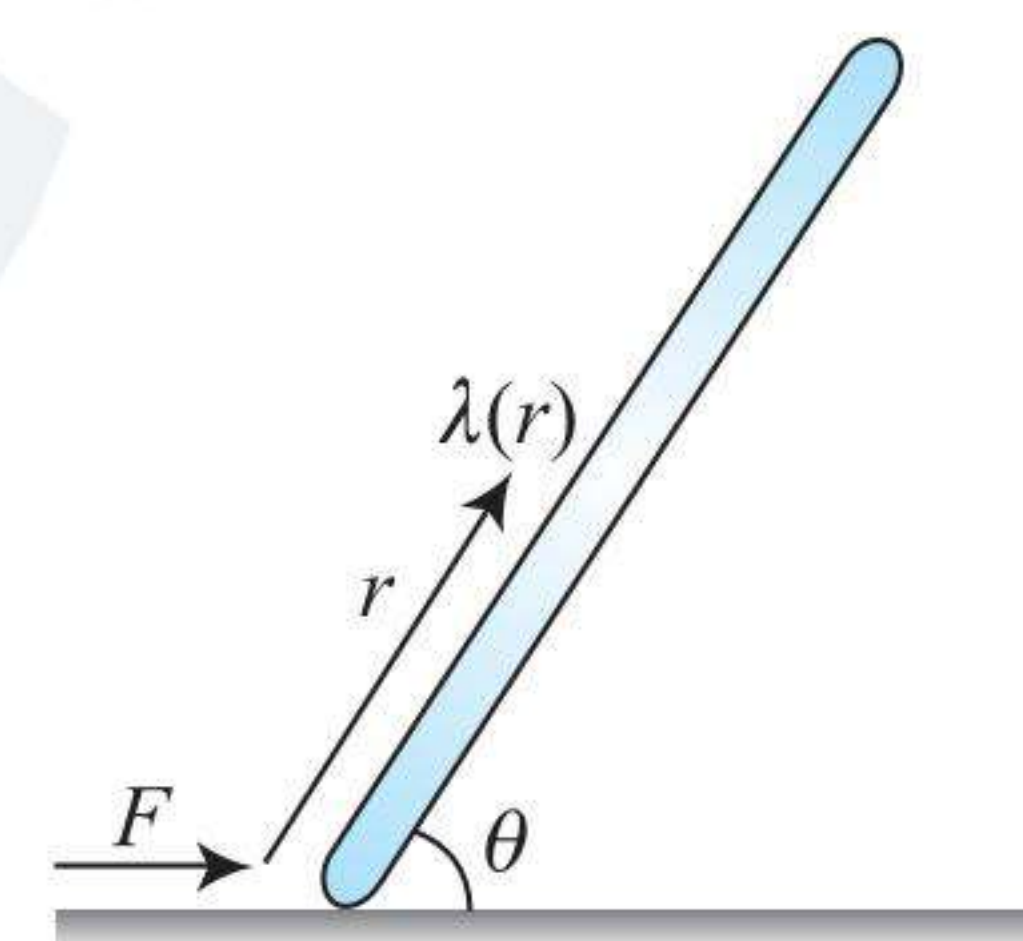
- (2) For the toppling, the minimum value of coefficient of friction is $\frac{2}{\sqrt{3}}$.

- (3) If $\mu = 2\mu_{\min}$, and applied force $F = \frac{mg}{\sqrt{3}}$ then angular acceleration of the body is $\frac{6}{17} \frac{g}{a}$.

- (4) If $\mu = 2\mu_{\min}$, and applied force $F = \frac{mg}{\sqrt{3}}$, then angular acceleration of the body is $\frac{6}{5} \frac{g}{a}$.

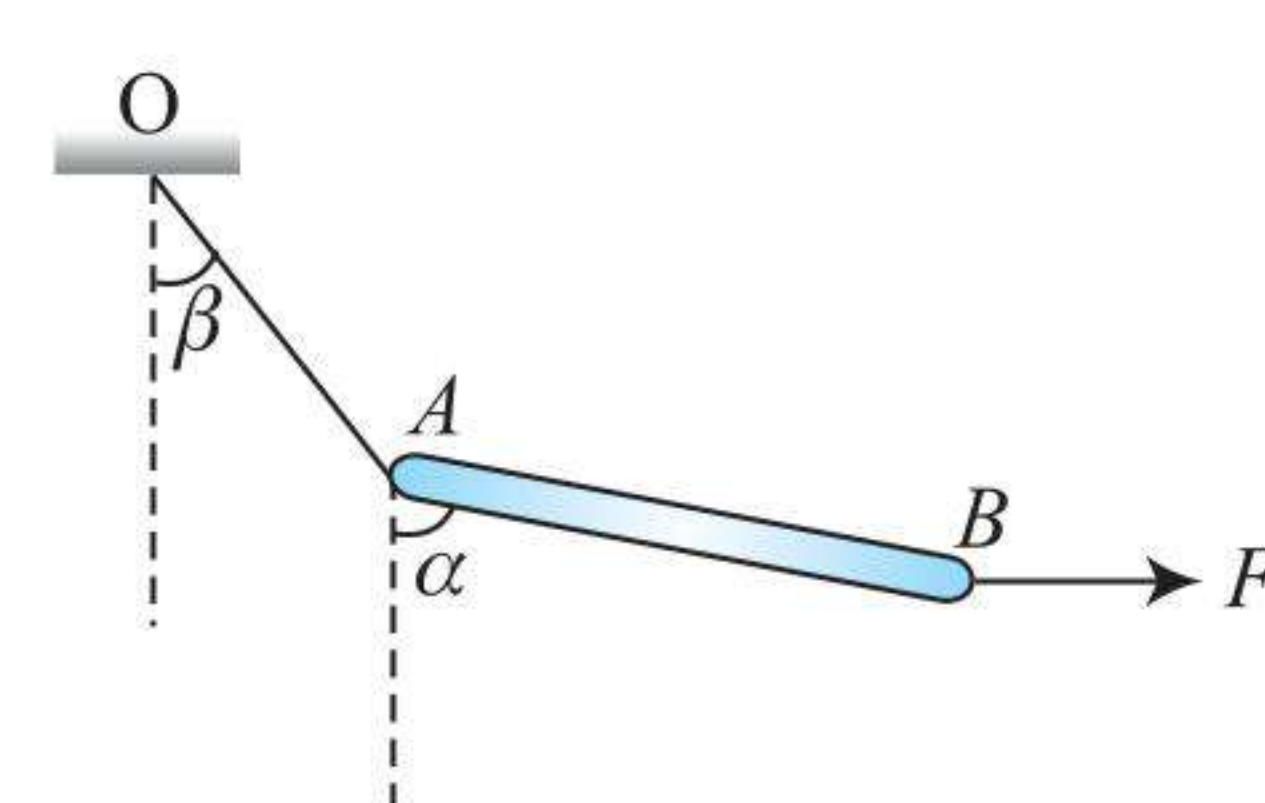
16. A rod, placed in vertical plane, is being pushed along a smooth horizontal surface by a horizontal force $F = \frac{2mg}{3}$.

Mass per unit length of the rod is given $\lambda(r) = \left(\frac{m}{l^2}r\right)$, where r is distance from the action point of applied force F and l is the length of the rod. Then,



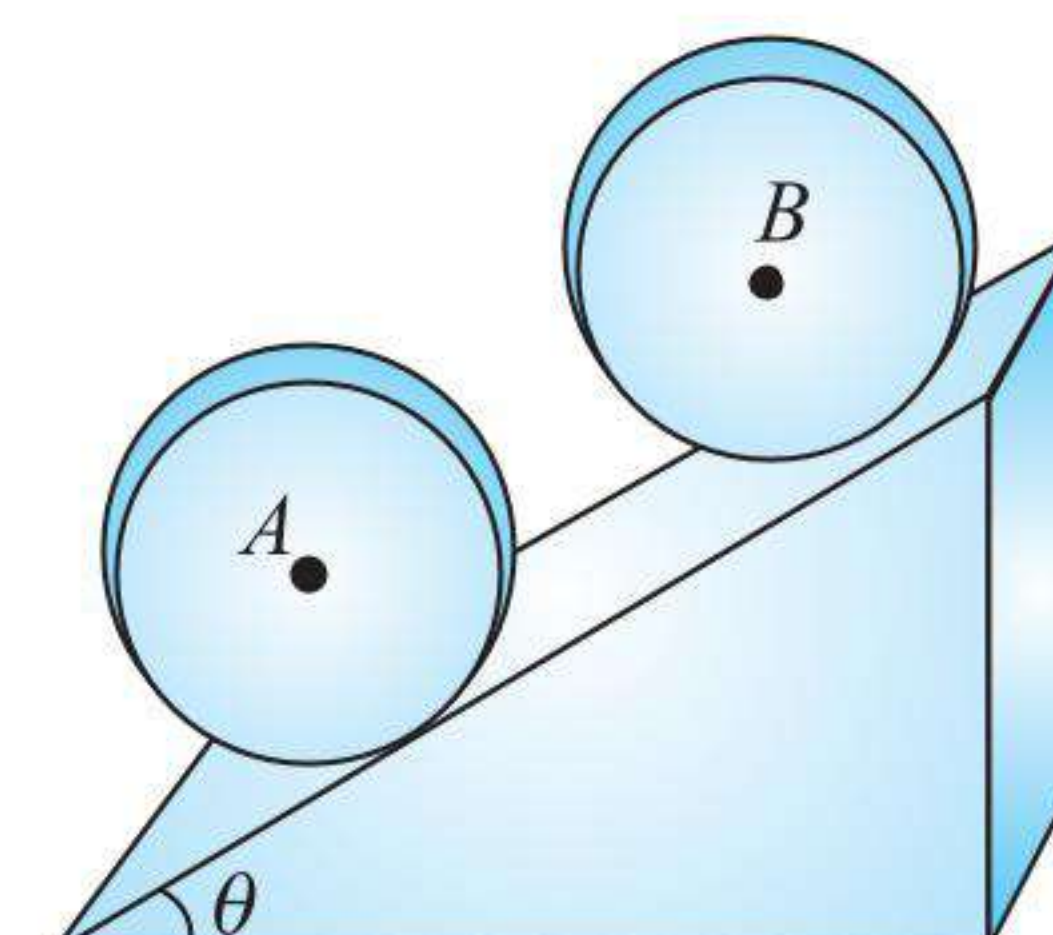
- (1) angle θ for which rod only translates is 37°
 (2) acceleration of the rod for which rod only translates is $\frac{4}{3}g$
 (3) normal force for which rod only translates is $\frac{m}{2}g$
 (4) normal force for which rod only translates is mg

17. A stick AB of mass M is tied at one end to a light string OA . A horizontal force $F = Mg$ is applied at end B of the stick and it remains in equilibrium in position shown. Then,



- (1) α is $\tan^{-1}(2)$
 (2) the value of β is 45°
 (3) tension in string OA is $\sqrt{2}Mg$
 (4) the value of α is $\tan^{-1}\left(\frac{1}{2}\right)$

18. Two cylinders A and B have been placed in contact on an incline. They remain in equilibrium. The dimensions of the two cylinders are same. Choose the correct statement.



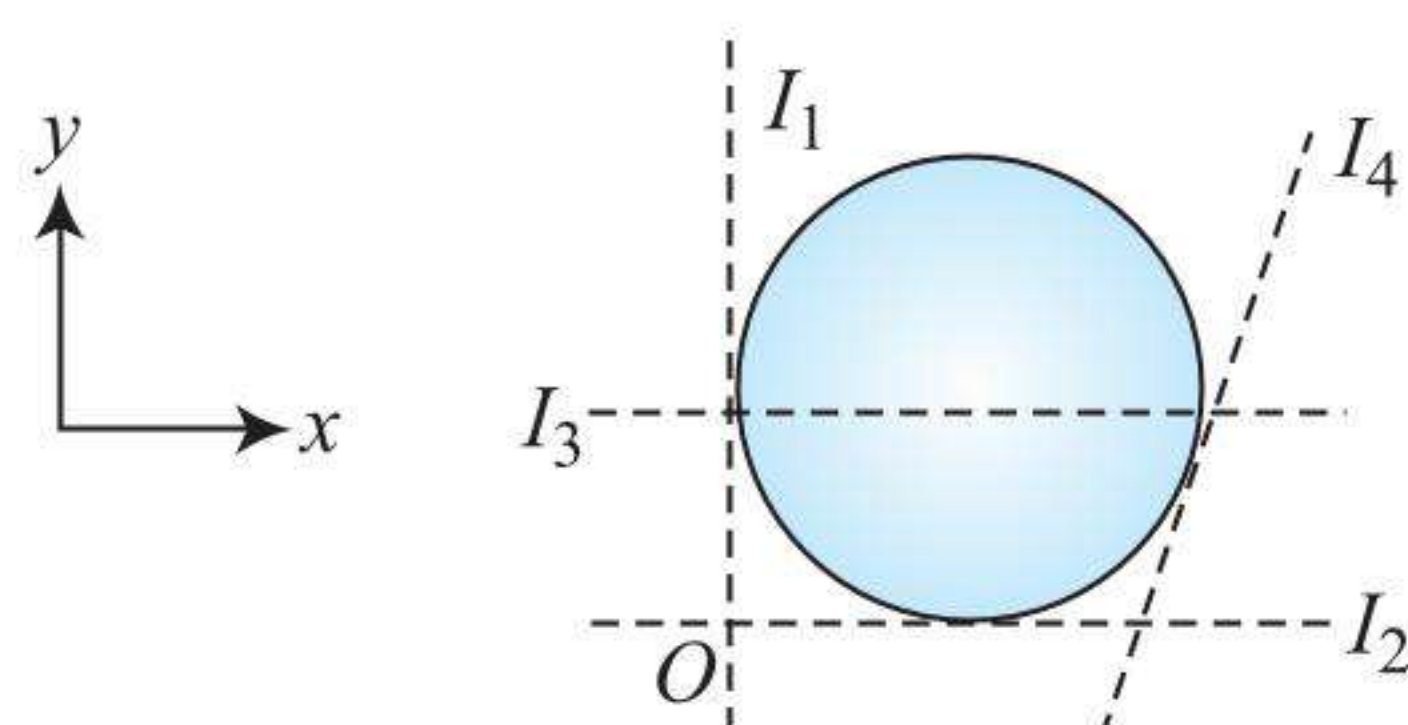
- (1) The magnitude of friction in all surfaces should be equal
 (2) The value of friction between the cylinder is

$$\left(\frac{M_A + M_B}{2} \right) g \sin \theta$$

- (3) For two cylinders to be in contact mass of cylinder B should be greater than the mass of cylinder A
 (4) The normal reaction between the cylinders should be

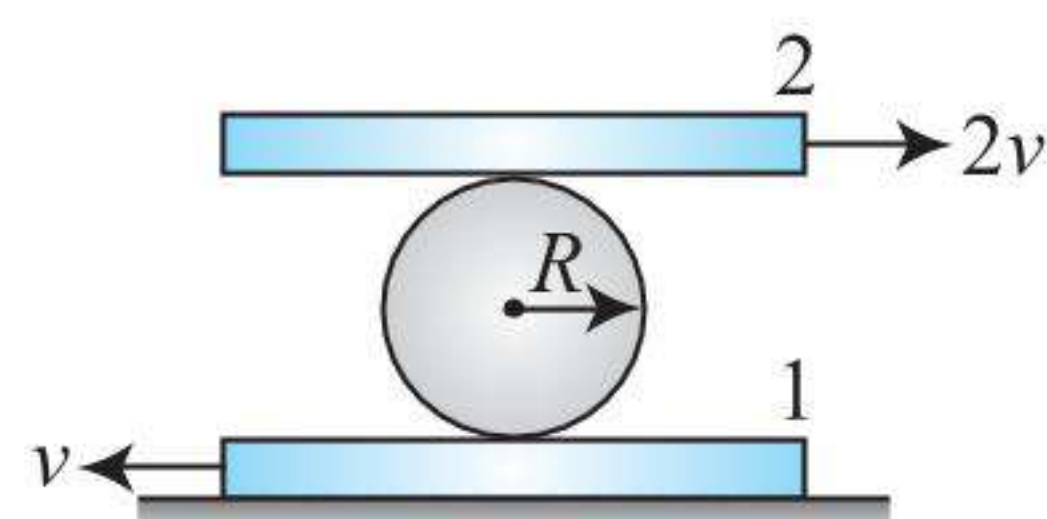
$$\left(\frac{M_A + M_B}{2} \right) g \sin \theta$$

19. A disc has radius R , mass m and has moment of inertia I_1, I_2, I_3 and I_4 about axis as shown in figure. Moment of inertia of disc about z -axis passing through O will be



- (1) $I_1 + I_2$ (2) $\frac{5}{2}mR^2$
 (3) $3mR^2$ (4) $I_1 + I_4$

20. A cylinder is rolling without sliding over two horizontal planks (surfaces) 1 and 2. If the velocities of the surfaces A and B are $-v\hat{i}$ and $2v\hat{i}$ respectively, find the

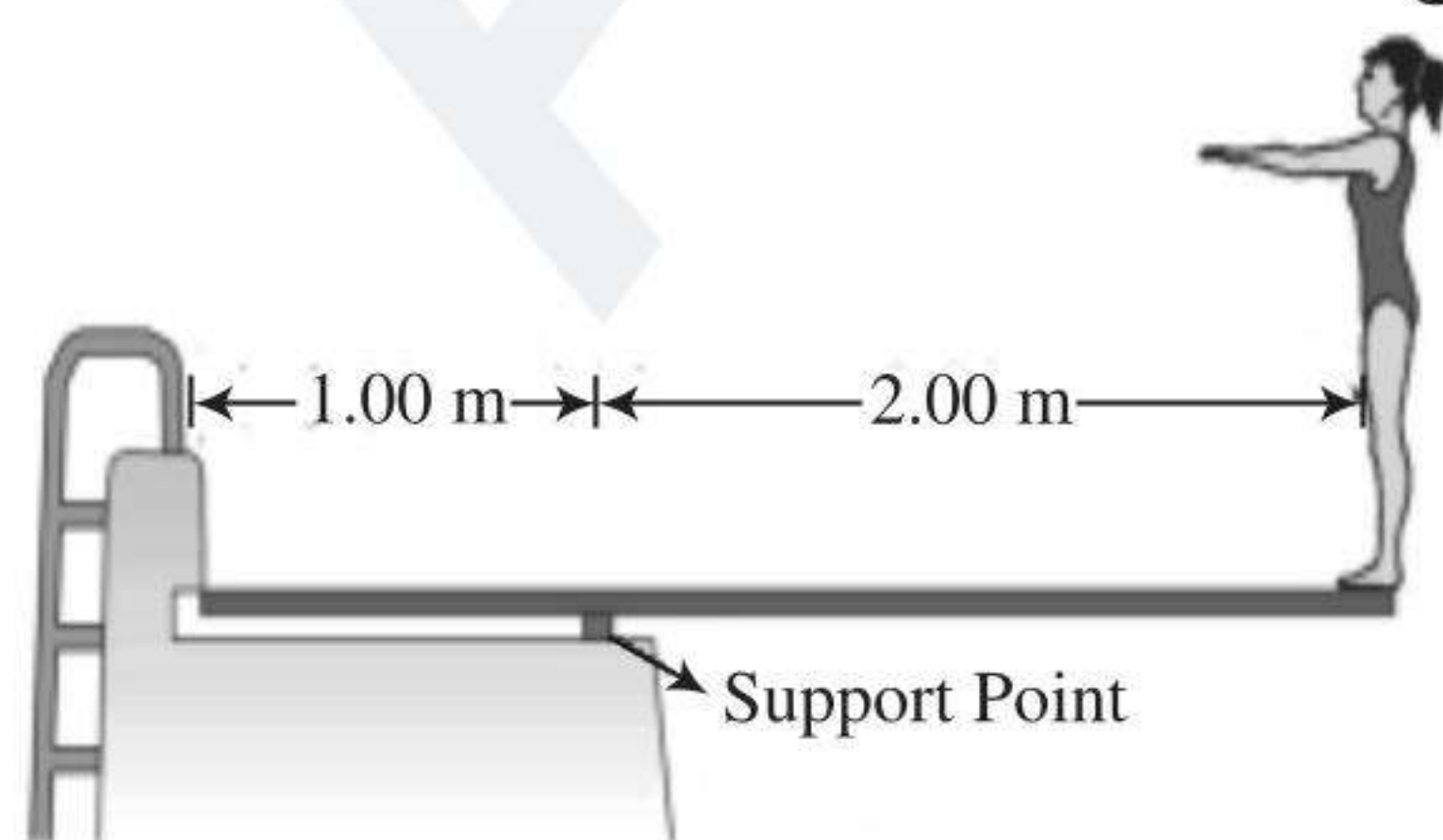


- (1) The instantaneous axis of rotation is located at a distance of $4R/3$ below the top of the disc
 (2) The instantaneous axis of rotation is located at a distance of $3R/2$ below the top of the disc
 (3) Angular velocity of the cylinder is $\frac{3v}{2R}$
 (4) Angular velocity of the cylinder is $\frac{5v}{3R}$

Linked Comprehension Type

For Problems 1–2

A diving board 3.00 m long is supported at a point 1.00 m from the end and a diver weighing 500 N stands at the free end. The diving board is of uniform cross section and weighs 280 N. Find

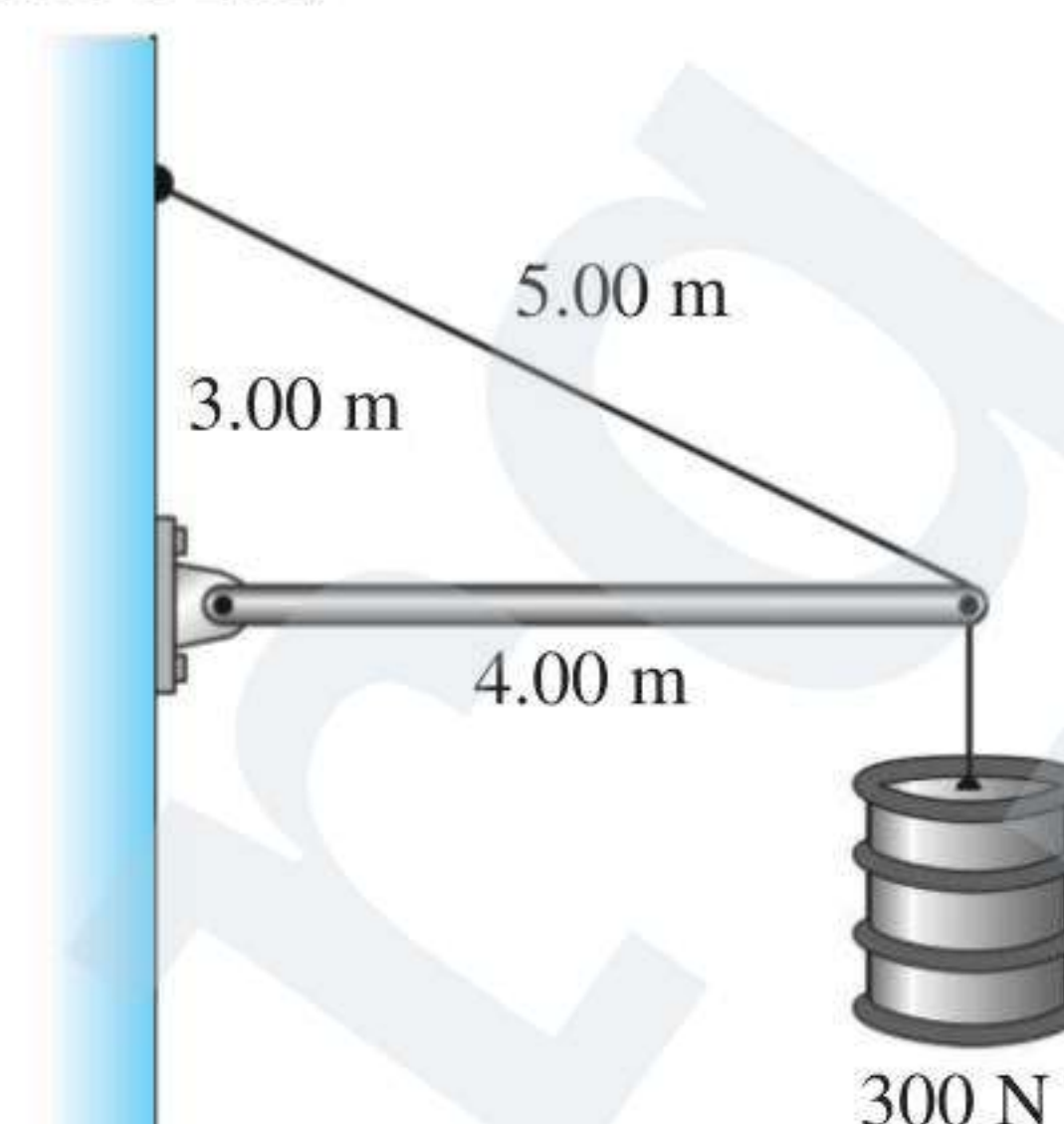


1. The forces at the support point
 (1) 780 N (2) 220 N
 (3) 1920 N (4) 1140 N

2. The force at the end that is held down.
 (1) 780 N (2) 220 N
 (3) 1920 N (4) 1140 N

For Problems 3–4

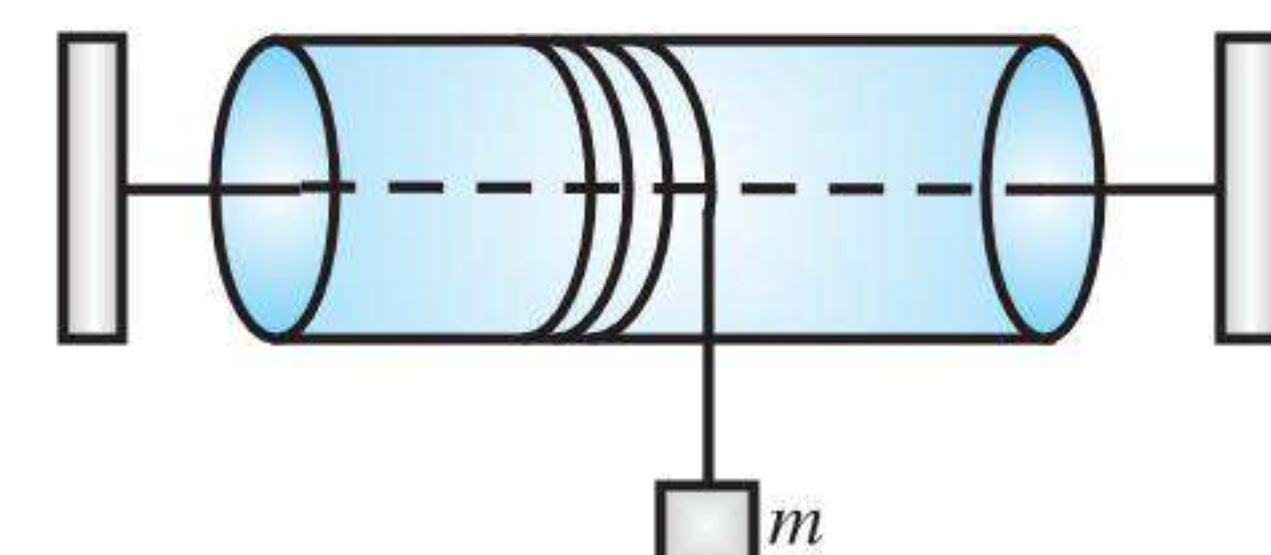
The horizontal beam in figure weighs 150 N, and its centre of gravity is at its centre. Find



3. The tension in the cable
 (1) 75 N (2) 500 N
 (3) 300 N (4) 625 N
 4. The horizontal and vertical components of the force exerted on the beam at the wall
 (1) Horizontal component is 500 N towards left and vertical component 75 N downwards
 (2) Horizontal component is 500 N towards right and vertical component 75 N upwards
 (3) Horizontal component is 625 N towards left and vertical component 150 N upwards
 (4) Horizontal component is 625 N towards right and vertical component 150 N downwards

For Problems 5–6

A cord is wound round the circumference of a solid cylinder of radius R and mass M . The axis of the cylinder is horizontal. A weight mg is attached to the end of the cord and falls from rest. After falling through a distance h .



5. The angular velocity of the cylinder will be

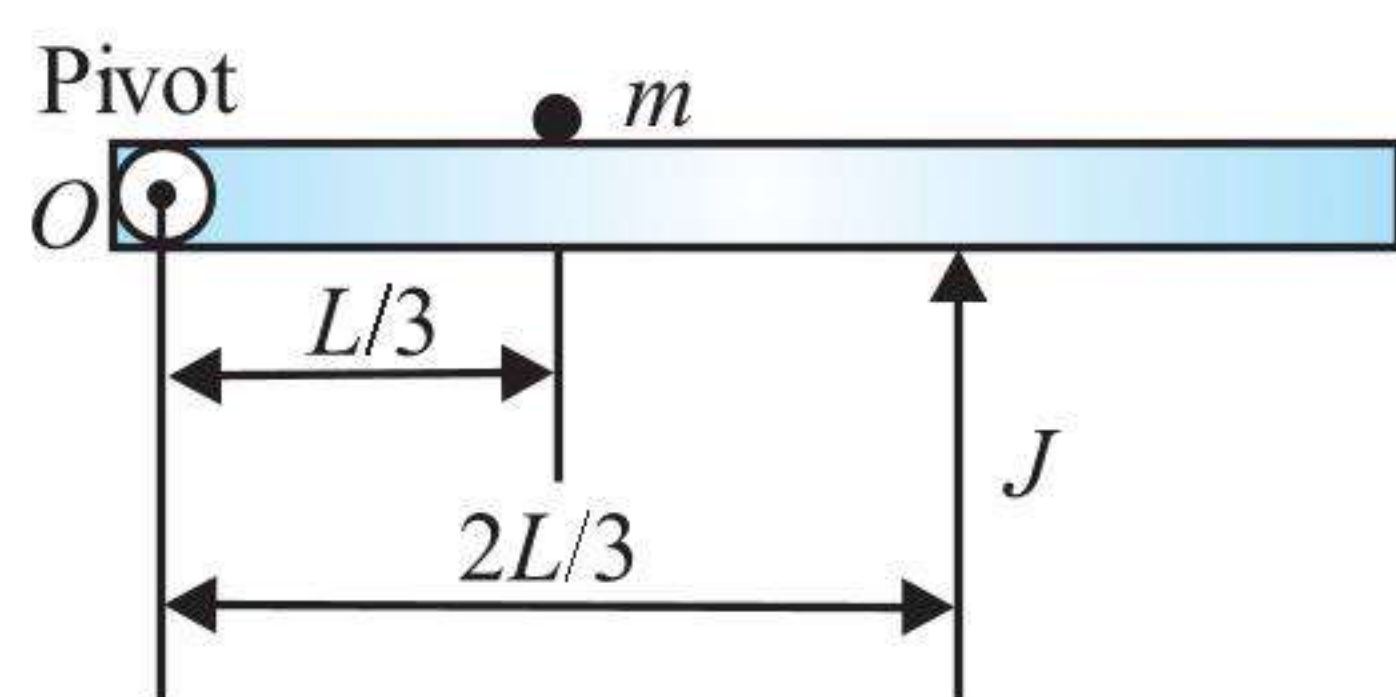
- (1) $\frac{2mg}{M+2m}$ (2) $\sqrt{\frac{2gh}{R^2}}$
 (3) $\sqrt{\frac{4mgh}{(M+2m)R^2}}$ (4) $\sqrt{\frac{4mgh}{M+2m}}$

6. If the mass starts from rest and falls a distance h , then its speed at that instant is

- (1) proportional to R (2) proportional to $\frac{1}{R}$
 (3) proportional to $\frac{1}{R^2}$ (4) independent of R

For Problems 7–9

A uniform rod of length L and mass M is lying on a frictionless horizontal plane and is pivoted at one of its ends as shown in figure. There is no friction at the pivot. An inelastic ball of mass m is fixed with the rod at a distance $L/3$ from O . A horizontal impulse J is given to the rod at a distance $2L/3$ from O in a direction perpendicular to the rod. Assume that the ball remains in contact with the rod after the collision and impulse J acts for a small time interval Δt .



Now answer the following questions:

7. Find the resulting instantaneous angular velocity of the rod after the impulse.

(1) $\frac{3J}{(m+3M)L}$ (2) $\frac{6J}{(m+3M)L}$

(3) $\frac{3J}{(3m+M)L}$ (4) $\frac{6J}{(3m+M)L}$

8. Find the impulse acted on the ball during the time interval Δt .

(1) $\frac{2MJ}{(3m+M)}$ (2) $\frac{2MJ}{(m+3M)}$

(3) $\frac{2mJ}{(3m+M)}$ (4) $\frac{2mJ}{(m+3M)}$

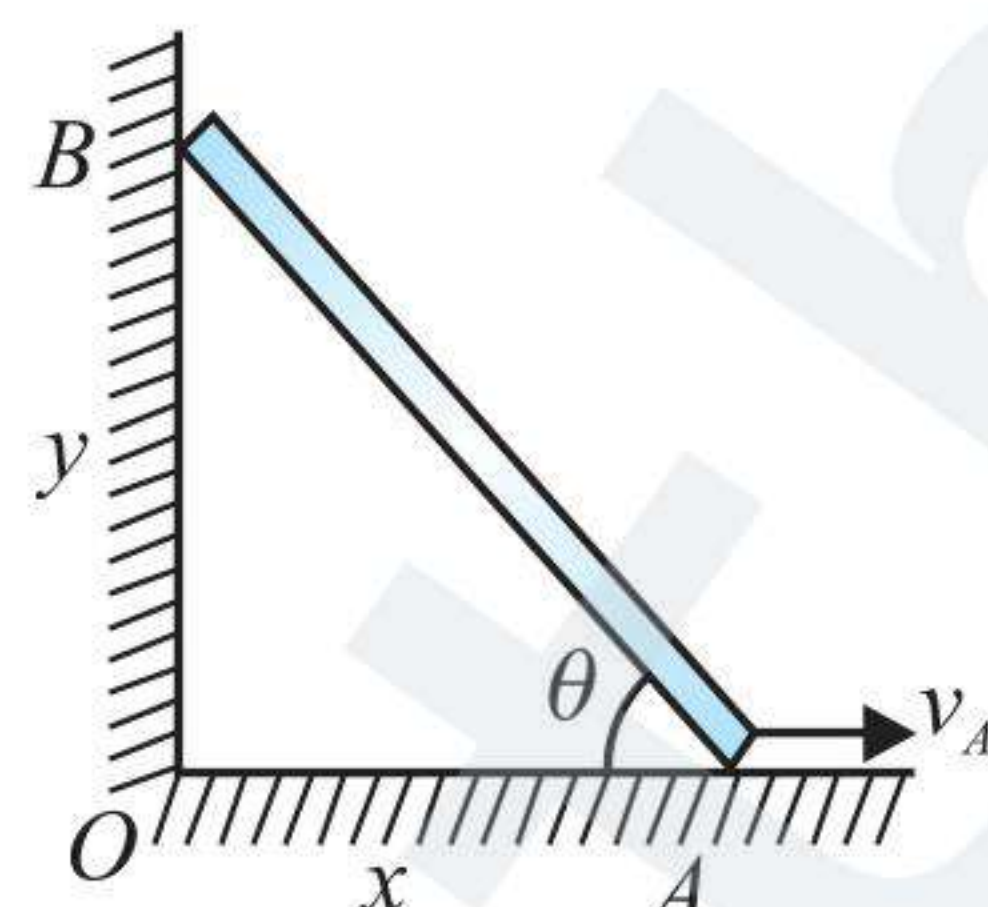
9. Find the magnitude of the impulse applied by the pivot during the time interval Δt .

(1) $\frac{mJ}{(m+3M)}$ (2) $\frac{mJ}{(3m+M)}$

(3) $\frac{MJ}{(m+3M)}$ (4) $\frac{MJ}{(3m+M)}$

For Problems 10–12

End A of a rod AB is being pulled on the floor with a constant velocity v_0 as shown. Taking the length of the rod as l , at an instant when the rod makes an angle 37° with the horizontal, calculate



10. The velocity of end B

(1) $\frac{3}{5} v_0$ (2) $\frac{4}{5} v_0$

(3) $\frac{5}{3} v_0$ (4) $\frac{5}{4} v_0$

11. The angular velocity of the end

(1) $\frac{5v_0}{3l}$ (2) $\frac{3v_0}{5l}$

(3) $\frac{5v_0}{4l}$ (4) $\frac{4v_0}{5l}$

12. The velocity of the CM of the rod

(1) $\frac{5}{7} v_0$ at $\tan^{-1} \frac{4}{3}$ below horizontal

(2) $\frac{5}{7} v_0$ at $\tan^{-1} \frac{3}{4}$ below horizontal

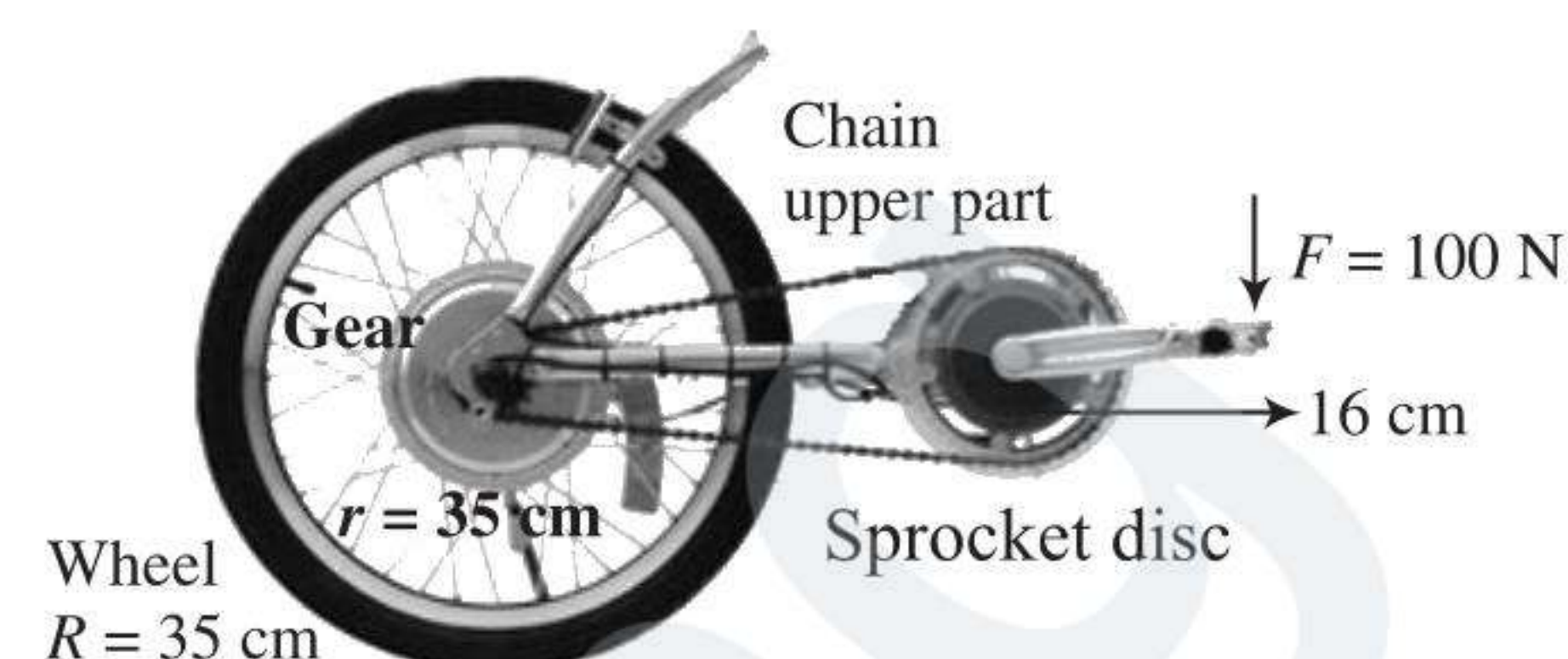
(3) $\frac{5}{6} v_0$ at $\tan^{-1} \frac{3}{4}$ below horizontal

(4) $\frac{5}{6} v_0$ at $\tan^{-1} \frac{4}{3}$ below horizontal

For Problems 13–17

A bicycle has pedal rods of length 16 cm connected to a sprocketed disc of radius 10 cm. The bicycle wheels are 70 cm in diameter and the chain runs over a gear of radius 4 cm. The speed of the cycle is constant and the cyclist applies 100 N force that is always perpendicular to the pedal rod, as shown in the figure. Assume

tension in the lower part of chain is negligible. The cyclist is peddling at a constant rate of two revolutions per second. Assume that the force applied by other foot is zero when one foot is exerting 100 N force. Neglect friction within cycle parts and the rolling friction.



13. The tension in the upper portion of the chain is equal to

(1) 100 N (2) 120 N

(3) 160 N (4) 240 N

14. Net torque on the rear wheel of the bicycle is equal to

(1) zero (2) 16 Nm

(3) 6.4 Nm (4) 4.8 Nm

15. The power delivered by the cyclist is equal to

(1) 280 W (2) 100 W

(3) 64π W (4) 32 W

16. The speed of the bicycle is

(1) 6.4π m s⁻¹ (2) 3.5π m s⁻¹

(3) 2.8π m s⁻¹ (4) 5.6π m s⁻¹

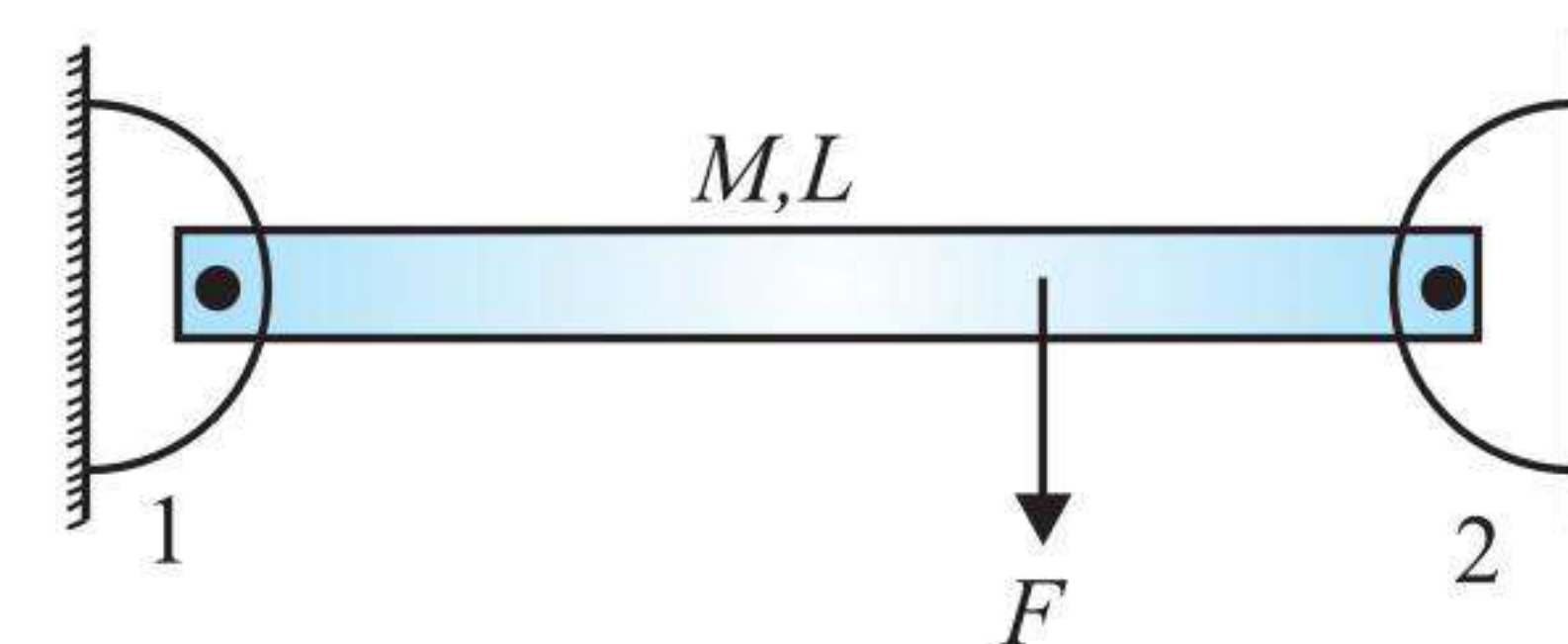
17. The net force of the friction on the rear wheel due to the road is

(1) 100 N (2) 62 N

(3) 32.6 N (4) 18.3 N

For Problems 18–20

A uniform rod of mass $M = 2$ kg and length L is suspended by two smooth hinges 1 and 2 as shown in figure. A force $F = 4$ N is applied downward at a distance $L/4$ from hinge 2. Due to the application of force F , hinge 2 breaks. At this instant, applied force F is also removed. The rod starts to rotate downward about hinge 1. ($g = 10$ m/s²)



18. The reaction at hinge 1, before hinge 2 breaks, is

(1) 24 N (2) 12 N

(3) 11 N (4) 10 N

19. The reaction at hinge 1, just after breaking of hinge 2, is

(1) 20 N (2) 10 N

(3) 5 N (4) 0

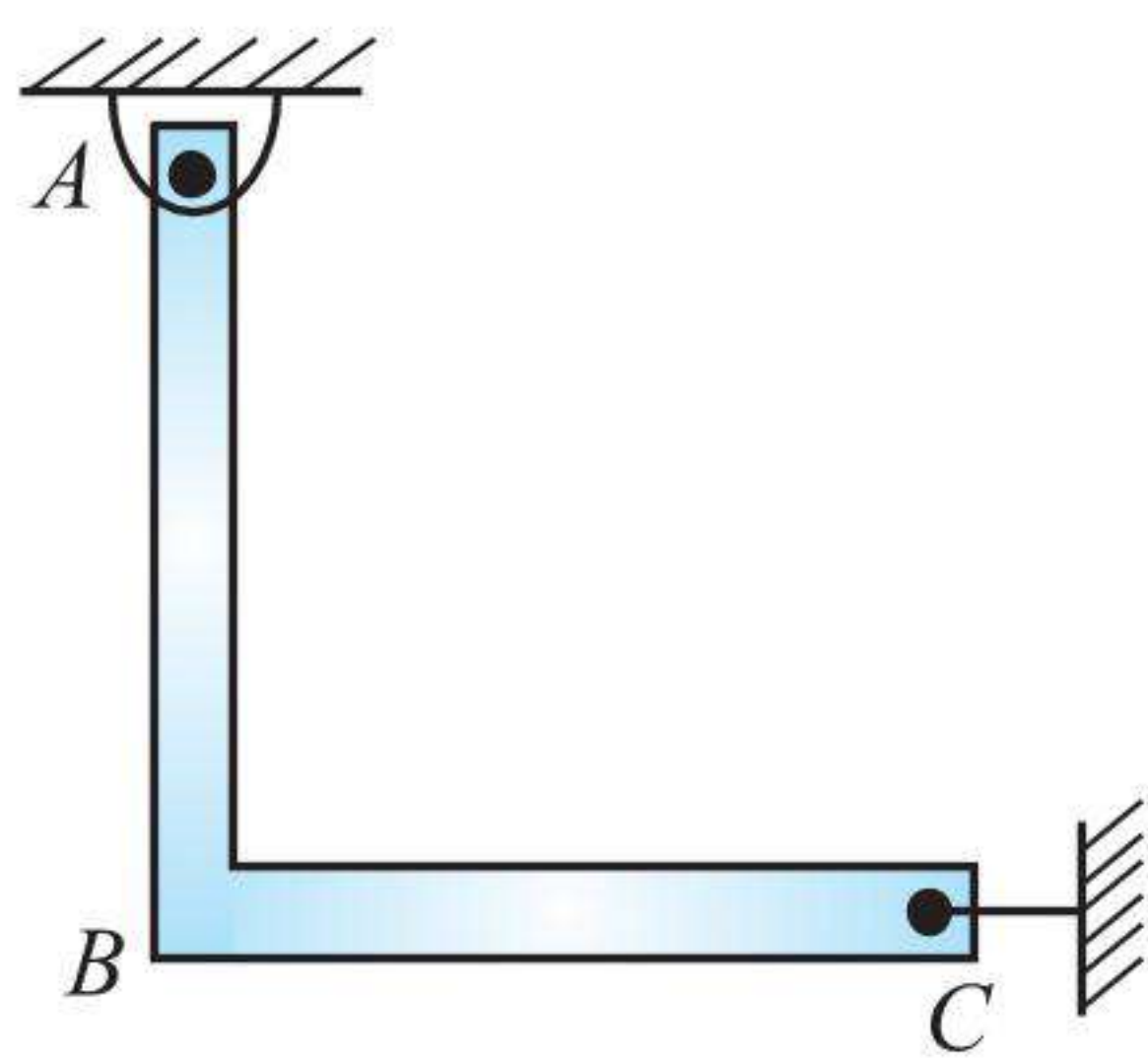
20. The acceleration of the end point of the rod of small mass dm at the end point of the rod, when the rod becomes vertical is

(1) 30 m/s² (2) 20 m/s²

(3) 10 m/s² (4) 0

For Problems 21–23

An L shaped uniform rod of mass $2M$ and length $2L$ ($AB = BC = L$) is held as shown in figure with a string fixed between C and wall so that AB is vertical and BC is horizontal. There is no friction between the hinge and the rod at A.



21. Find the tension in the string.

- (1) $\frac{Mg}{3}$ (2) $\frac{Mg}{4}$
 (3) Mg (4) $\frac{Mg}{2}$

22. What will be the reaction between hinge and rod at point A?

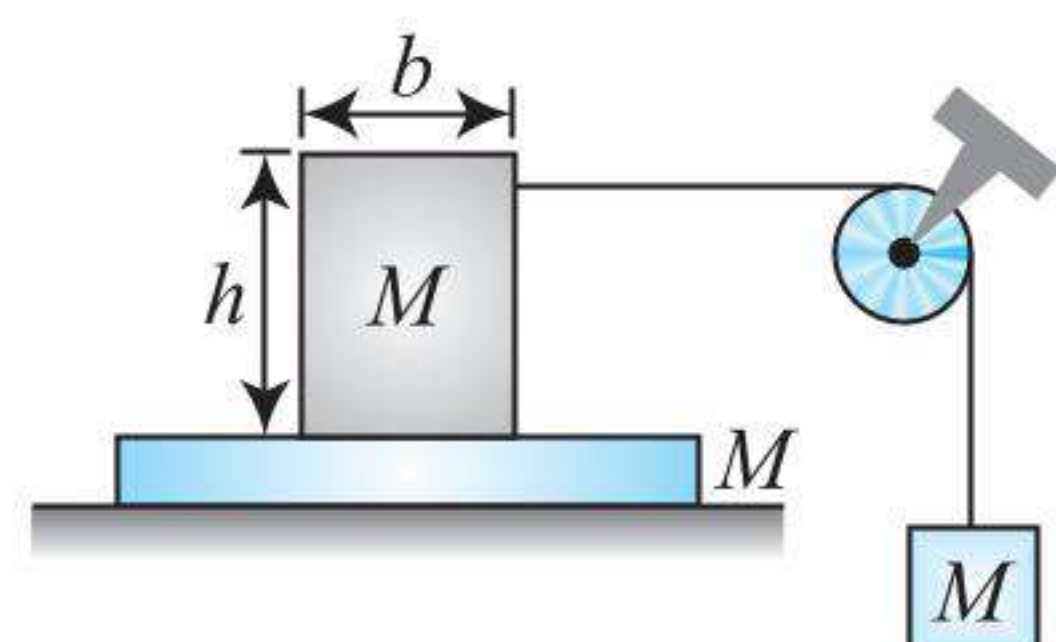
- (1) $\frac{\sqrt{65}Mg}{4}$ (2) $2Mg$
 (3) $\frac{\sqrt{17}Mg}{4}$ (4) $\frac{\sqrt{17}Mg}{2}$

23. If the string is burnt, find the angle between AB and the vertical at equilibrium position.

- (1) $\tan^{-1}\left(\frac{1}{3}\right)$ (2) $\tan^{-1}\left(\frac{1}{4}\right)$
 (3) $\tan^{-1}(3)$ (4) $\tan^{-1}\left(\frac{1}{2}\right)$

For Problems 24–25

A block of mass $M = 4$ kg of height h' and breadth b is placed on a rough plank of same mass M . A light inextensible string is connected to the upper end of the block and passed through a light smooth pulley as shown in figure. A mass $m = 1$ kg is hung to the other end of the string.



24. What should be the minimum value of coefficient of friction between the block and the plank so that, there is no slipping between the block and the wedge?

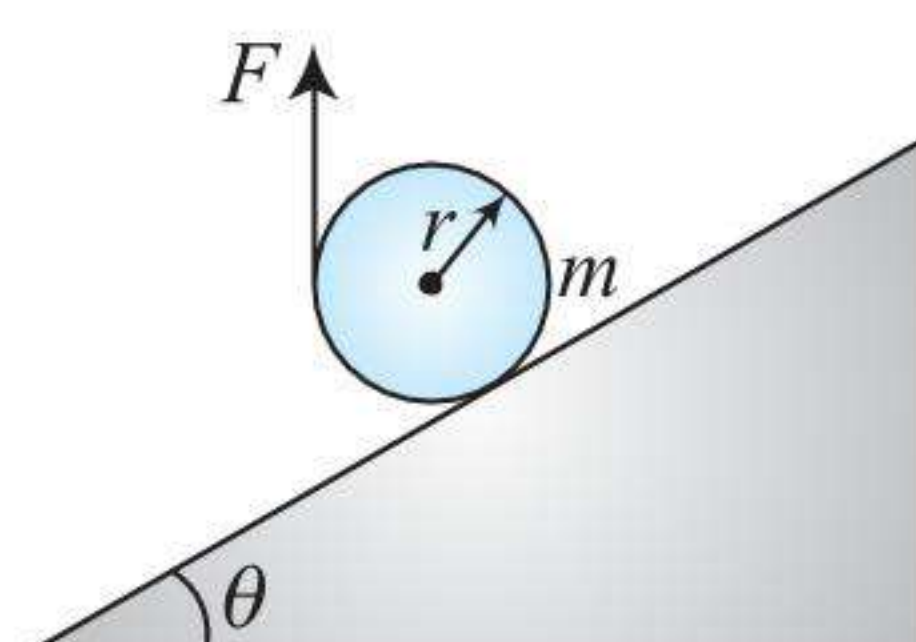
- (1) $\frac{3}{5}$ (2) $\frac{5}{9}$
 (3) $\frac{4}{5}$ (4) $\frac{1}{9}$

25. Find the minimum value of b/h so that the block does not topple over the plank, friction is absent between the plank and the ground.

- (1) $\frac{2}{5}$ (2) $\frac{4}{9}$
 (3) $\frac{4}{5}$ (4) $\frac{5}{9}$

For Problems 26–28

A solid cylinder of mass m is kept in balance on a fixed incline of angle $\alpha = 37^\circ$ with the help of a thread fastened to its jacket. The cylinder does not slip.



26. What force F is required to keep the cylinder in balance when the thread is held vertically?

- (1) $\frac{mg}{2}$ (2) $\frac{3mg}{4}$
 (3) $\frac{3mg}{8}$ (4) None of these

27. The value of minimum coefficient of static friction between cylinder and incline in the case mentioned in previous case:

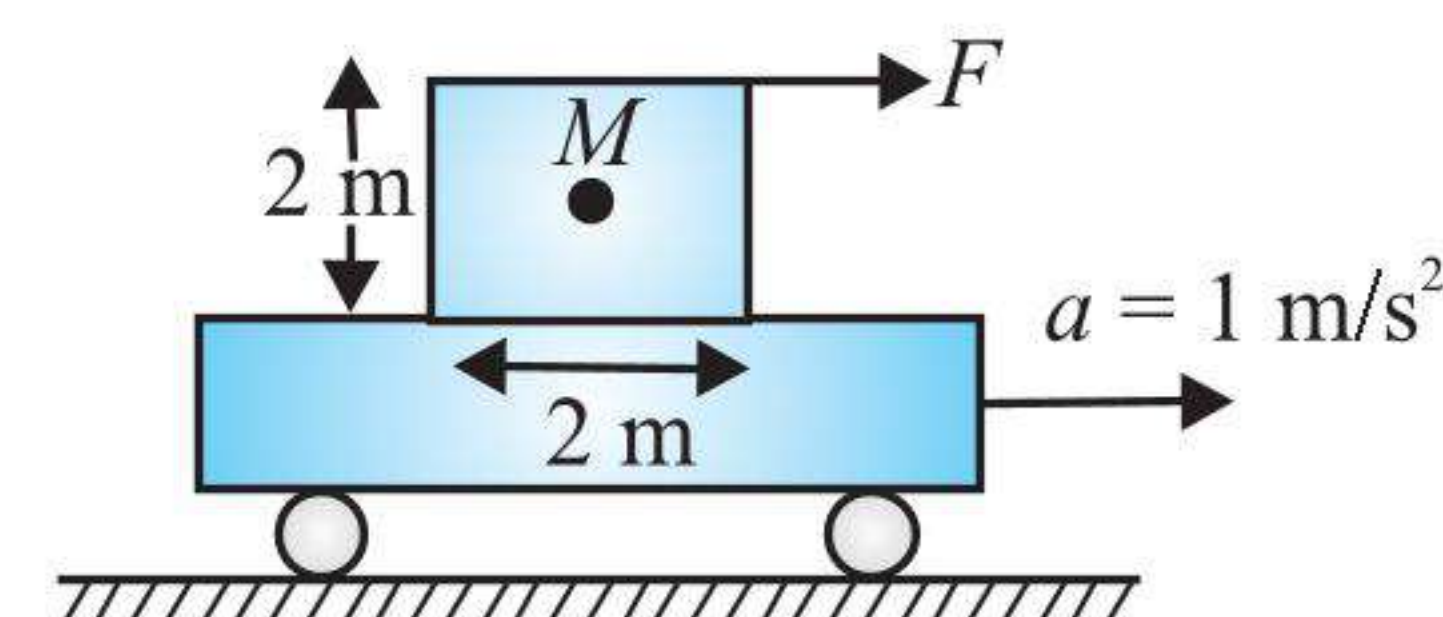
- (1) $\frac{3}{5}$ (2) $\frac{2}{3}$
 (3) $\frac{3}{4}$ (4) $\frac{4}{5}$

28. In what direction should the thread be pulled to minimise the force required to hold the cylinder? What is the magnitude of this force?

- (1) $\frac{3mg}{10}$ parallel to incline (2) Vertical $\frac{3mg}{8}$
 (3) Horizontal $\frac{mg}{5}$ (4) None of these

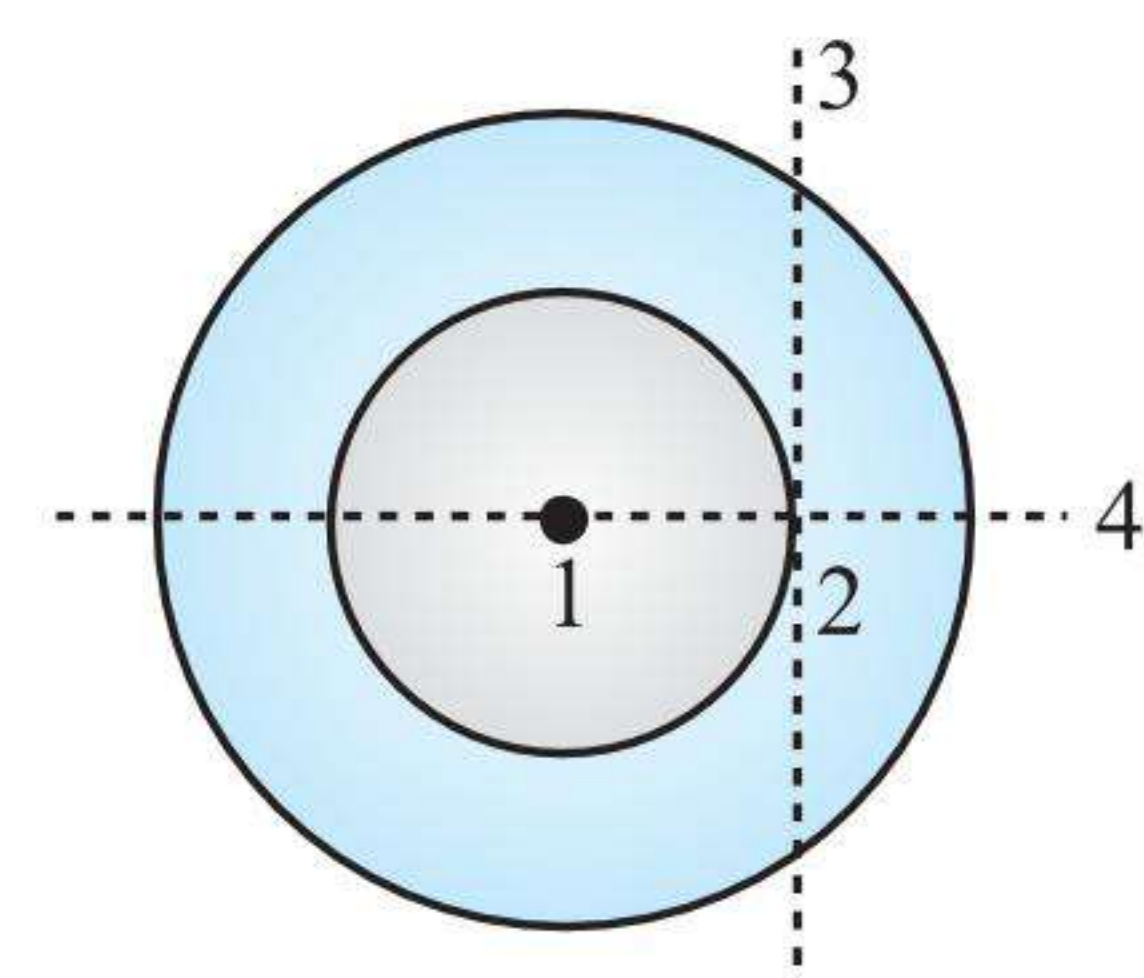
Matrix Match Type

1. A cubical block having square cross section of side 2 m and of mass $M = 10$ kg is resting over a platform moving at constant acceleration $a = 1$ m/s². Coefficient of friction between the block and the platform is $\mu = 0.1$. A force F acts at the top of the cube as shown in figure. Now match Column I with Column II.



Column I	Column II
i. $F = 0$	a. Block neither topples nor slips over the platform
ii. $F = 45$ N	b. Block topples but does not slip over the platform
iii. $F = 15$ N	c. Block slips but does not topple over the platform
iv. $F = 25$ N	d. Block slips as well as topples on the platform

2. From a uniform disc of mass M and radius R , a concentric disc of radius $R/2$ is cut out. For the remaining annular disc: I_1 is the moment of inertia about axis '1', I_2 about '2', I_3 about '3' and I_4 about '4'. Axes '1' and '2' are perpendicular to the disc and '3' and '4' are in the plane of the disc. Axes '2', '3' and '4' intersect at a common point. Match the following

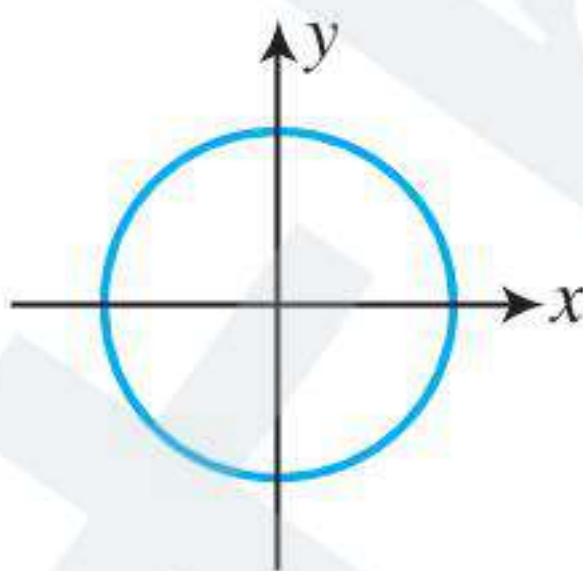


Column I	Column II
i. I_1 is equal to	a. $\frac{21}{32} MR^2$
ii. I_2 is equal to	b. $I_1/2$
iii. $I_3 + I_4$ is equal to	c. $\frac{15}{32} MR^2$
iv. $I_2 - I_3$ is equal to	d. none of these

3. In Column I, information about the force(s) acting on a body is mentioned, while in Column II information about the motion of a body is given. Match the entries in Column II with the entries in Column I.

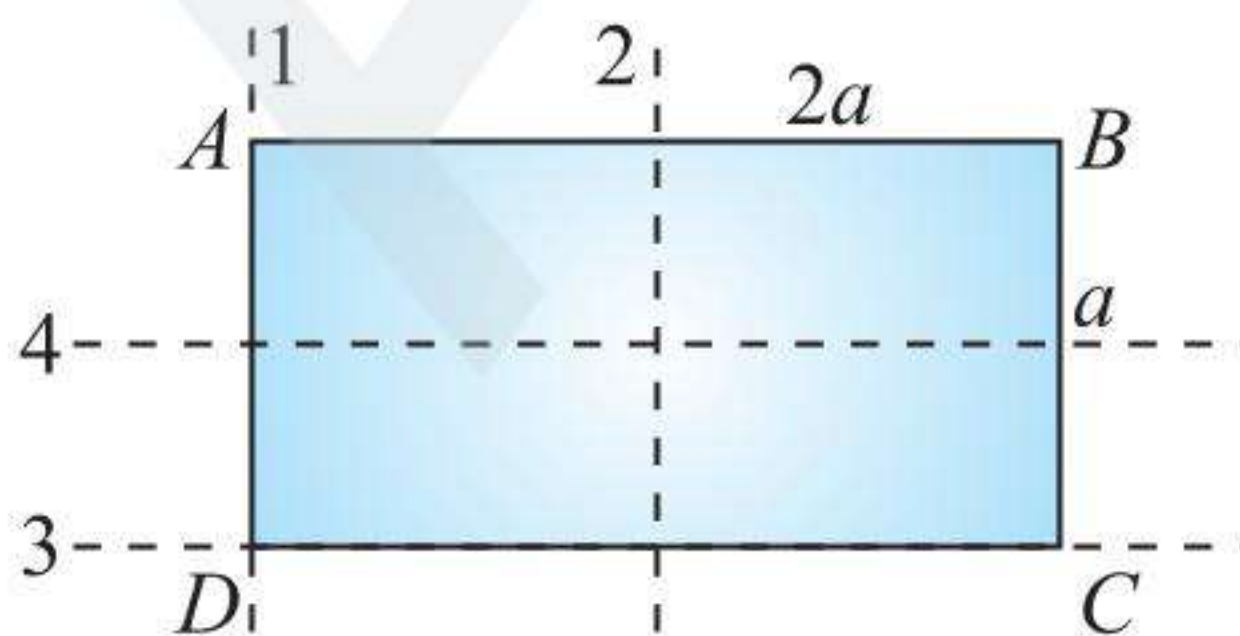
Column I	Column II
i. A single force through centre of mass	a. Rotational motion
ii. Equal and opposite forces separated by non-zero distance	b. Translational motion
iii. Equal and opposite forces acting at the same point	c. No motion
iv. A single force not through centre of mass	d. Centre of mass performs curvilinear motion

4. A body of Mass M and radius R is as shown in figure. The body can be a ring, a disc, a hollow sphere or a solid sphere. I_x , I_y and I_z are the moments of inertia about x , y and z axis respectively. Origin coincides with centre of mass of the body.



Column I	Column II
i. $I_x = I_y = I_z$	a. possible for ring or disc
ii. $I_x = I_y + I_z$	b. possible for hollow sphere or solid sphere
iii. $I_y = I_z + I_x$	c. possible for all of the bodies mentioned
iv. $I_z = I_x + I_y$	d. not possible for any of the body mentioned

5. A rectangular slab ABCD have dimensions $a \times 2a$ as shown in figure. Match the following two columns.

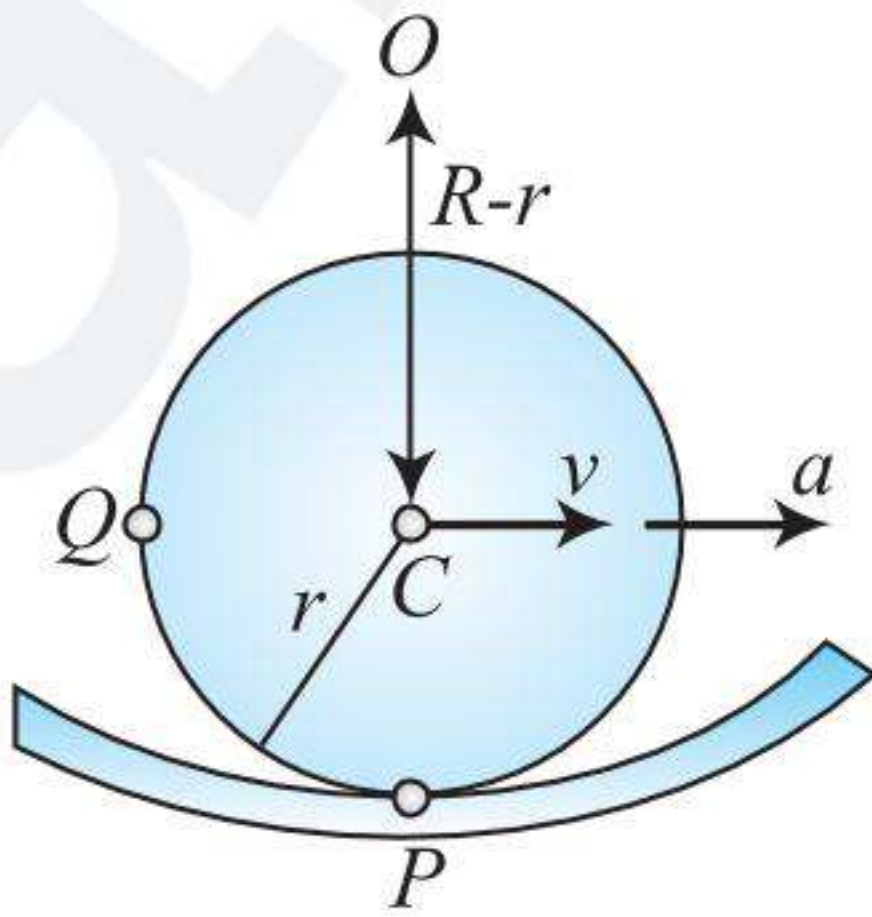


Column I	Column II
i. Radius of gyration about axis-1	a. $\frac{a}{\sqrt{12}}$

ii. Radius of gyration about axis-2	b. $\frac{2a}{\sqrt{3}}$
iii. Radius of gyration about axis-3	c. $\frac{a}{\sqrt{3}}$
iv. Radius of gyration about axis-4	d. None

- (1) i – b, ii – a, iii – c, iv – c
(2) i – b, ii – c, iii – c, iv – a
(3) i – d, ii – b, iii – a, iv – c
(4) i – c, ii – a, iii – d, iv – b

6. A body of radius r rolls without sliding on a curved surface of radius of curvature R as shown in the figure. Match the Columns:

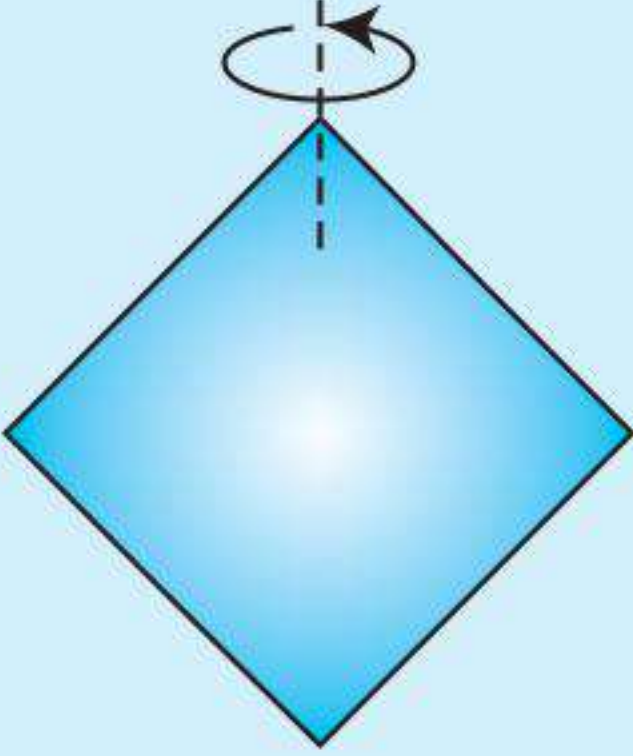
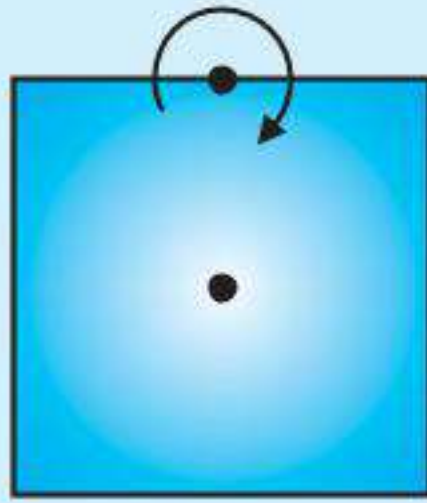


Column I	Column II
i. ω_{CO}	a. $\frac{v}{r}$
ii. ω_{CP}	b. $\frac{v}{R-r}$
iii. α_{PC}	c. $\frac{a}{R-r}$
iv. α_{CO}	d. $\frac{a}{r}$

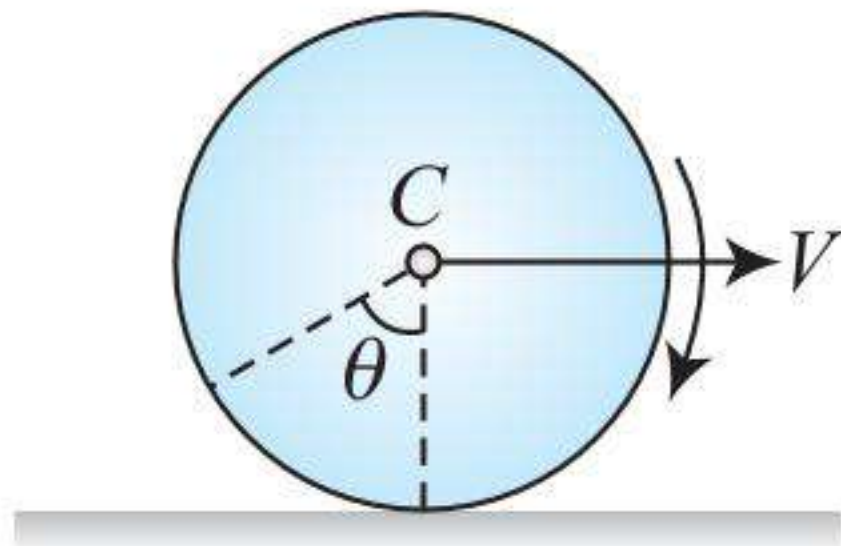
- (1) i – b, ii – a, iii – d, iv – c
(2) i – a, ii – b, iii – c, iv – d
(3) i – a, ii – b, iii – d, iv – c
(4) i – b, ii – a, iii – c, iv – d

7. Square frame is made of four uniform rods. Mass of each rod is m and length l .

Column I (Axis of rotation)	Column II (Moment of inertia about axis of rotation)
i.	a. $\frac{2}{3} ml^2$
ii.	b. $\frac{5}{3} ml^2$

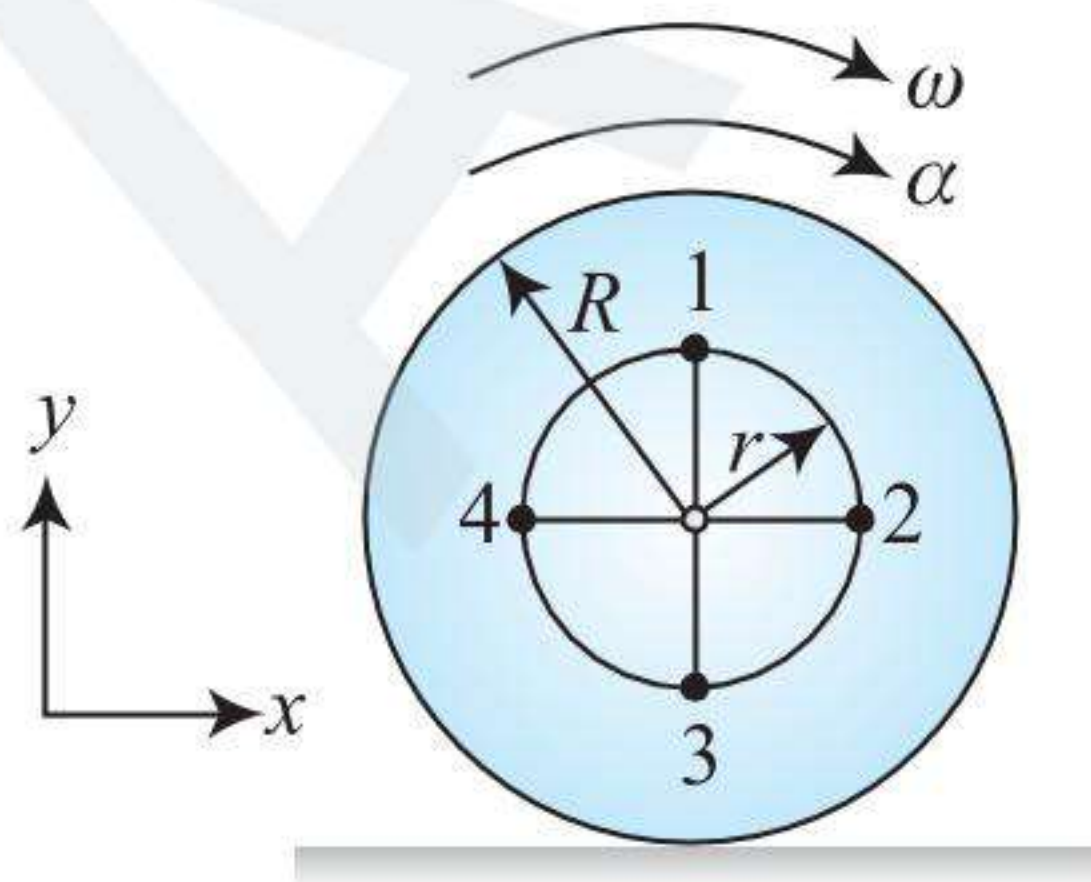
iii.		c. $\frac{7}{3}ml^2$
iv.		d. $\frac{8}{3}ml^2$

- (1) i – b, ii – a, iii – c, iv – c
(2) i – b, ii – a, iii – d, iv – c
(3) i – d, ii – b, iii – a, iv – c
(4) i – c, ii – a, iii – d, iv – b
8. A disc rolls on ground without slipping. Velocity of centre of mass is v . There is a point P on circumference of disc at angle θ . Suppose v_p is the speed of this point. Then, match the following:



Column I	Column II
i. If $\theta = 60^\circ$	a. $v_p = \sqrt{2}v$
ii. If $\theta = 90^\circ$	b. $v_p = v$
iii. If $\theta = 120^\circ$	c. $v_p = 2v$
iv. If $\theta = 18^\circ$	d. $v_p = \sqrt{3}v$

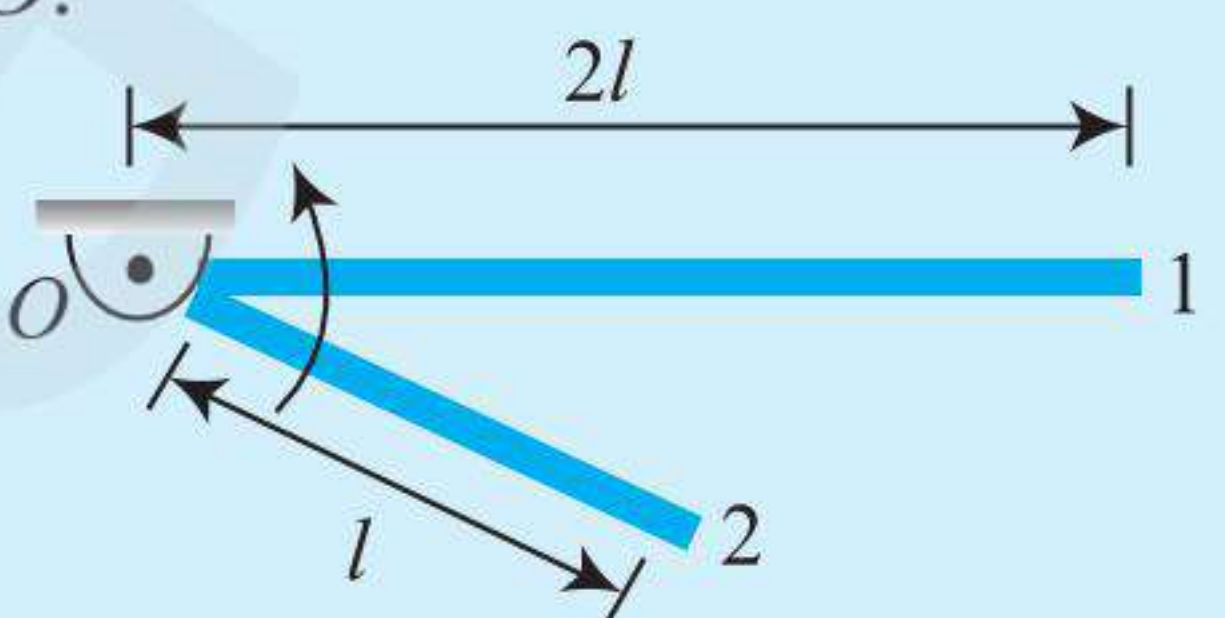
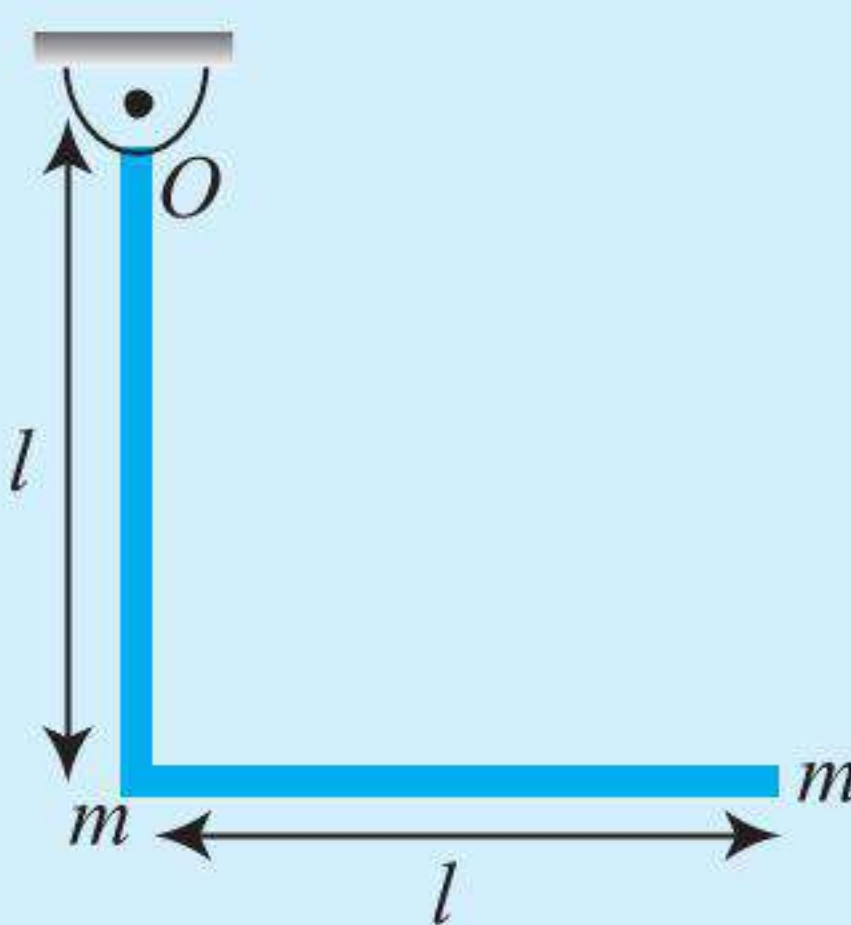
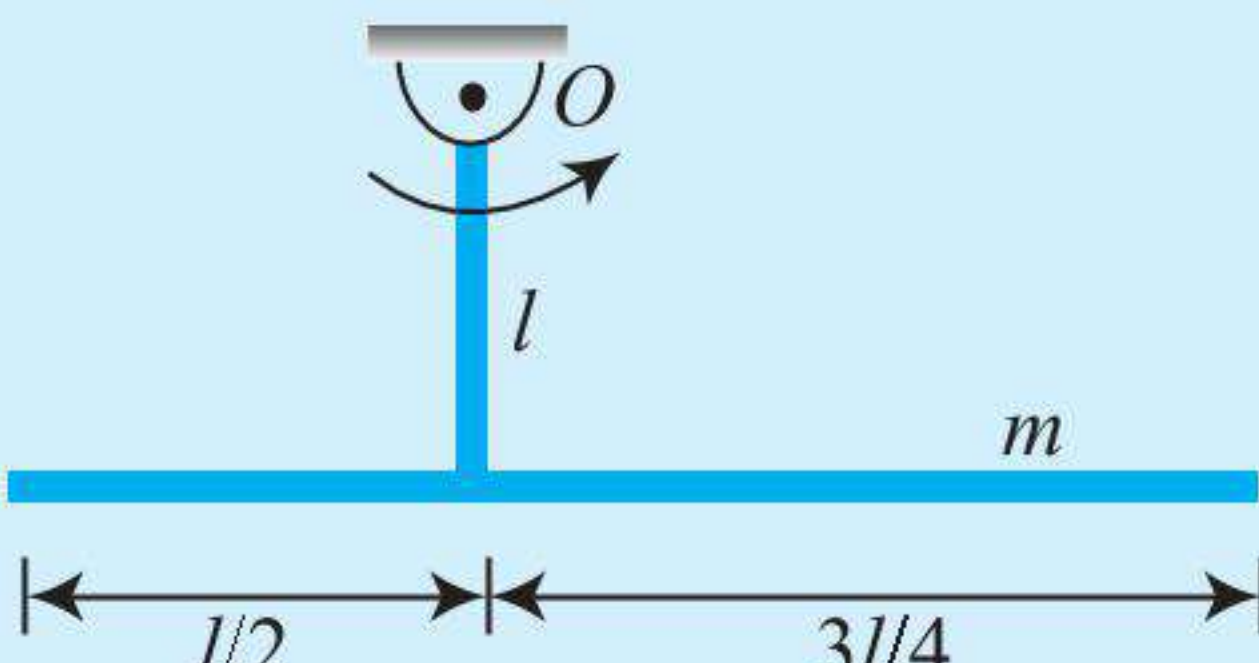
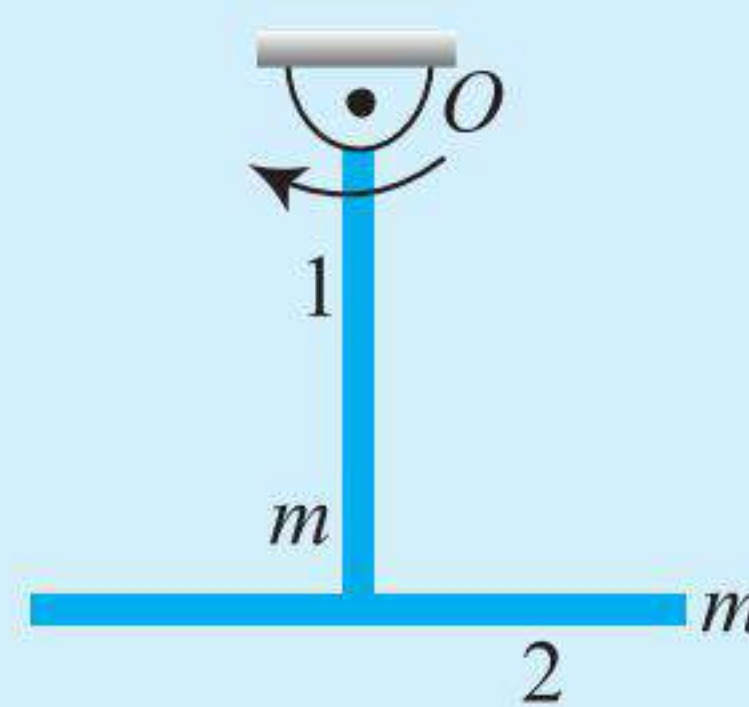
- (1) i – b, ii – a, iii – c, iv – d
(2) i – b, ii – a, iii – d, iv – c
(3) i – b, ii – d, iii – a, iv – c
(4) i – a, ii – b, iii – d, iv – c
9. A disc of radius R is rolling without slipping with an angular acceleration α , on a horizontal plane. Four points are marked at the end of horizontal and vertical diameter of a circle of radius r ($< R$) on the disc. If horizontal and vertical direction are chosen as x and y axis as shown in the figure, then acceleration of points 1, 2, 3 and 4 are $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ and $\vec{a}_4$ respectively, at the moment when angular velocity of the disc is ω . Match the following



Column I	Column II
i. $\vec{a}_1$	a. $(R\alpha - r\alpha)\hat{i} + (r\omega^2)\hat{j}$
ii. $\vec{a}_2$	b. $(R\alpha + r\alpha)\hat{i} - (r\omega^2)\hat{j}$

iii. $\vec{a}_3$	c. $(R\alpha - r\omega^2)\hat{i} - (r\alpha)\hat{j}$
iv. $\vec{a}_4$	d. $(R\alpha + r\omega^2)\hat{i} + (r\alpha)\hat{j}$
	e. None of these

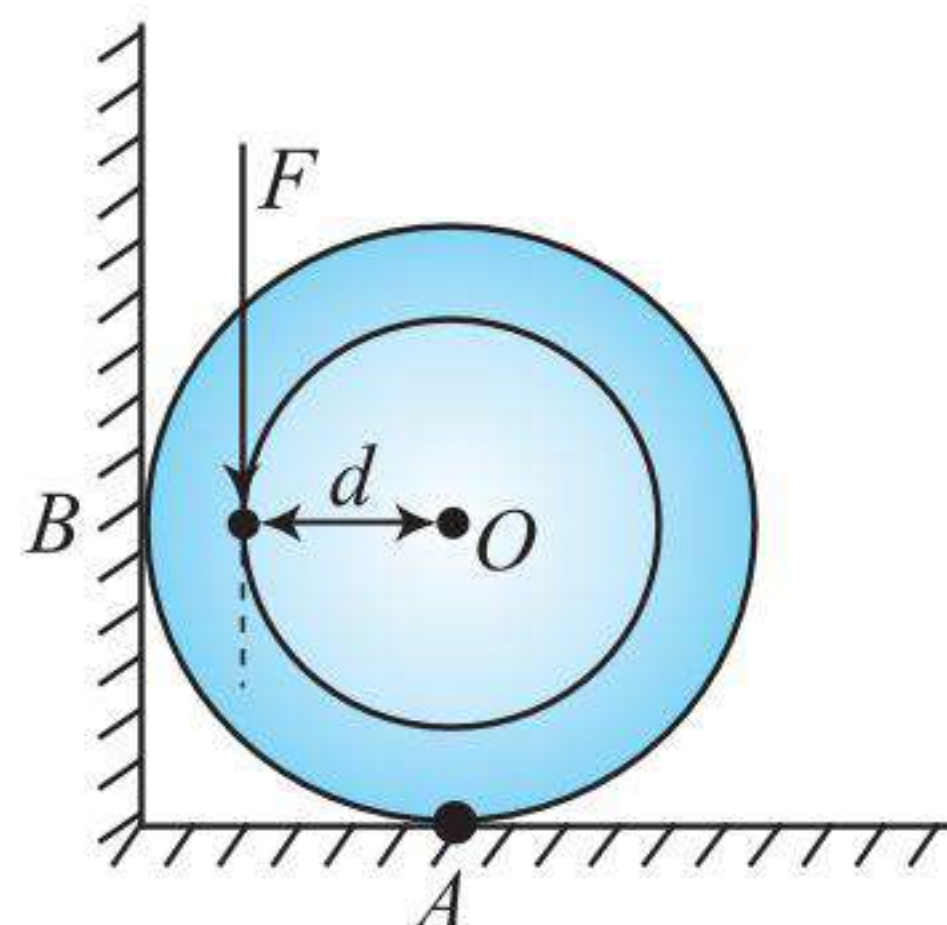
- (1) i – b, ii – a, iii – c, iv – c
(2) i – b, ii – c, iii – a, iv – d
(3) i – d, ii – b, iii – a, iv – c
(4) i – c, ii – a, iii – d, iv – b
10. For the situations given in column-I match the corresponding values of moment of inertia about an axis passing through O .

Column I (Axis of rotation)	Column II (Moment of inertia about axis of rotation)
i. A rod of mass m length $3l$ is bent at O . 	a. $\frac{7}{3}ml^2$
ii. L-shaped composite rod comprising two identical rods each of mass m and length l . 	b. $\frac{17}{12}ml^2$
iii. A rod of mass m length l is rotating about an axis passing through O . 	c. ml^2
iv. Two identical rods each of mass m and length l are welded as shown. 	d. $\frac{55}{48}ml^2$

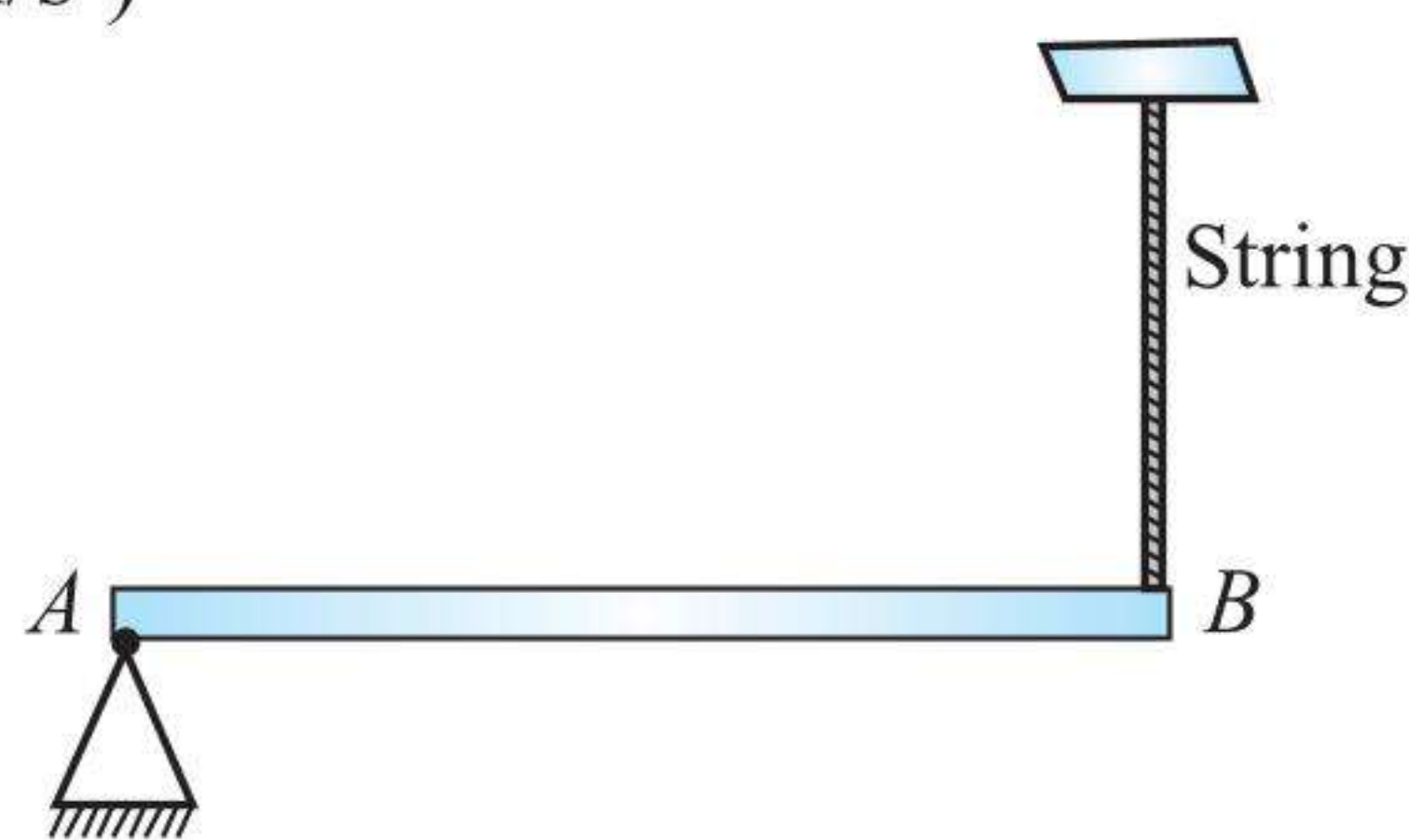
- (1) i – b, ii – a, iii – c, iv – c
(2) i – b, ii – a, iii – d, iv – c
(3) i – d, ii – b, iii – a, iv – c
(4) i – c, ii – a, iii – d, iv – b

Numerical Value Type

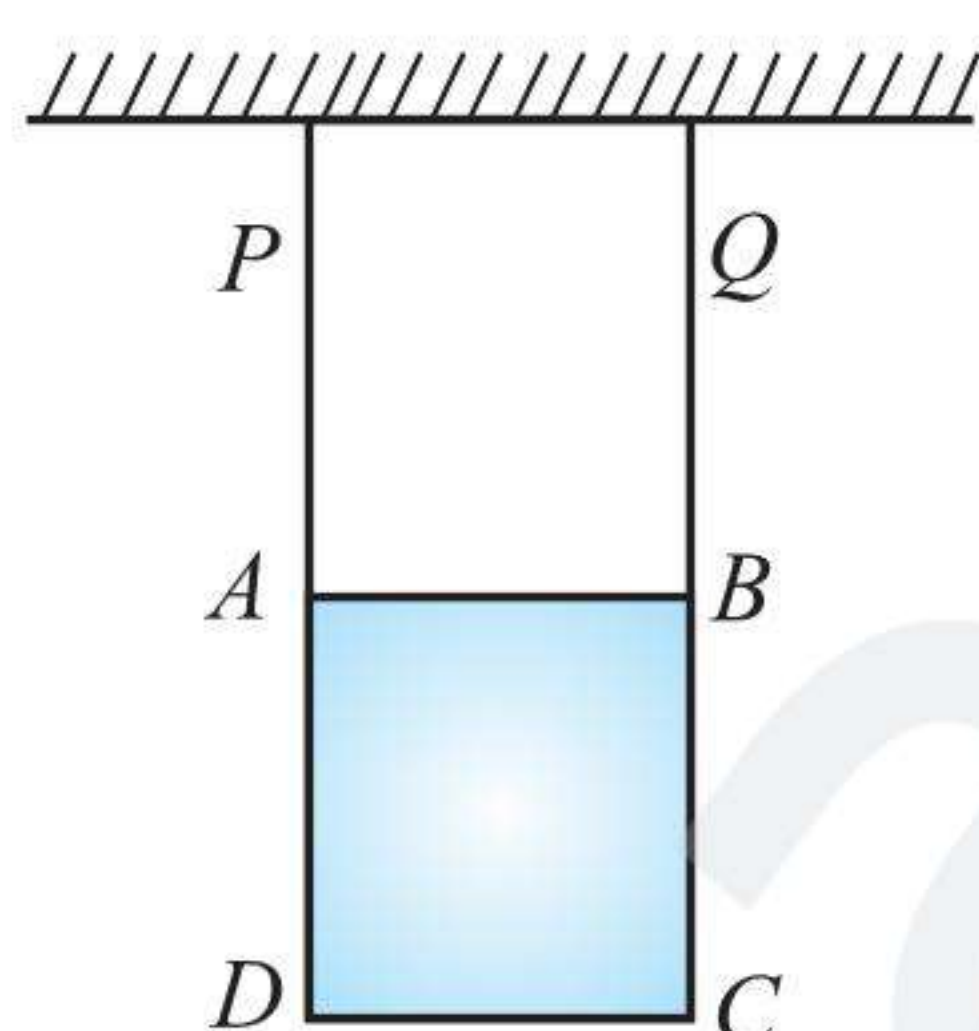
1. A solid cylinder with $r = 0.1$ m and mass $M = 2$ kg is placed such that it is in contact with the vertical and a horizontal surface as shown in figure. The coefficient of static friction is $\mu = (1/3)$ for both the surfaces. Find the distance d (in cm) from the centre of the cylinder at which a force $F = 40$ N should be applied vertically so that the cylinder just starts rotating in anticlockwise direction.



2. A uniform rod AB of mass 2 kg is hinged at one end A . The rod is kept in the horizontal position by a massless string tied to point B . Find the reaction of the hinge (in N) on end A of the rod at the instant when string is cut. ($g = 10$ m/s²)



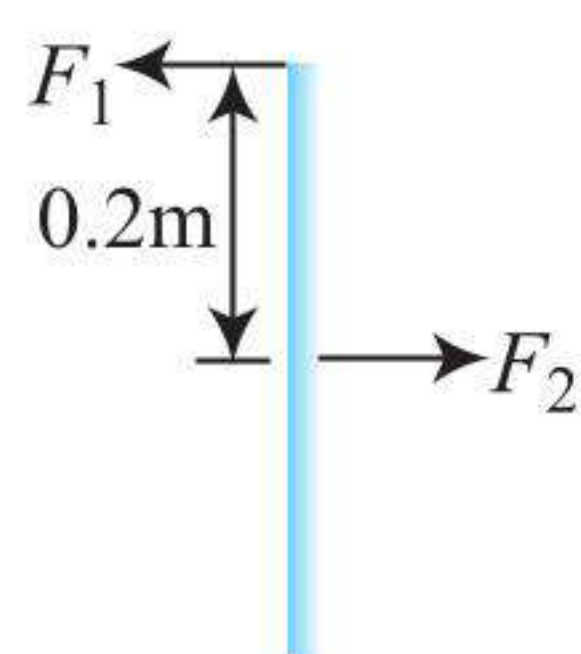
3. A square plate $ABCD$ of mass m and side l is suspended with the help of two ideal strings P and Q as shown. Determine the acceleration (in m/s²) of corner A of the square just at the moment the string Q is cut. ($g = 10$ m/s²)



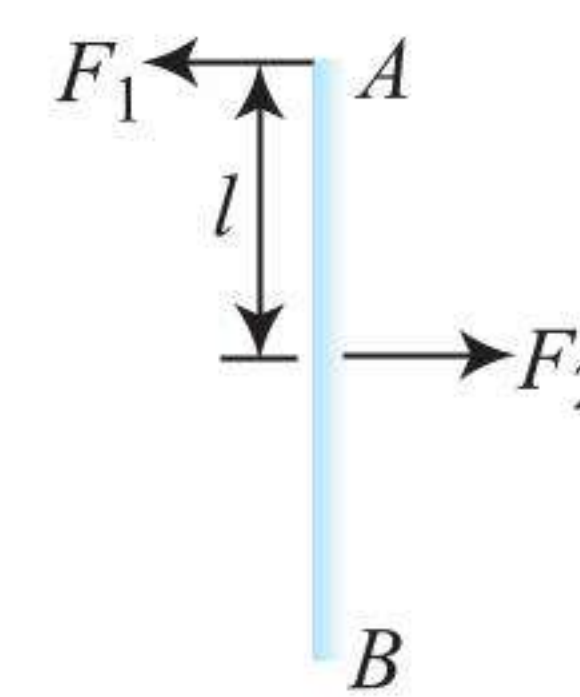
4. A uniform cylinder rests on a cart as shown. The coefficient of static friction between the cylinder and the cart is 0.5 . If the cylinder is 4 cm in diameter and 10 cm in height, then what is the minimum acceleration (in ms⁻²) of the cart needed to cause the cylinder to tip over?



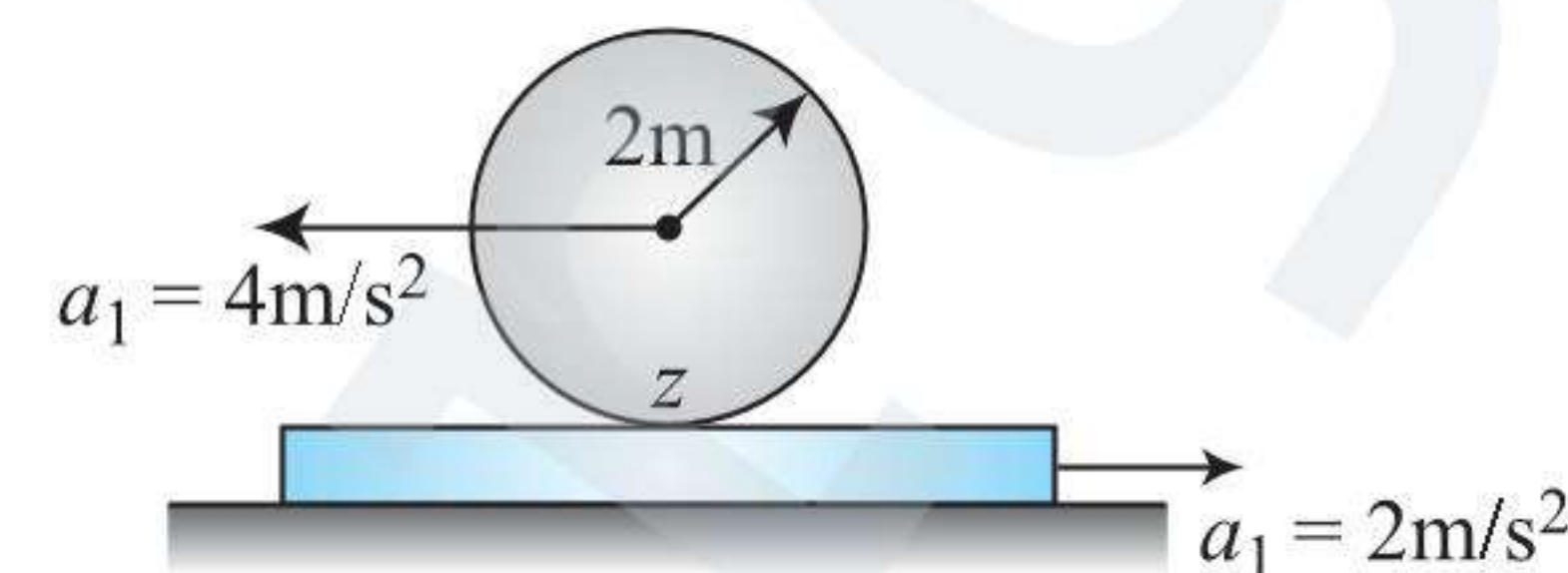
5. A thin uniform rod AB of mass 1 kg moves translationally with acceleration $a = 2$ m/s² due to two antiparallel forces F_1 and F_2 . The distance between the points at which these forces are applied is equal to $l = 0.2$ m. If force $F_1 = 8$ N, then find the length of the rod in meter.



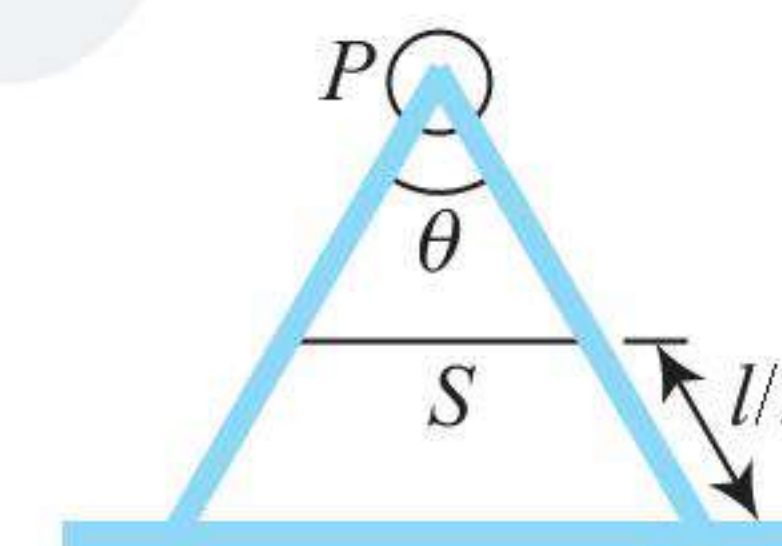
6. A thin uniform rod AB of mass $m = 1$ kg moves translationally with acceleration $a = 2$ m/s² due to two antiparallel forces F_1 and F_2 . The distance between the points at which these forces are applied is equal to $l = 20$ cm. Besides, it is known that $F_2 = 5$ N. Find the length of the rod in m.



7. In figure, a sphere of radius 2 m rolls on a plank. The accelerations of the sphere and the plank are indicated. Find the value of α in rad/s².

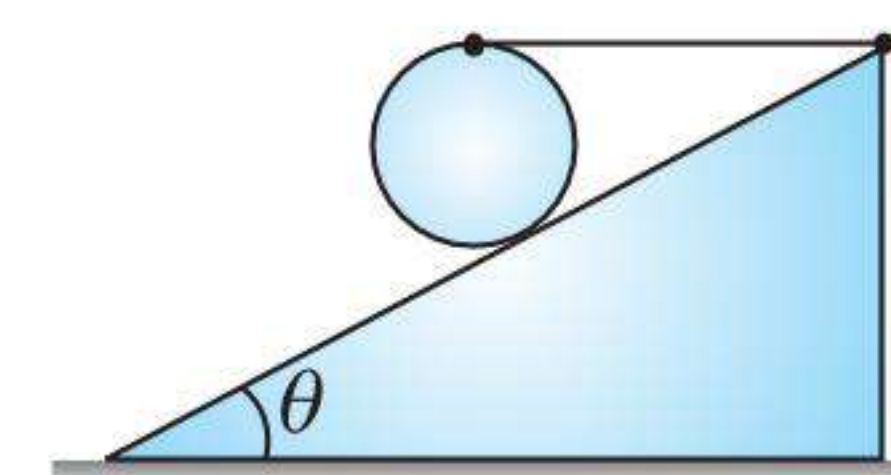


8. Two rods each of mass $m = 4$ kg and length $l = 1$ m are in equilibrium being smoothly hinged at P . Find the tension in the connecting string as shown in figure.

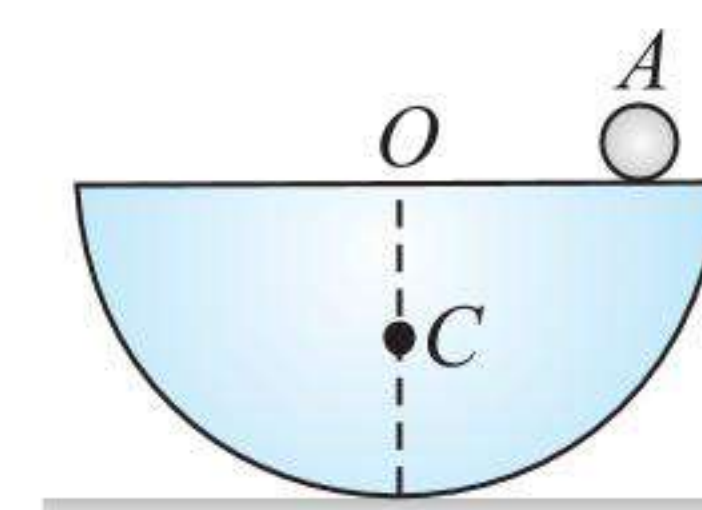


9. A cylinder is in rotational equilibrium on an inclined plane, inclined at angle $\theta = 53^\circ$. Find the minimum μ between the wooden log and the inclined plane.

(Given: $\sin 53^\circ = \frac{4}{5}$ and $\cos 53^\circ = \frac{3}{5}$)

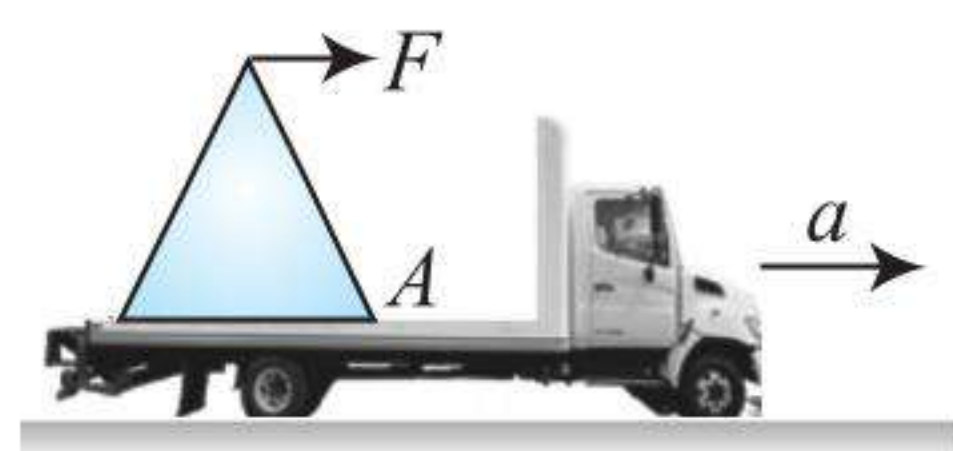


10. A uniform hemisphere with radius $R = 10$ m is placed on a horizontal smooth surface. Its centre of mass C is below the centre O . An object 'A' is placed on the flat surface of the hemisphere at point O . The mass of the object is $1/8^{\text{th}}$ of mass of the hemisphere. The coefficient of friction between the object and the surface of the hemisphere is $\mu = 0.2$. Now the object 'A' is displaced slowly along the flat surface of the hemisphere towards its outer rim. Find the maximum distance (in m) of the object from the centre O without slipping on the hemisphere.

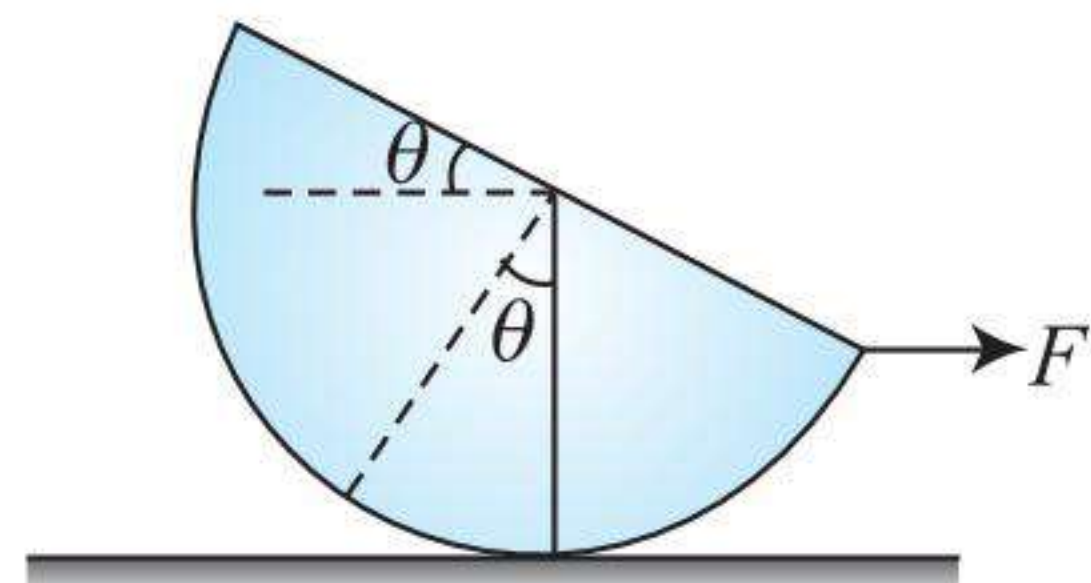


11. Figure show a uniform block of mass $m = \sqrt{\frac{3}{13}}$ kg having square base and equilateral triangular cross section placed at the edge of the floor of a truck which is under constant acceleration $a = \frac{g\sqrt{3}}{4}$ m/s² to the right. A horizontal force on the apex of the triangle F is just enough to just tilt the block. Find the force (in newton) on the block at point A

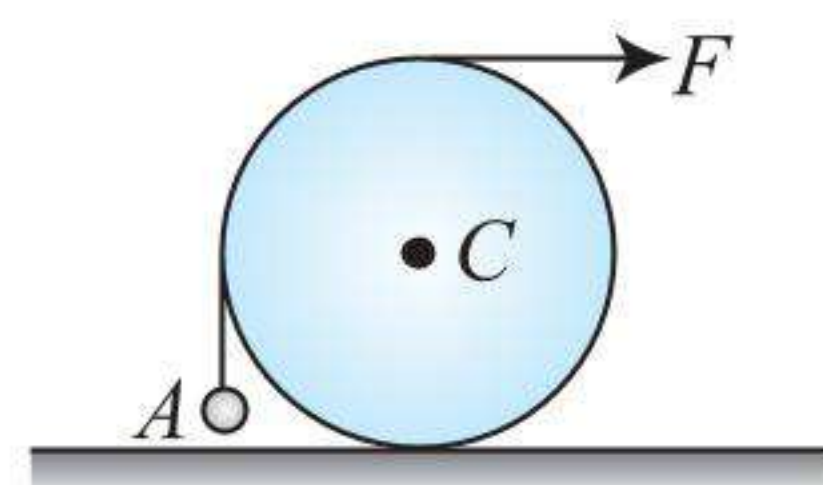
where the block is in contact with the step edge when F is applied. (Take $g = 10 \text{ m/s}^2$)



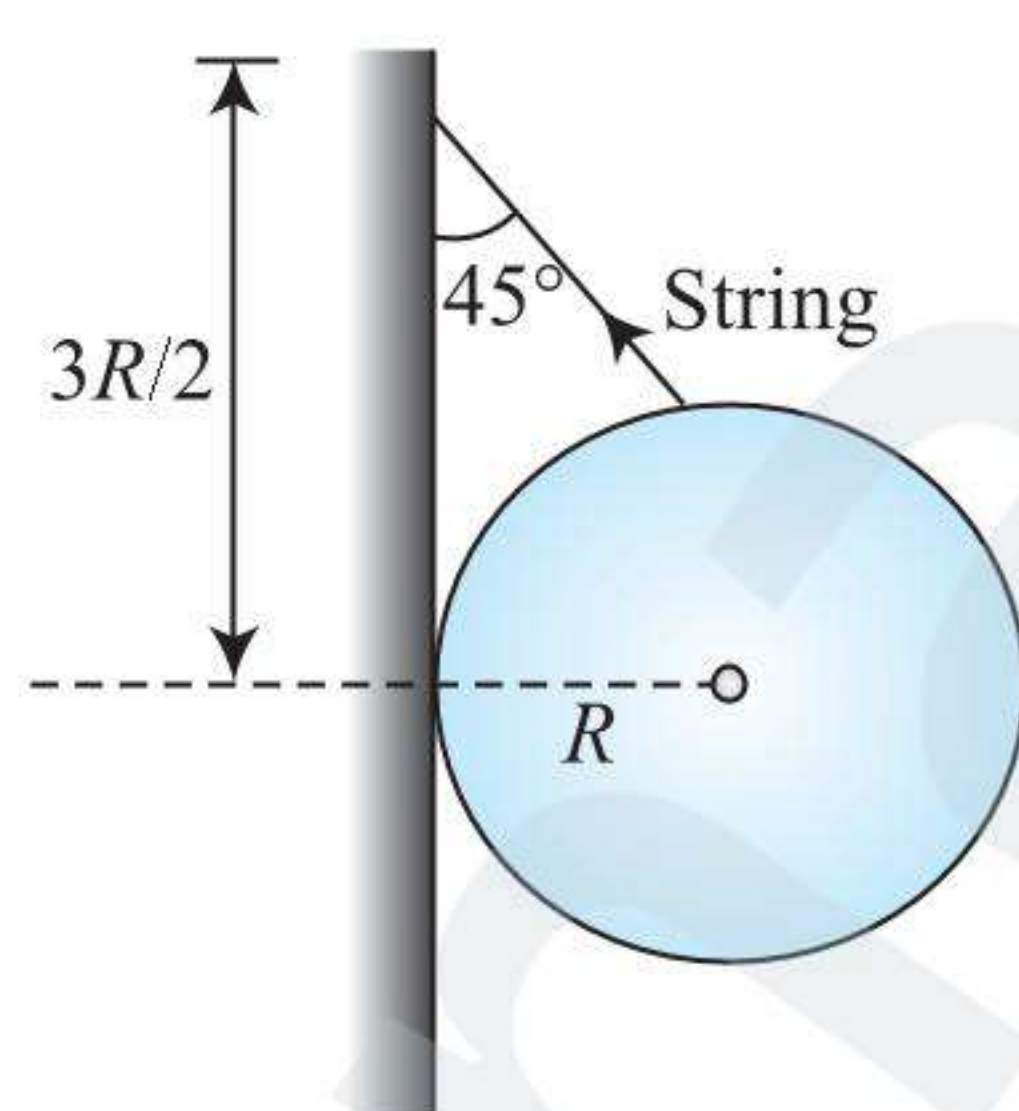
12. A semicircular disc of radius R and mass M is pulled by a horizontal force F so that it moves with uniform velocity. The coefficient of friction between disc and ground is $\mu = \frac{4}{3\pi}$. Find the value of θ in degree.



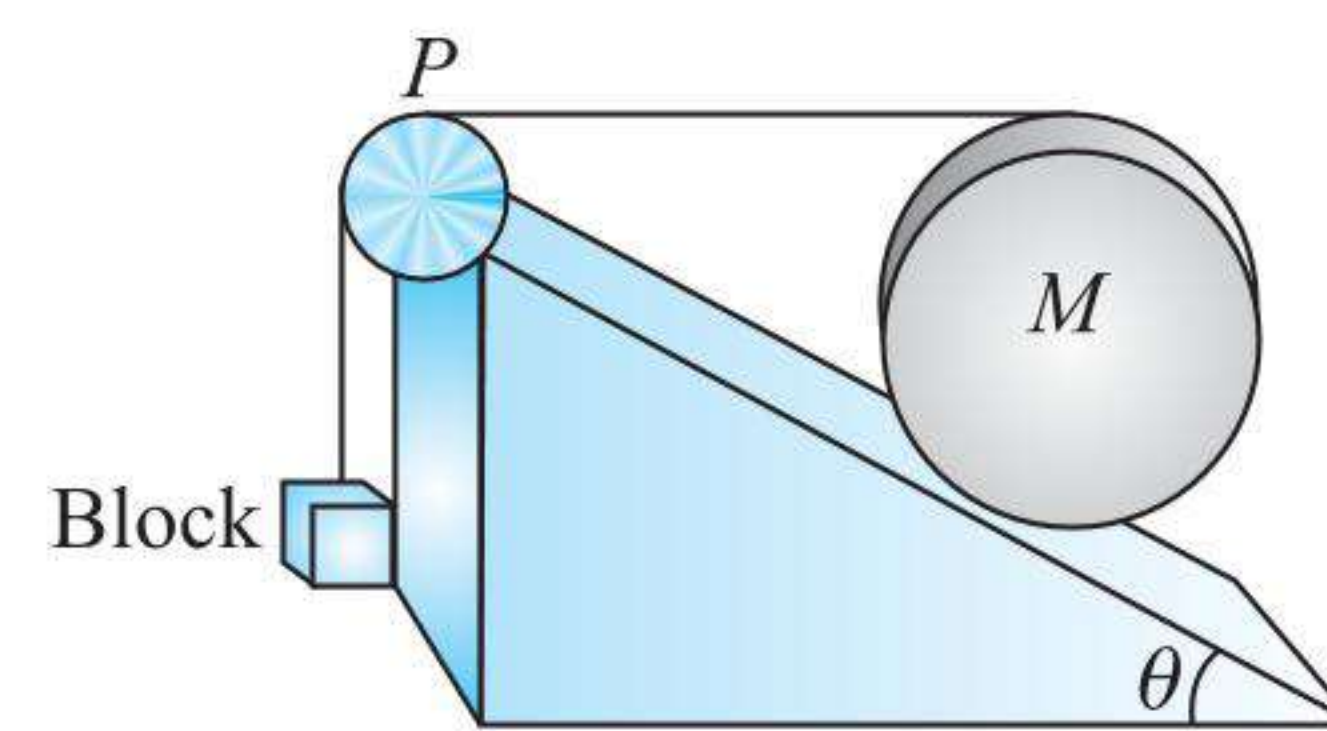
13. A cylinder C rests on a horizontal surface. A small particle of mass $m = 5 \text{ kg}$ is held in equilibrium connected to an overhanging string as shown. The other end of the massless string is being pulled horizontally by a force F as shown. Find F (in N).



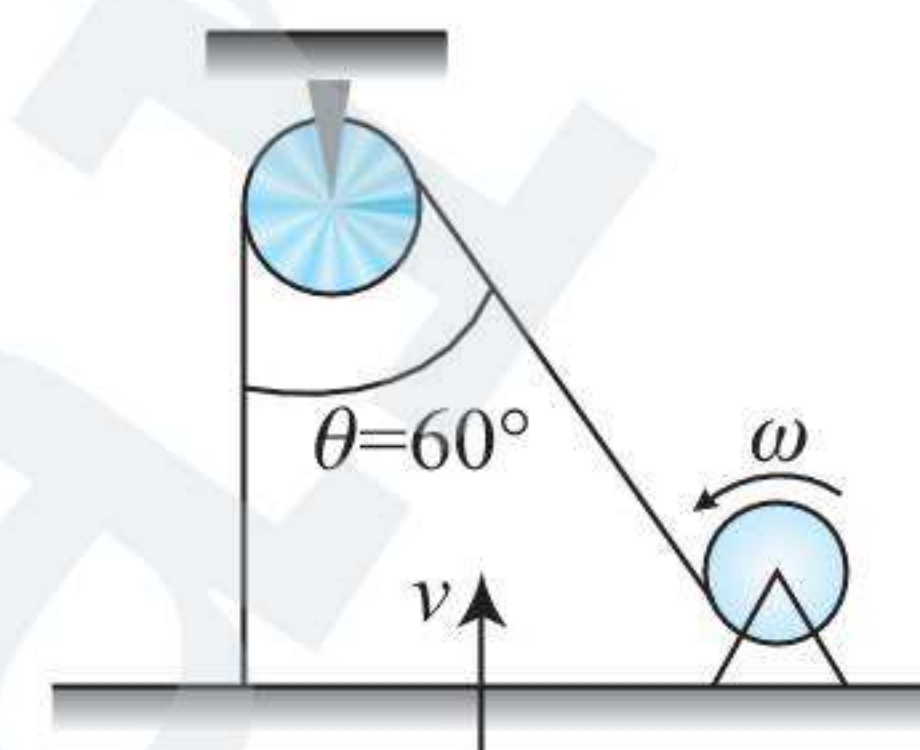
14. A sphere of radius R is supported by a rope attached to the wall. The rope makes an angle $\theta = 45^\circ$ with respect to the wall. The point where the rope is attached to the wall is at a distance of $3R/2$ from the point where the sphere touches the wall. Find the minimum coefficient of friction (μ) between the wall and the sphere for this equilibrium to be possible.



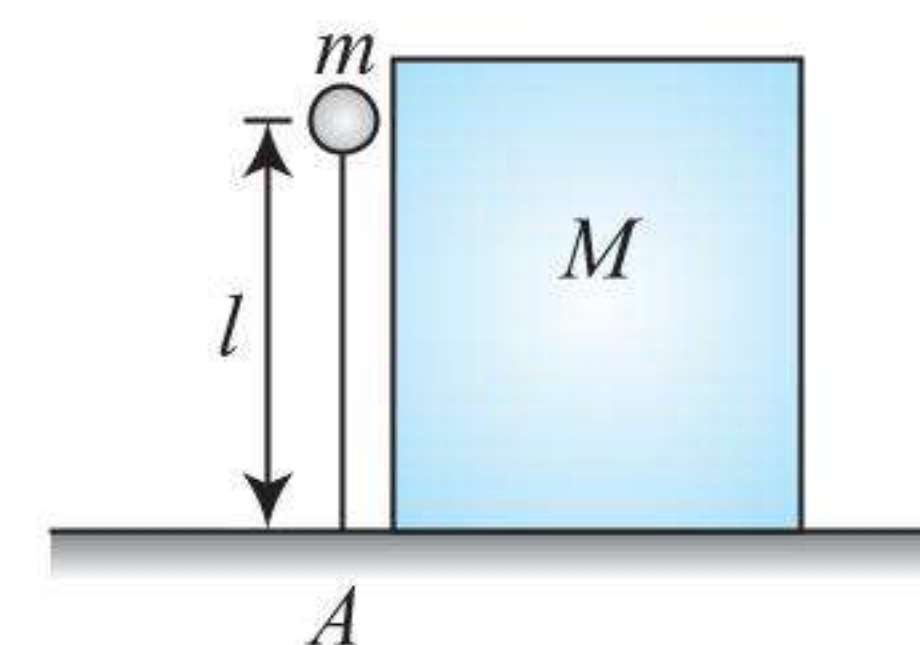
15. A wheel of radius $R = 1 \text{ m}$ is rolling without sliding uniformly on a horizontal surface. Find the radius of curvature of the path (in m), of a point on its circumference when it is at highest point in its path.
16. In the arrangement shown in figure the cylinder of mass $M = 9 \text{ kg}$ is at rest on an incline. The plane is inclined at angle 37° with horizontal. The string between the cylinder and the pulley (P) is horizontal. Find the minimum the mass of the block (in kg) to keep the system in equilibrium.



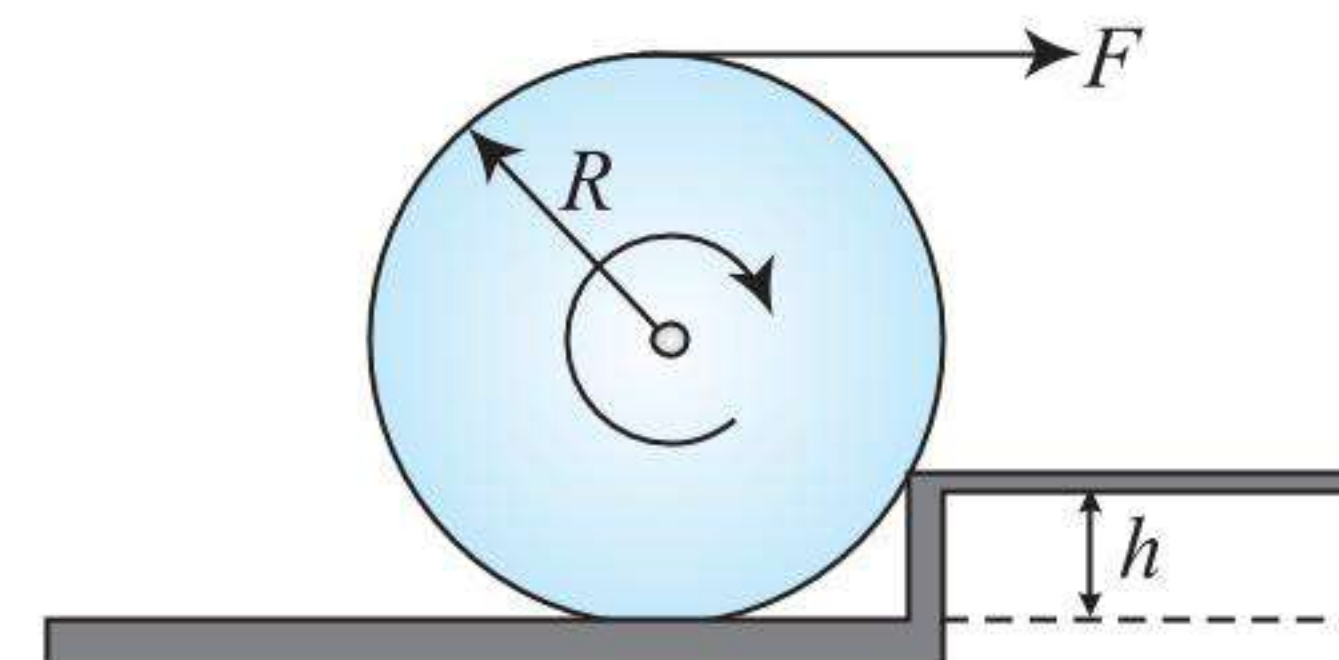
17. In the instant shown in the figure, the board is moving up vertically with constant velocity (v). The drum winds up at a constant rate $\omega = 15 \text{ rad/s}$. If the radius of the drum is $R = 10 \text{ cm}$ and the board always remains horizontal. Find its velocity of board at this moment (in m/s).



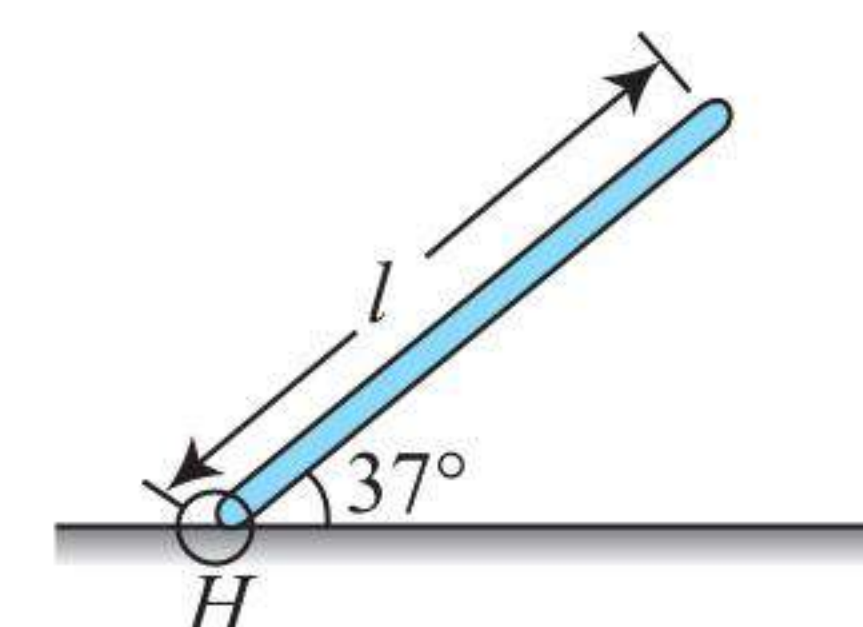
18. A weightless rod of length l with a small load of mass m at the end is hinged at point A as shown and occupies a strictly vertical position, touching a block of mass M . A light jerk sets the system in motion. For what mass ratio M/m will the rod form an angle $\theta = \pi/6$ with the horizontal at the moment of the separation from the block?



19. A cylinder of mass 15 kg and radius R is to be raised onto a horizontal step of height $h = \frac{R}{5}$ as shown in figure. A rope is wrapped around the cylinder and pulled horizontally with force F . Assuming the cylinder does not slip on the step, find the minimum force F (in N) necessary to raise the cylinder.



20. A uniform rod of mass $m = 2\sqrt{10} \text{ kg}$ and length l can rotate in vertical plane about a smooth horizontal axis hinged at point H . Find angular acceleration a of the rod just after it is released from initial position making an angle of 37° with horizontal from rest? Find force (in N) exerted by the hinge just after the rod is released from rest.

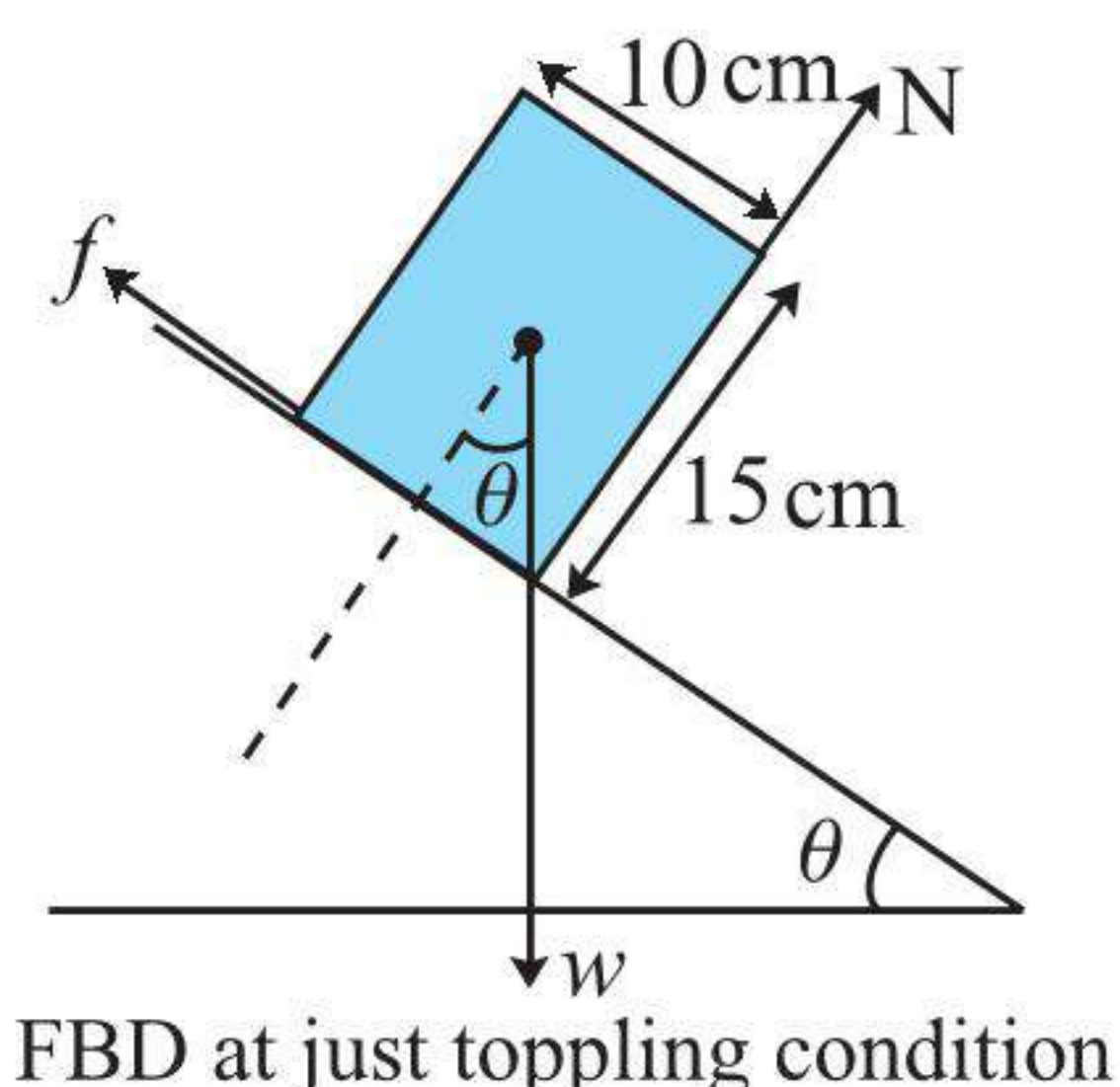


Archives

JEE ADVANCED

Single Correct Answer Type

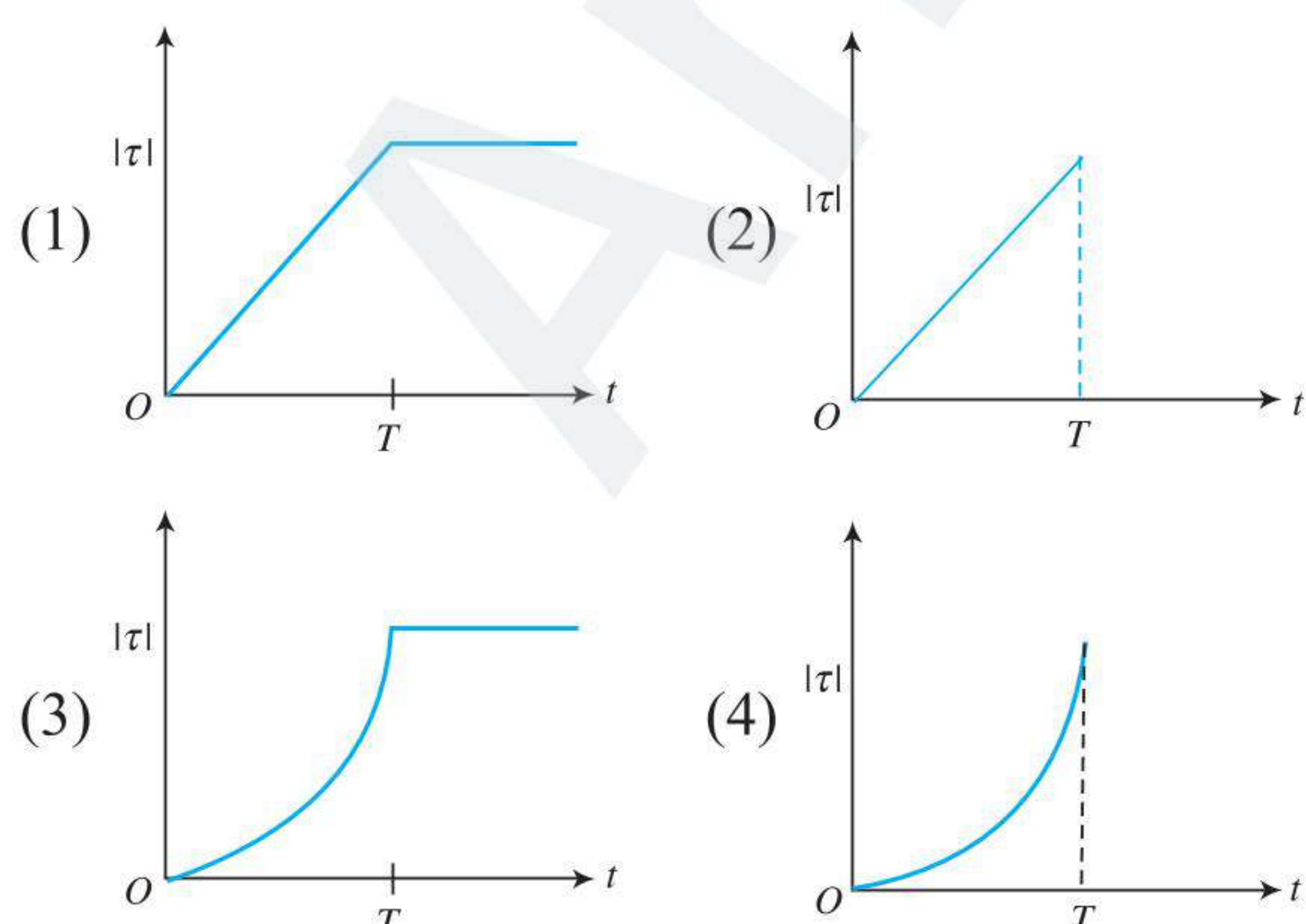
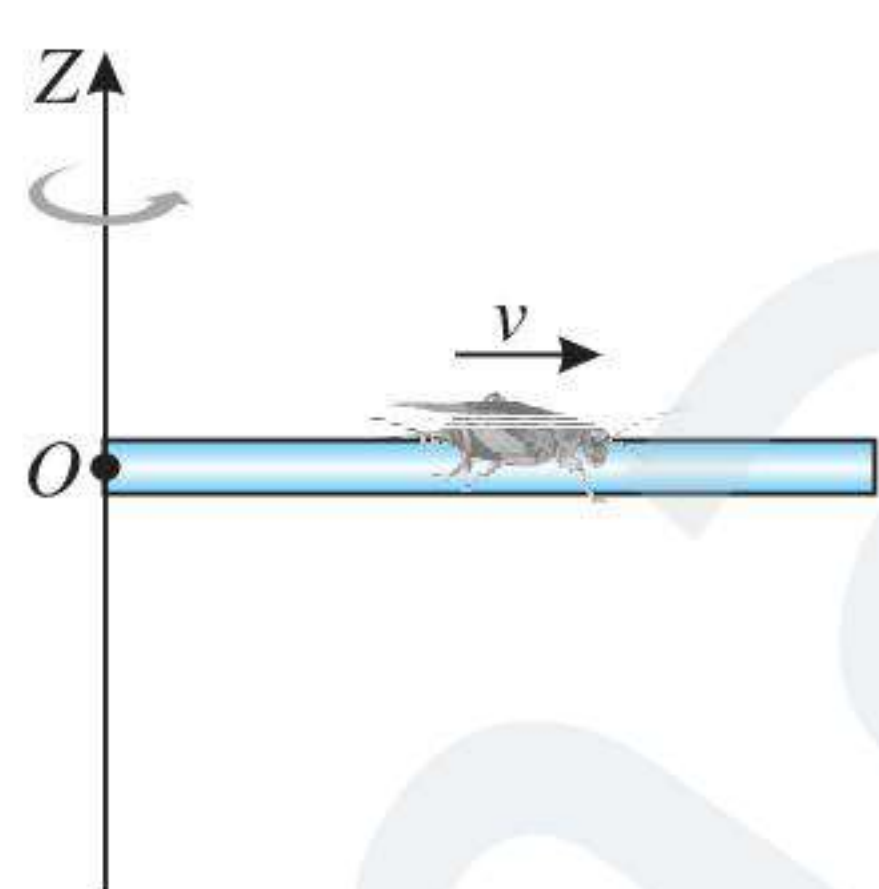
1. A block of base $10 \text{ cm} \times 10 \text{ cm}$ and height 15 cm is kept on an inclined plane. The coefficient of friction between them is 3. The inclination θ of this inclined plane from the horizontal plane is gradually increased from 0° . Then



- (1) at $\theta = 30^\circ$, the block will start sliding down the plane
- (2) the block will remain at rest on the plane up to certain θ and then it will topple
- (3) at $\theta = 60^\circ$, the block will start sliding down the plane and continue to do so at higher angles
- (4) at $\theta = 60^\circ$, the block will start sliding down the plane and on further increasing θ , it will topple at certain θ

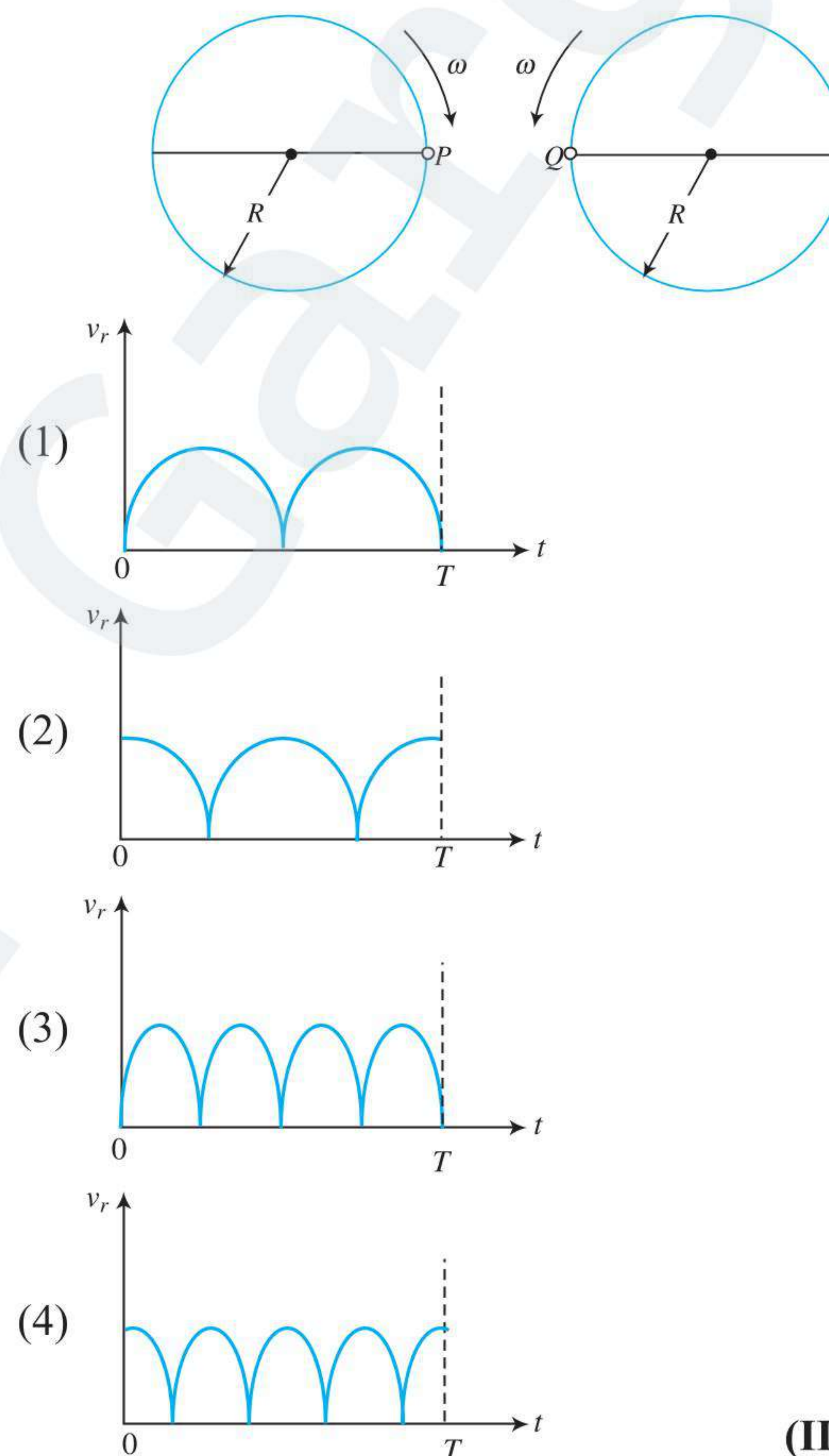
(IIT-JEE 2009)

2. A thin uniform rod, pivoted at O , is rotating in the horizontal plane with constant angular speed ω , as shown in the figure. At time $t = 0$, a small insect starts from O and moves with constant speed v with respect to the rod towards the other end. It reaches the end of the rod at $t = T$ and stops. The angular speed of the system remains ω throughout. The magnitude of the torque ($\vec{\tau}$) on the system about O , as a function of time is best represented by which plot?



(IIT-JEE 2012)

3. Two identical discs of same radius R are rotating about their axes in opposite directions with the same constant angular speed ω . The discs are in the same horizontal plane. At time $t = 0$, the points P and Q are facing each other as shown in the figure. The relative speed between the two points P and Q is v_r . In one time period (T) of rotation of the discs, v_r as a function of time is best represented by



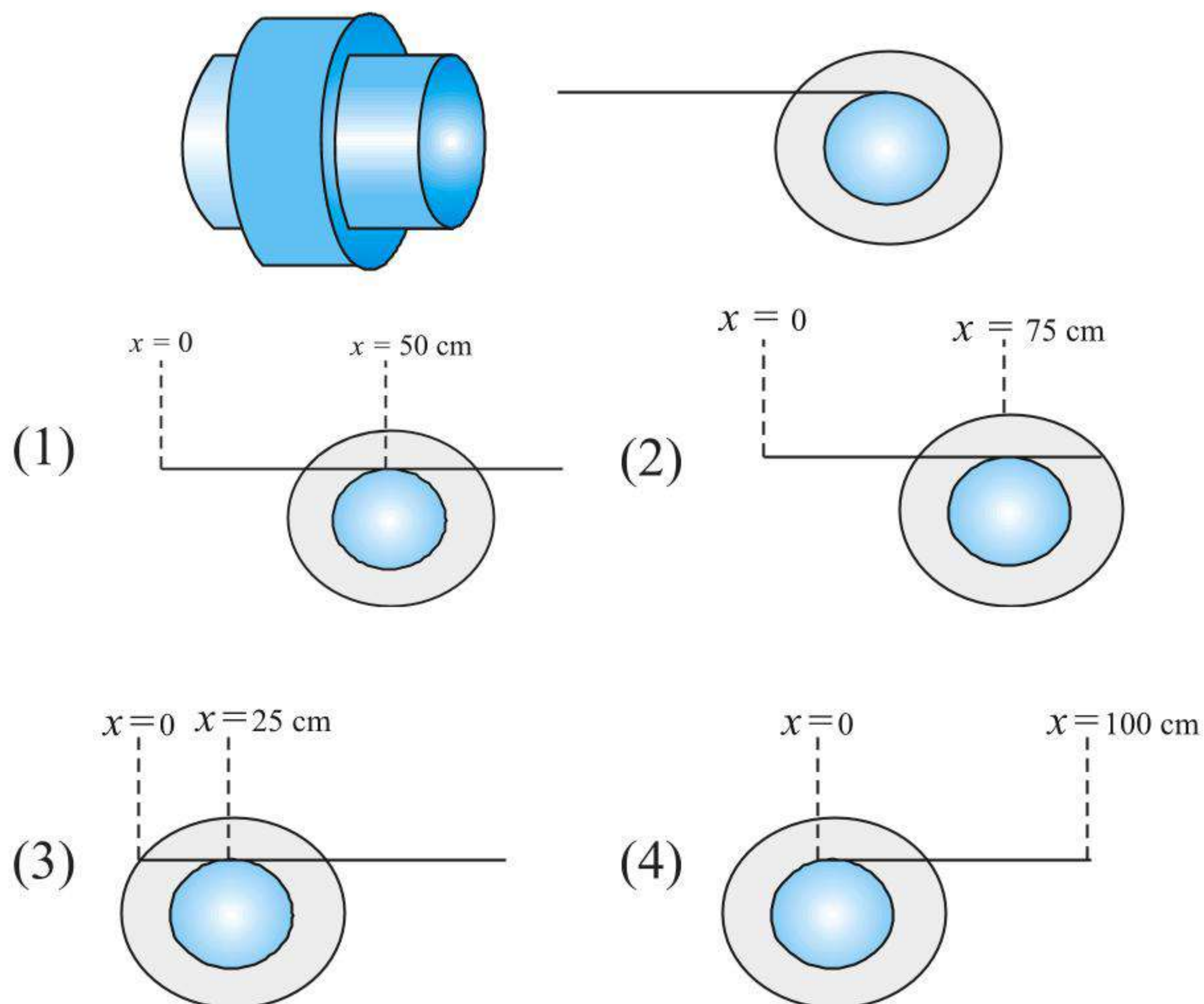
(IIT-JEE 2012)

4. A uniform wooden stick of mass 1.6 kg and length l rests in an inclined manner on a smooth, vertical wall of height h ($h < l$) such that a small portion of the stick extends beyond the wall. The reaction force of the wall on the stick is perpendicular to the stick. The stick makes an angle of 30° with the wall and the bottom of the stick is on a rough floor. The reaction of the wall on the stick is equal in magnitude to the reaction of the floor on the stick. The ratio h/l and the frictional force f at the bottom of the stick are ($g = 10 \text{ ms}^{-2}$)

- (1) $\frac{h}{l} = \frac{\sqrt{3}}{16}, f = \frac{16\sqrt{3}}{3} \text{ N}$
- (2) $\frac{h}{l} = \frac{3}{16}, f = \frac{16\sqrt{3}}{3} \text{ N}$
- (3) $\frac{h}{l} = \frac{3\sqrt{3}}{16}, f = \frac{8\sqrt{3}}{3} \text{ N}$
- (4) $\frac{h}{l} = \frac{3\sqrt{3}}{16}, f = \frac{16\sqrt{3}}{3} \text{ N}$

(JEE Advanced 2016)

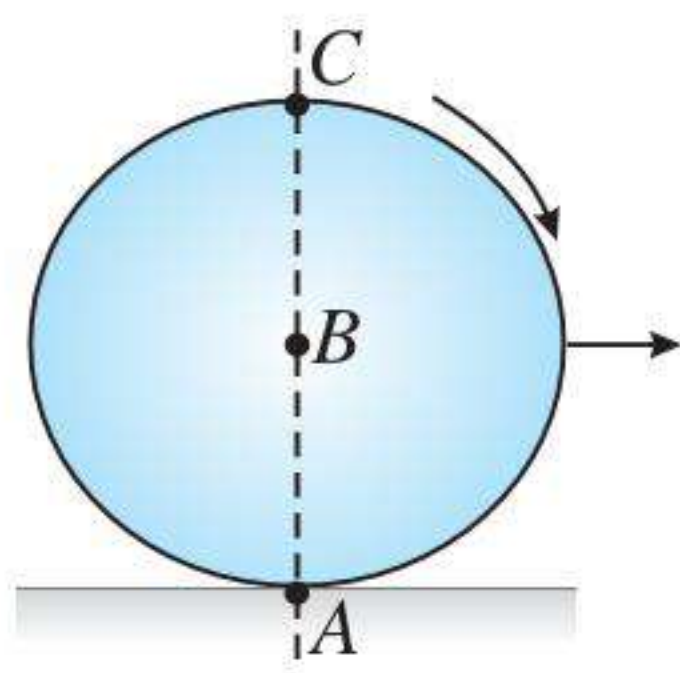
5. A small roller of diameter 20 cm has an axle of diameter 10 cm (see figure below on the left). It is on a horizontal floor and a meter scale is positioned horizontally on its axle with one edge of the scale on top of the axle (see figure below on the right). The scale is now pushed slowly on the axle so that it moves without slipping on the axle, and the roller starts rolling without slipping. After the roller has moved 50 cm, the position of the scale will look like (figures are schematic and not drawn to scale)



(JEE Advanced 2020)

Multiple Correct Answers Type

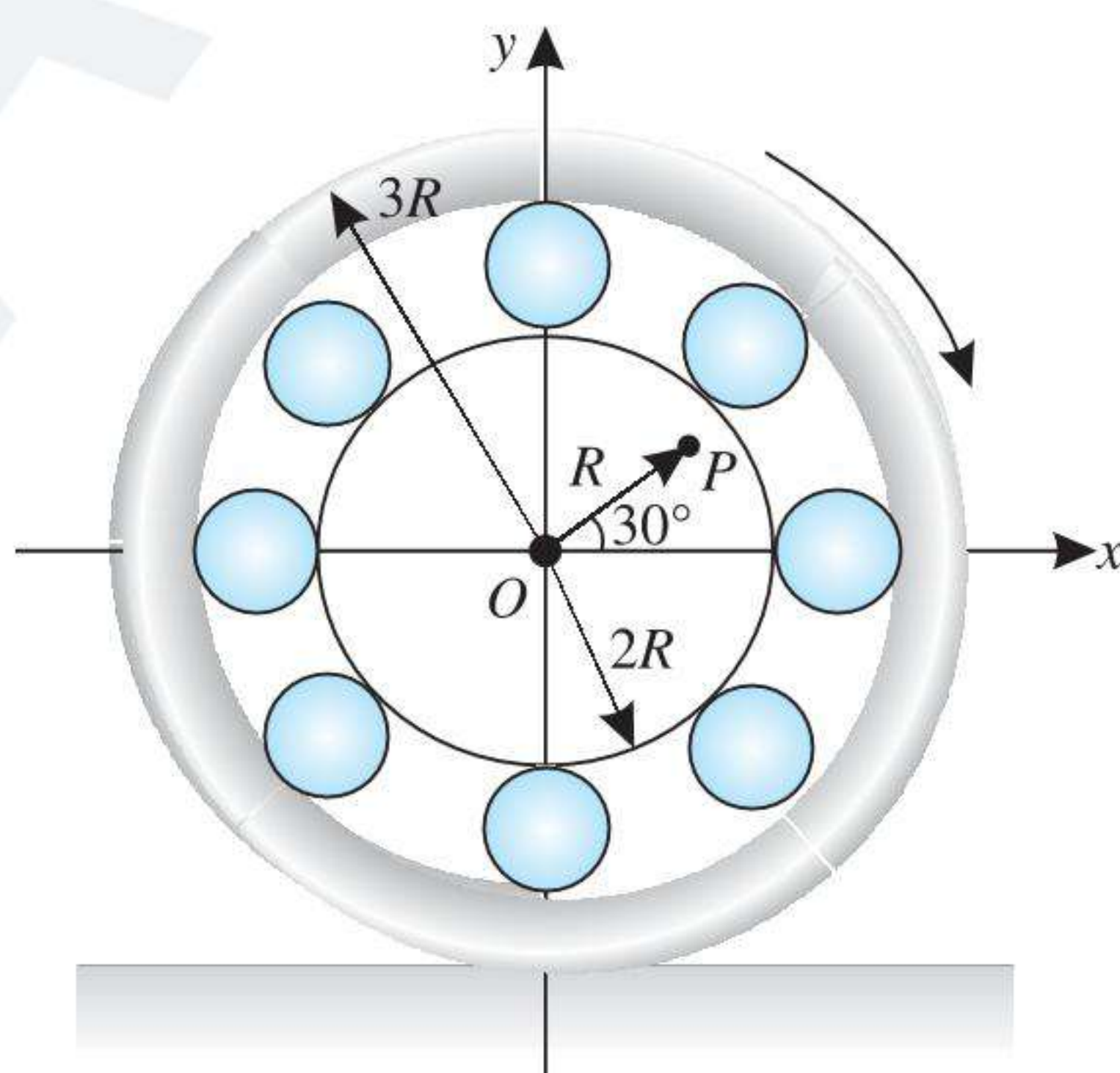
1. A sphere is rolling without slipping on a fixed horizontal plane surface. In the figure, A is the point of contact, B is the centre of the sphere and C is its topmost point. Then,



- (1) $\vec{V}_C - \vec{V}_A = 2(\vec{V}_B - \vec{V}_C)$ (2) $\vec{V}_C - \vec{V}_A = \vec{V}_B - \vec{V}_A$
 (3) $|\vec{V}_C - \vec{V}_A| = 2|\vec{V}_B - \vec{V}_C|$ (4) $|\vec{V}_C - \vec{V}_A| = 4|\vec{V}_B|$

(IIT-JEE 2009)

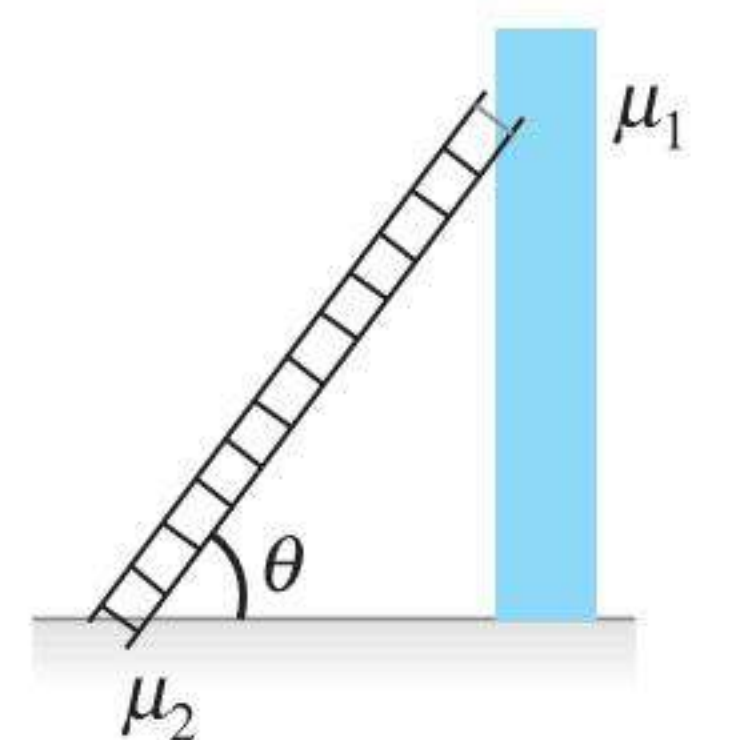
2. The figure shows a system consisting of (i) a ring of outer radius $3R$ rolling clockwise without slipping on a horizontal surface with angular speed ω and (ii) an inner disc of radius $2R$ rotating anti-clockwise with angular speed $\omega/2$. The ring and disc are separated by frictionless ball bearings. The system is in the x - z plane. The point P on the inner disc is at distance R from the origin, where OP makes an angle of 30° with the horizontal. Then with respect to the horizontal surface,



- (1) The point O has a linear velocity $3R\omega\hat{i}$
 (2) The point P has a linear velocity $\frac{11}{4}R\omega\hat{i} + \frac{\sqrt{3}}{4}R\omega\hat{k}$
 (3) The point P has a linear velocity $\frac{13}{4}R\omega\hat{i} + \frac{\sqrt{3}}{4}R\omega\hat{k}$
 (4) The point P has a linear velocity $\left(3 - \frac{\sqrt{3}}{4}\right)R\omega\hat{i} + \frac{1}{4}R\omega\hat{k}$

(IIT-JEE 2012)

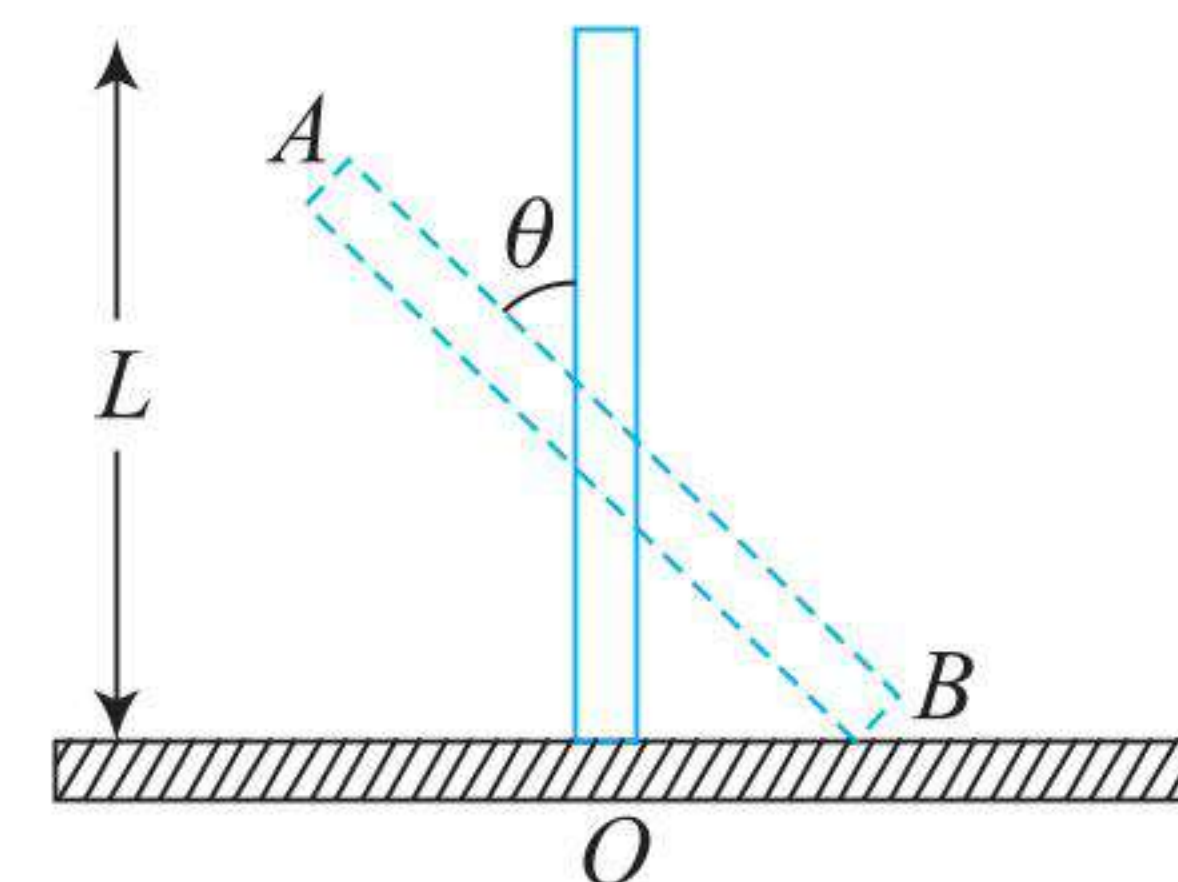
3. In the figure, a ladder of mass m is shown leaning against a wall. It is in static equilibrium making an angle θ with the horizontal floor. The coefficient of friction between the wall and the ladder is μ_1 and that between the floor and the ladder is μ_2 . The normal reaction of the wall on the ladder is N_1 and that of the floor is N_2 . If the ladder is about to slip, then



- (1) $\mu_1 = 0$ $\mu_2 \neq 0$ and $N_2 \tan \theta = mg/2$
 (2) $\mu_1 \neq 0$ $\mu_2 = 0$ and $N_1 \tan \theta = mg/2$
 (3) $\mu_1 \neq 0$ $\mu_2 \neq 0$ and $N_2 = \frac{mg}{1 + \mu_1\mu_2}$
 (4) $\mu_1 = 0$ $\mu_2 \neq 0$ and $N_1 \tan \theta = mg/2$

(JEE Advanced 2014)

4. A rigid uniform bar AB of length L is slipping from its vertical position on a frictionless floor (as shown in the figure). At some instant of time, the angle made by the bar with the vertical is θ . Which of the following statements about its motion is/are correct?



- (1) When the bar makes an angle θ with the vertical, the displacement of its midpoint from the initial position is proportional to $(1 - \cos \theta)$
 (2) The midpoint of the bar will fall vertically downward
 (3) Instantaneous torque about the point in contact with the floor is proportional to $\sin \theta$
 (4) The trajectory of the point A is a parabola

(JEE Advanced 2017)

5. Consider a body of mass 1.0 kg at rest at the origin at time $t = 0$. A force $\vec{F} = (\alpha t\hat{i} + \beta\hat{j})$ is applied on the body, where $\alpha = 1.0 \text{ N s}^{-1}$ and $\beta = 1.0 \text{ N}$. The torque acting on the body about the origin at time $t = 1.0 \text{ s}$ is $\vec{\tau}$. Which of the following statements is (are) true?

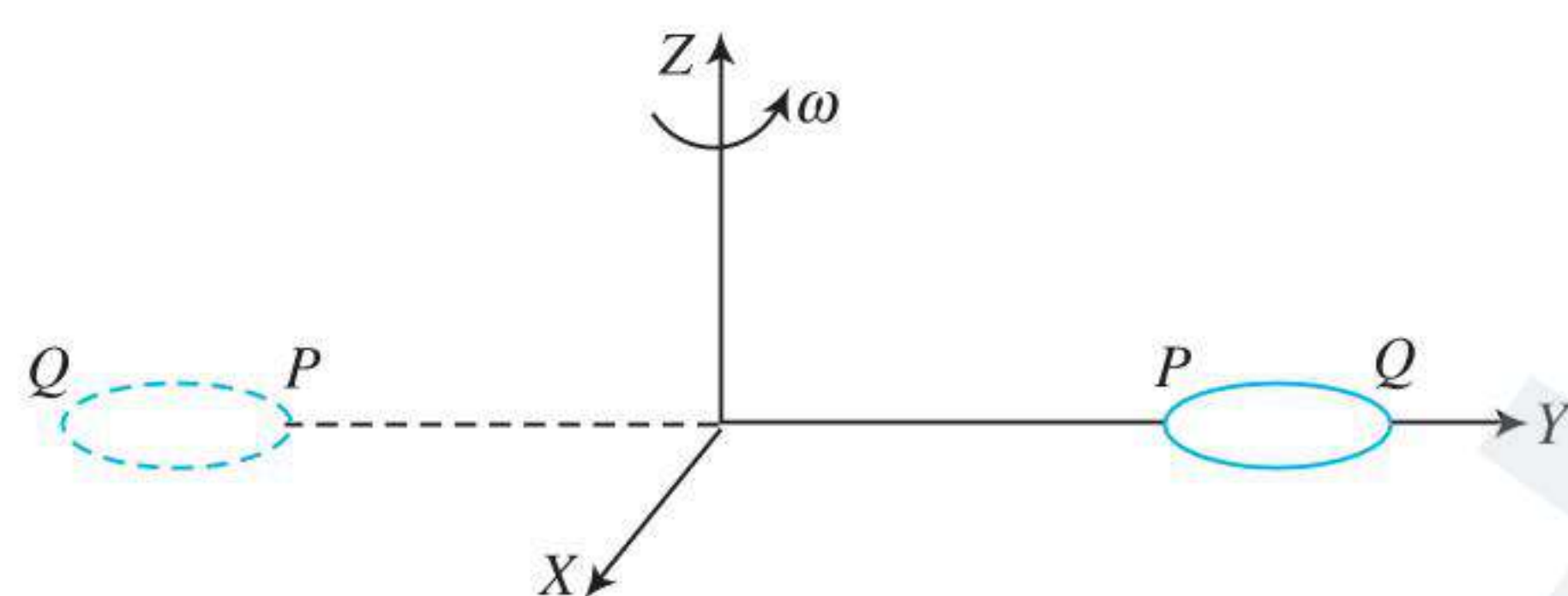
- (1) $|\vec{\tau}| = \frac{1}{3} \text{ Nm}$
 (2) The torque $\vec{\tau}$ is in the direction of the unit vector $+\hat{k}$

- (3) The velocity of the body at $t = 1$ s is $\vec{v} = \frac{1}{2}(\hat{i} + 2\hat{j}) \text{ ms}^{-1}$
- (4) The magnitude of displacement of the body at $t = 1$ s is $\frac{1}{6} \text{ m}$

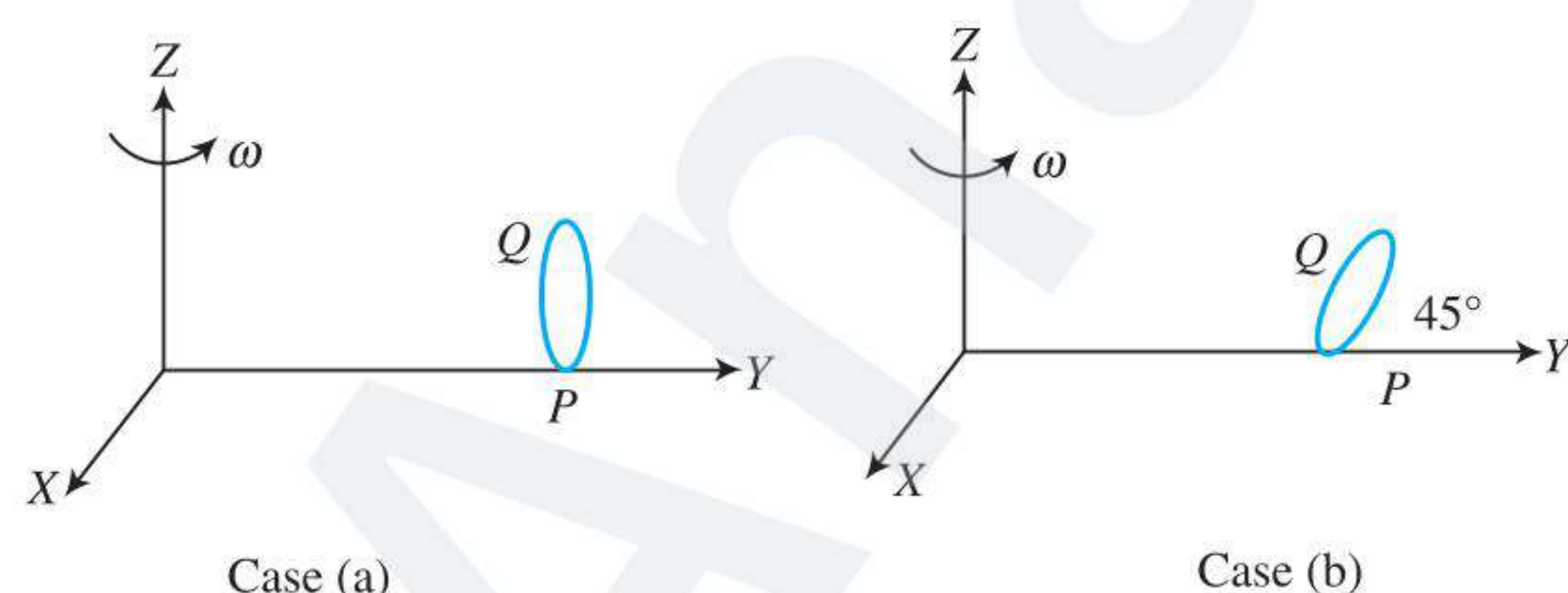
(JEE Advanced 2018)

Linked Comprehension Type**For Problems 1–2**

The general motion of a rigid body can be considered to be a combination of (i) a motion... of its centre of mass about an axis, and (ii) its motion about an instantaneous axis passing through the centre of mass. These axes need not be stationary. Consider, for example, a thin uniform disc welded (rigidly fixed) horizontally at its rim to a massless stick, as shown in the figure. When the disc-stick system is rotated about the origin on a horizontal frictionless plane with angular speed ω , the motion at any instant can be taken as a combination of (i) a rotation of the centre of mass of the disc about the z -axis, and (ii) a rotation of the disc through an instantaneous vertical axis passing through its centre of mass (as is seen from the changed orientation of points P and Q). Both these motions have the same angular speed ω in this case.



Now consider two similar systems as shown in the figure: case (a) the disc with its face vertical and parallel to x - z plane; case (b) the disc with its face making an angle of 45° with x - y plane and its horizontal diameter parallel to x -axis. In both the cases, the disc is welded at point P , and the system are rotated with constant angular speed ω about the z -axis.



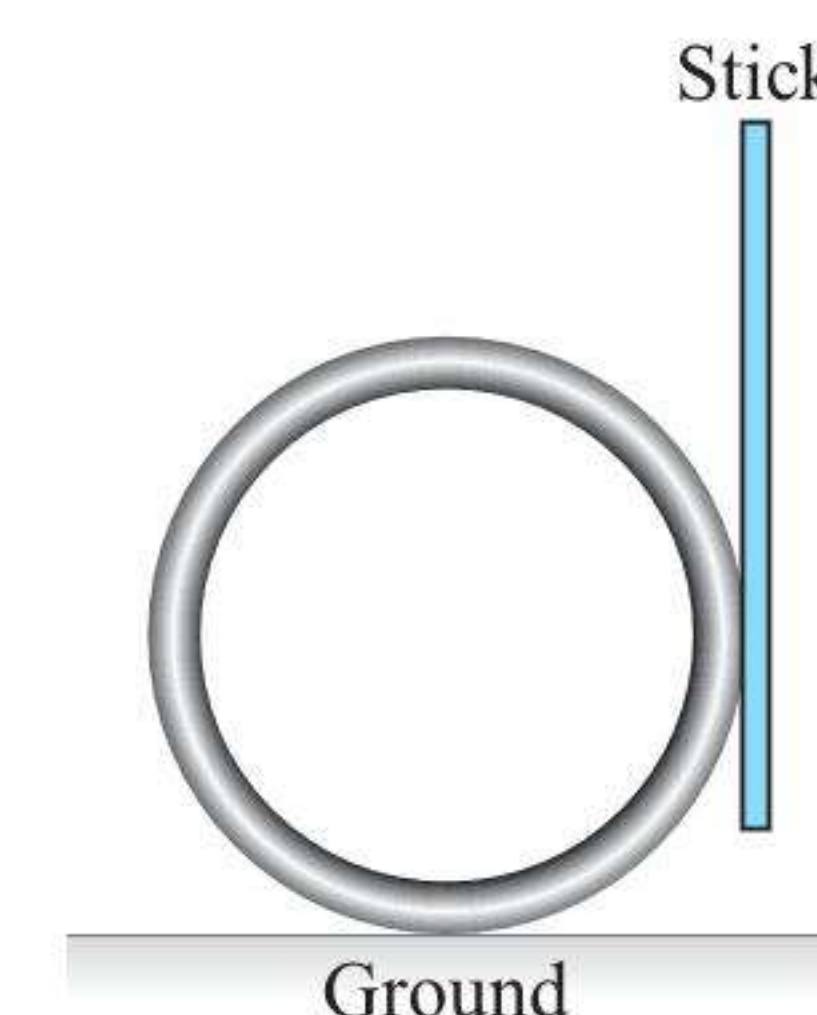
(IIT-JEE 2012)

- Which of the following statements regarding the angular speed about the instantaneous axis (passing through the centre of mass) is correct?
 - It is $\sqrt{2}\omega$ for both the cases
 - It is ω for case (a); and $\omega/\sqrt{2}$ for case (b)
 - It is ω for case (a); and $\sqrt{2}\omega$ for case (b)
 - It is ω for both the cases

- Which of the following statements about the instantaneous axis (passing through the centre of mass) is correct?
 - It is vertical for both the cases (a) and (b)
 - It is vertical for case (a); and is at 45° to the x - z plane and lies in the plane of the disc for case (b)
 - It is horizontal for case (a); and is at 45° to the x - z plane and is normal to the plane of the disc for case (b)
 - It is vertical for case (a); and is at 45° to the x - z plane and is normal to the plane of the disc for case (b)

Numerical Value Type

- A boy is pushing a ring of mass 2 kg and radius 0.5 m with a stick as shown in the figure. The stick applies a force of 2 N on the ring and rolls it without slipping with an acceleration of 0.3 m/s^2 . The coefficient of friction between the ground and the ring is large enough that rolling always occurs and the coefficient of friction between the stick and the ring is $(P/10)$. The value of P is

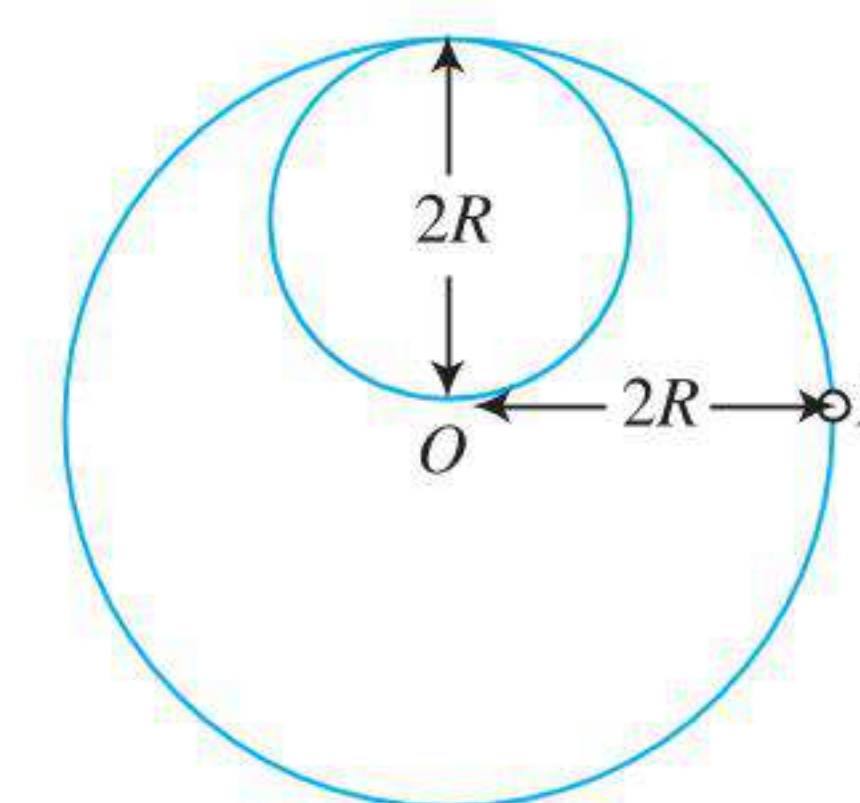


(IIT-JEE 2011)

- Four solid spheres each of diameter $\sqrt{5} \text{ cm}$ and mass 0.5 kg are placed with their centers at the corners of a square of side 4 cm. The moment of inertia of the system about the diagonal of the square is $N \times 10^{-4} \text{ kg-m}^2$, then N is

(IIT-JEE 2011)

- A lamina is made by removing a small disc of diameter $2R$ from a bigger disc of uniform mass density and radius $2R$, as shown in the figure. The moment of inertia of this lamina about axes passing through O and P is I_O and I_P , respectively. Both these axes are perpendicular to the plane of the lamina. The ratio I_P/I_O to the nearest integer is

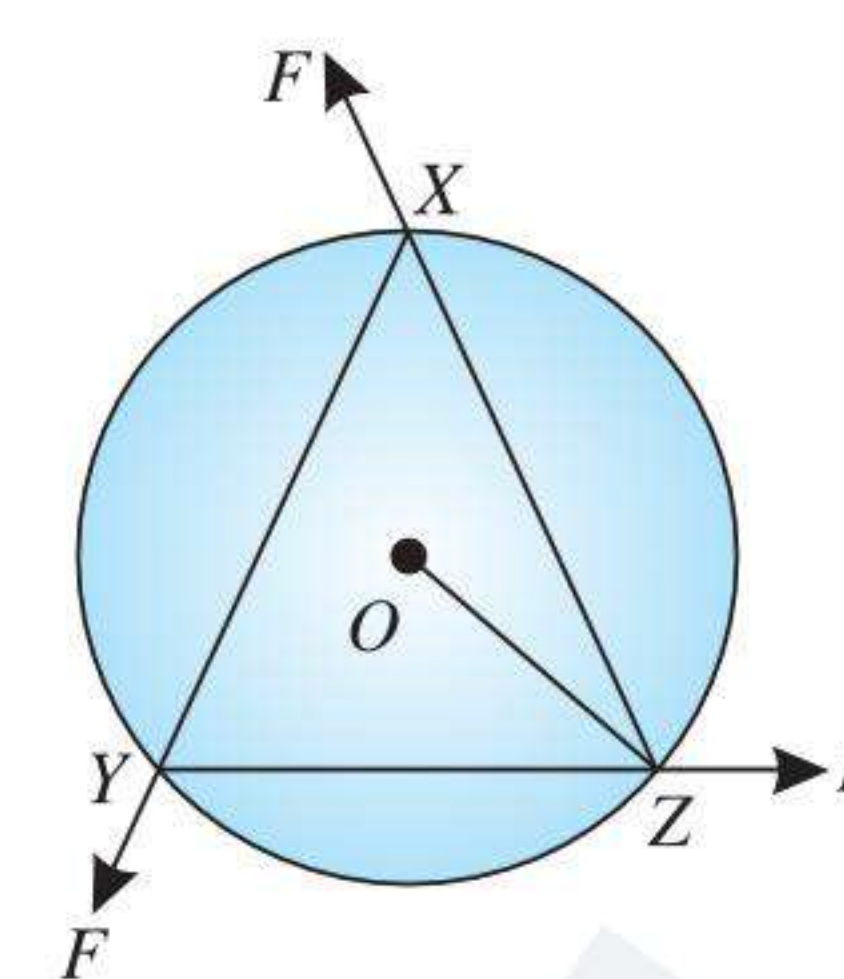


(IIT-JEE 2012)

- A uniform circular disc of mass 50 kg and radius 0.4 m is rotating with an angular velocity of 10 rad/s about its own axis, which is vertical. Two uniform circular rings, each of mass 6.25 kg and radius 0.2 m, are gently placed symmetrically on the disc in such a manner that they are touching each other along the axis of the disc and are horizontal. Assume that the friction is large enough such that the rings are at rest relative to the disc and the system rotates about the original axis. The new angular velocity (in rad s^{-1}) of the system is

(JEE Advanced 2013)

5. A uniform circular disc of mass 1.5 kg and radius 0.5 m is initially at rest on a horizontal frictionless surface. Three forces of equal magnitude $F = 0.5$ N are applied simultaneously along the three sides of an equilateral triangle XYZ with its vertices on the perimeter of the disc (see figure). One second after applying the forces, the angular speed of the disc in rad s^{-1} is



(JEE Advanced 2014)

Answers Key

EXERCISES

Single Correct Answer Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (4) | 2. (2) | 3. (4) | 4. (3) | 5. (4) |
| 6. (3) | 7. (2) | 8. (2) | 9. (2) | 10. (3) |
| 11. (2) | 12. (2) | 13. (3) | 14. (4) | 15. (1) |
| 16. (2) | 17. (1) | 18. (4) | 19. (4) | 20. (2) |
| 21. (3) | 22. (3) | 23. (3) | 24. (2) | 25. (2) |
| 26. (3) | 27. (4) | 28. (3) | 29. (1) | 30. (1) |
| 31. (4) | 32. (3) | 33. (4) | 34. (3) | 35. (3) |
| 36. (3) | 37. (3) | 38. (3) | 39. (3) | 40. (4) |
| 41. (3) | 42. (2) | 43. (4) | 44. (3) | 45. (3) |
| 46. (2) | 47. (2) | 48. (2) | 49. (1) | 50. (2) |
| 51. (2) | 52. (2) | 53. (2) | 54. (2) | 55. (2) |
| 56. (2) | 57. (3) | 58. (2) | 59. (3) | 60. (4) |
| 61. (1) | 62. (2) | 63. (1) | 64. (2) | 65. (3) |
| 66. (2) | 67. (3) | 68. (4) | 69. (3) | 70. (1) |
| 71. (4) | 72. (2) | 73. (3) | 74. (4) | 75. (1) |
| 76. (4) | 77. (1) | 78. (2) | 79. (4) | |

Multiple Correct Answers Type

- | | | |
|-----------------|-----------------|-----------------|
| 1. (2),(3) | 2. (3),(4) | 3. (1),(2),(3) |
| 4. (1),(2),(3) | 5. (1),(2),(3) | 6. (1),(3) |
| 7. (3),(4) | 8. (1),(2),(3) | 9. (1),(4) |
| 10. (1),(3) | 11. (2),(4) | 12. (2),(3) |
| 13. (1),(2),(4) | 14. (1),(3) | 15. (1),(3) |
| 16. (1),(2),(3) | 17. (1),(2),(3) | 18. (1),(2),(3) |
| 19. (1),(2),(4) | 20. (1),(3) | |

Linked Comprehension Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (3) | 2. (4) | 3. (4) | 4. (2) | 5. (3) |
| 6. (4) | 7. (2) | 8. (4) | 9. (1) | 10. (2) |
| 11. (1) | 12. (4) | 13. (3) | 14. (1) | 15. (3) |

- | | | | | |
|---------|---------|---------|---------|---------|
| 16. (2) | 17. (4) | 18. (3) | 19. (3) | 20. (1) |
| 21. (4) | 22. (4) | 23. (1) | 24. (4) | 25. (2) |
| 26. (3) | 27. (3) | 28. (1) | | |

Matrix Match Type

- i. $\rightarrow$ a.; ii. $\rightarrow$ d.; iii. $\rightarrow$ a.; iv. $\rightarrow$ c.
- i. $\rightarrow$ c.; ii. $\rightarrow$ a.; iii. $\rightarrow$ a.; iv. $\rightarrow$ b.
- i. $\rightarrow$ b.; ii. $\rightarrow$ a.; iii. $\rightarrow$ c.; iv. $\rightarrow$ a., b., d.
- i. $\rightarrow$ b.; ii. $\rightarrow$ d.; iii. $\rightarrow$ d.; iv. $\rightarrow$ a.

- | | | | | |
|---------|--------|--------|--------|--------|
| 5. (2) | 6. (1) | 7. (3) | 8. (2) | 9. (2) |
| 10. (4) | | | | |

Numerical Value Type

- | | | | | |
|-------------|------------|-------------|-------------|-------------|
| 1. (6) | 2. (5) | 3. (6) | 4. (4) | 5. (2.00) |
| 6. (1.0) | 7. (3.00) | 8. (30.00) | 9. (0.50) | 10. (6.00) |
| 11. (5.00) | 12. (30) | 13. (25.00) | 14. (0.50) | 15. (4.00) |
| 16. (30.00) | 17. (1.00) | 18. (4) | 19. (50.00) | 20. (40.00) |

ARCHIVES

JEE Advanced

Single Correct Answer Type

- | | | | | |
|--------|--------|--------|--------|--------|
| 1. (2) | 2. (2) | 3. (1) | 4. (4) | 5. (2) |
|--------|--------|--------|--------|--------|

Multiple Correct Answers Type

- | | | |
|----------------|------------|------------|
| 1. (2),(3) | 2. (1),(2) | 3. (3),(4) |
| 4. (1),(2),(3) | 5. (1),(3) | |

Linked Comprehension Type

- | | |
|--------|--------|
| 1. (4) | 2. (1) |
|--------|--------|

Numerical Value Type

- | | | | | |
|--------|--------|--------|--------|--------|
| 1. (4) | 2. (9) | 3. (3) | 4. (8) | 5. (2) |
|--------|--------|--------|--------|--------|

Chapter 3

Concept Application Exercises

Exercise 3.1

1. The angular velocity of B with respect to A ,

$$\omega = \omega_{B,A} = \frac{v_{B,A}}{l} = \frac{0.20 + 0.30}{1} = 0.50 \text{ rad/s}$$

From right hand rule, it is along $+z$ direction, hence in vector form $\vec{\omega} = 0.50 \hat{k} \text{ rad/s}$

2. $v_P = v_C - R\omega = 10 - \frac{20}{100} \times 10 = 8 \text{ m/s}$

$$v_Q = v_C + R\omega = 10 + 2 = 12 \text{ m/s}$$

3. Let the velocity at Q be v' . As rod is a rigid body,

$$v' \cos \beta = v \cos \alpha$$

or
$$v' = \frac{v \cos \alpha}{\cos \beta}$$

$$\omega = \frac{v' \sin \beta + v \sin \alpha}{l} = \frac{1}{l} \left(\frac{v \cos \alpha \sin \beta}{\cos \beta} + v \sin \alpha \right)$$

$$= \frac{v \sin(\alpha + \beta)}{l \cos \beta}$$

4. Let us assume that, the mid-point of the rod has velocity $\vec{v}_C$. Then, we have

$$\vec{v}_A = \vec{v}_{AC} + \vec{v}_C \quad \dots(i)$$

$$\vec{v}_B = \vec{v}_{BC} + \vec{v}_C \quad \dots(ii)$$

Adding Eqs. (i) and (ii), we have

$$\vec{v}_A + \vec{v}_B = \vec{v}_{AC} + \vec{v}_{BC} + 2\vec{v}_C$$

Substituting $\vec{v}_{AC} = +\vec{\omega}_{AC} \times \vec{r}_{AC}$

and $\vec{v}_{BC} = +\vec{\omega}_{BC} \times \vec{r}_{BC}$, we get

$$\vec{v}_A + \vec{v}_B = \vec{\omega}_{AC} \times \vec{r}_{AC} + \vec{\omega}_{BC} \times \vec{r}_{BC} + 2\vec{v}_C$$

Substituting $\vec{\omega}_{AC} = \vec{\omega}_{BC} = \vec{\omega}$ (angular velocity of the rod), we have,

$$\vec{v}_A + \vec{v}_B = \vec{\omega} \times (\vec{r}_{AC} + \vec{r}_{BC}) + 2\vec{v}_C$$

Since C is the mid-point of the rod, $\vec{r}_{AC} = -\vec{r}_{BC}$. Then we have,

$$\vec{v}_C = \frac{\vec{v}_A + \vec{v}_B}{2}$$

This tells us that when any two points A and B , (say;) of a rigid body have velocities $\vec{v}_A$ and $\vec{v}_B$ respectively, the mid-point C (but not necessarily the centre of mass) of the line joining the points will have a velocity $\vec{v}_C = \frac{\vec{v}_A + \vec{v}_B}{2}$.

5. (a) Two particles of a rigid body must have same velocity component along the line joining them.

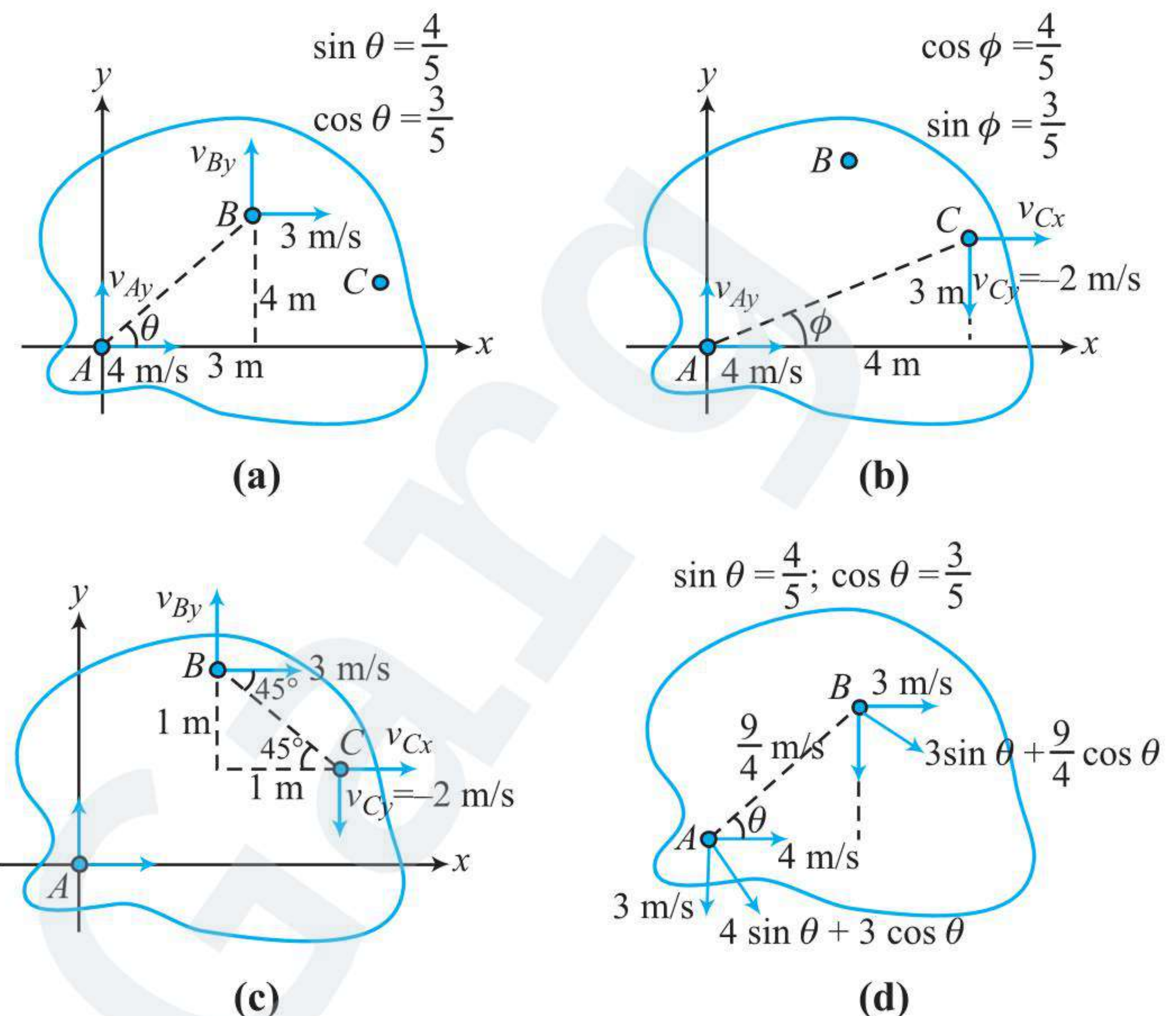
Velocity of 'A' along AB = velocity of 'B' along AB

$$4 \cos \theta + v_{Ay} \sin \theta = 3 \cos \theta + v_{By} \sin \theta$$

$$4 \times \frac{3}{5} + v_{Ay} \times \frac{4}{5} = 3 \times \frac{3}{5} + v_{By} \times \frac{4}{5}$$

$$12 + 4v_{Ay} = 9 + 4v_{By}$$

$$\frac{3}{4} = v_{By} - v_{Ay} \quad \dots(i)$$



Similarly, for A and C

Velocity of 'A' along AC = velocity of 'C' along AC

$$4 \cos \phi + v_{Ay} \sin \phi = v_{Cx} \cos \phi - v_{Cy} \sin \phi$$

$$4 \times \frac{4}{5} + v_{Ay} \times \frac{3}{5} = v_{Cx} \times \frac{4}{5} - 2 \times \frac{3}{5}$$

$$4v_{Cx} - 3v_{Ay} = 22 \quad \dots(ii)$$

and for B and C ,

$$3 \cos 45^\circ - v_{By} \cos 45^\circ = 2 \cos 45^\circ + v_{Cx} \cos 45^\circ$$

$$v_{By} + v_{Cx} = 1 \quad \dots(iii)$$

Solving we get

$$v_{Ay} = -3 \text{ m/s}; v_{By} = -\frac{9}{4} \text{ m/s}; v_{Cx} = \frac{13}{4} \text{ m/s}$$

$$\therefore \vec{v}_A = 4\hat{i} - 3\hat{j}; \vec{v}_B = 3\hat{i} - \frac{9}{4}\hat{j}; \vec{v}_C = \frac{13}{4}\hat{i} - 2\hat{j}$$

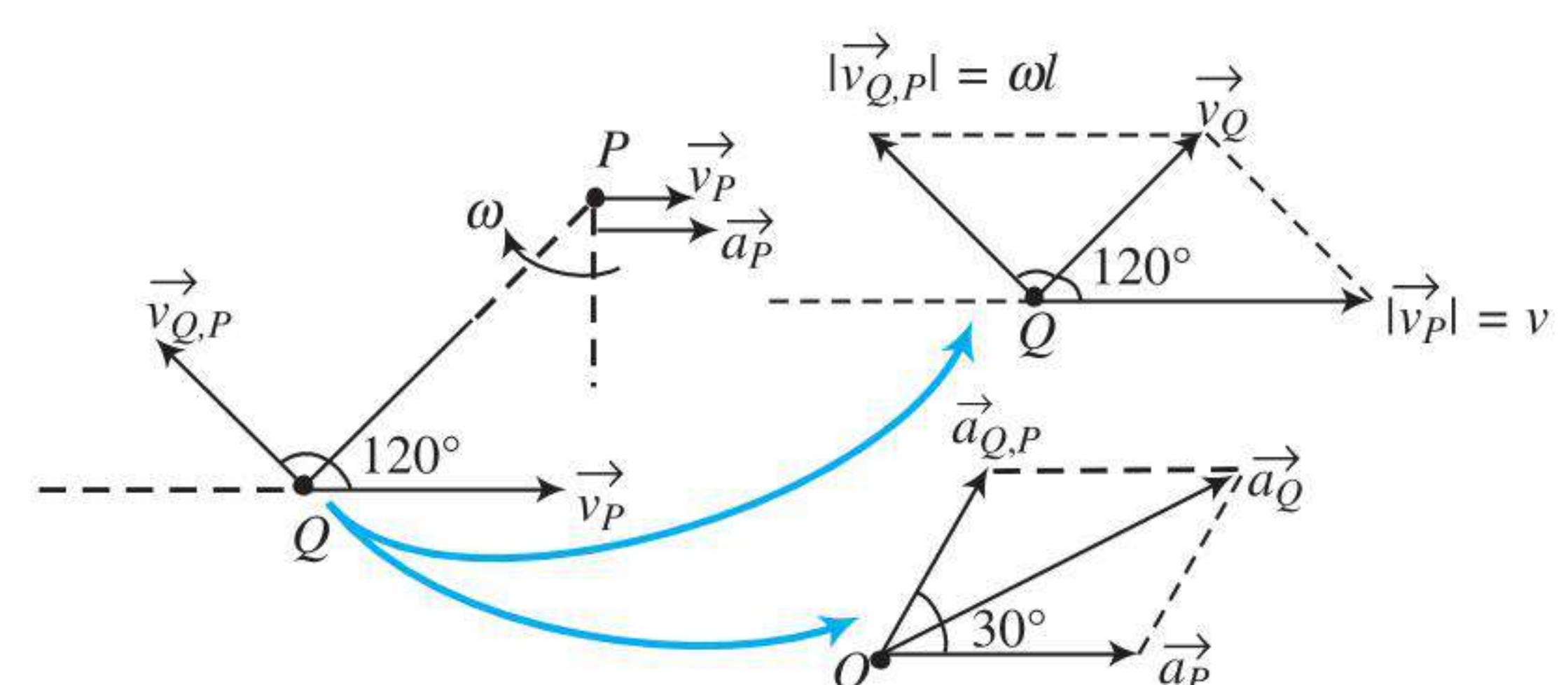
$$(b) \text{ Angular velocity } (\omega) = \frac{\text{Relative velocity of } A \text{ w.r.t. } B \perp \text{ to } AB}{AB}$$

$$= \frac{\text{Velocity of } B \perp \text{ to } AB - \text{velocity of } A \perp \text{ to } AB}{AB}$$

$$\omega = \frac{\left(3 \times \frac{4}{5} + \frac{9}{4} \times \frac{3}{5} \right) - \left(4 \times \frac{4}{5} + 3 \times \frac{3}{5} \right)}{5}$$

$$= \frac{1}{4} \text{ rad/s}$$

$$6. v_Q = |\vec{v}_{QP} + \vec{v}_P| = \sqrt{l^2 \omega^2 + v^2 + 2vl\omega \cos 120^\circ}$$



$$= \sqrt{(1 \times 2)^2 + 1^2 + 2(1)(2) \times \left(\frac{-1}{2}\right)} = \sqrt{3} \text{ ms}^{-1}$$

$$a_Q = |\vec{a}_{QP} + \vec{a}_P| = \sqrt{a_{QP}^2 + a_P^2 + 2a_{QP}a_P \cos 30^\circ}$$

where $a_{QP} = l\omega^2 = 1(2)^2 = 4 \text{ ms}^{-2}$ and $a_P = 1 \text{ ms}^{-2}$

$$a_Q = \sqrt{4^2 + 1^2 + 2.4 \times 1 \times \frac{\sqrt{3}}{2}} = \sqrt{17 + 4\sqrt{3}} \text{ ms}^{-2}$$

$$7. \vec{a}_B = \vec{a}_{BA} + \vec{a}_A = \vec{a}_{BA_t} + \vec{a}_{BA_r} + \vec{a}_A$$

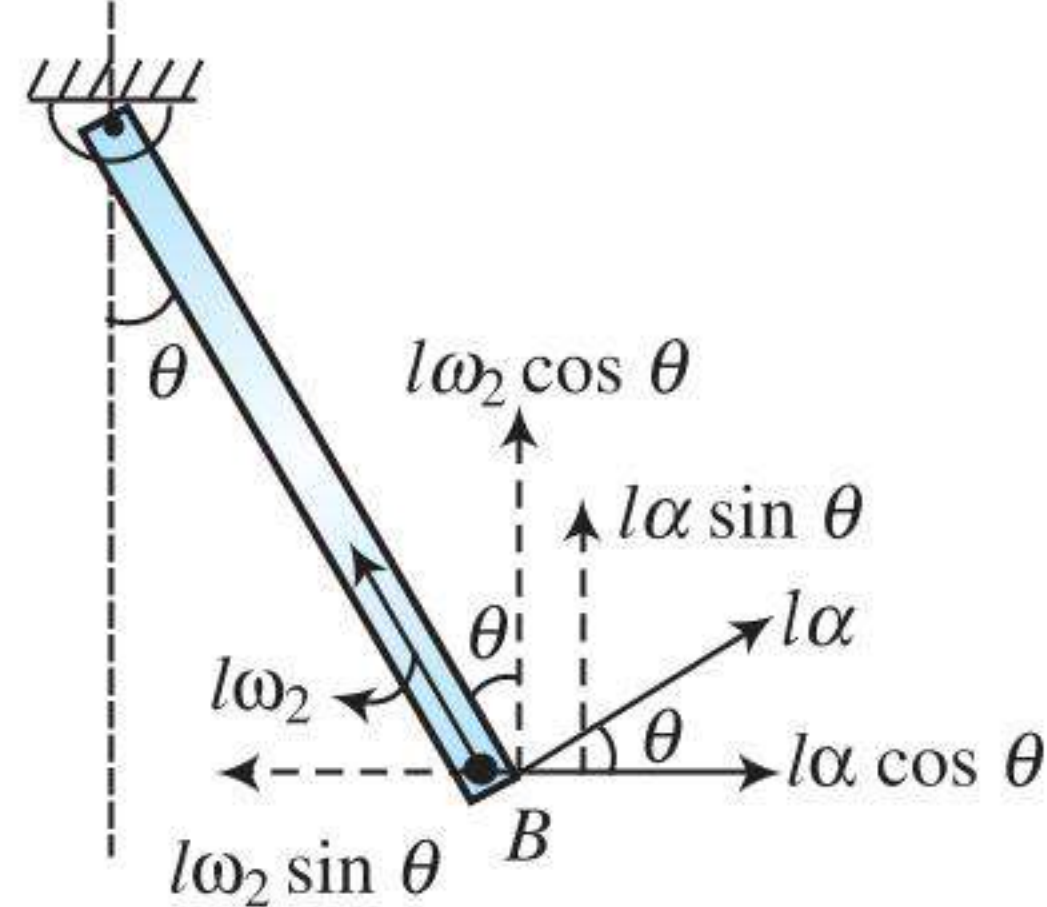
where $a_A = 0$, $a_{BA_t} = l\alpha$

and $a_{BA_r} = l\omega^2$

Then resolving the vectors in x and y directions, we have

$$a_{B_x} = l\alpha \cos \theta - l\omega^2 \sin \theta$$

$$\text{and } a_{B_y} = l\alpha \sin \theta + l\omega^2 \cos \theta$$



Exercise 3.2

1. In rolling, the angular velocity $\vec{\omega}$ and velocity of the center of a rolling body are related as

$$v_C = \omega R \Rightarrow \omega = \frac{v_C}{R} = \frac{25}{5} = 5 \text{ rad/s}$$

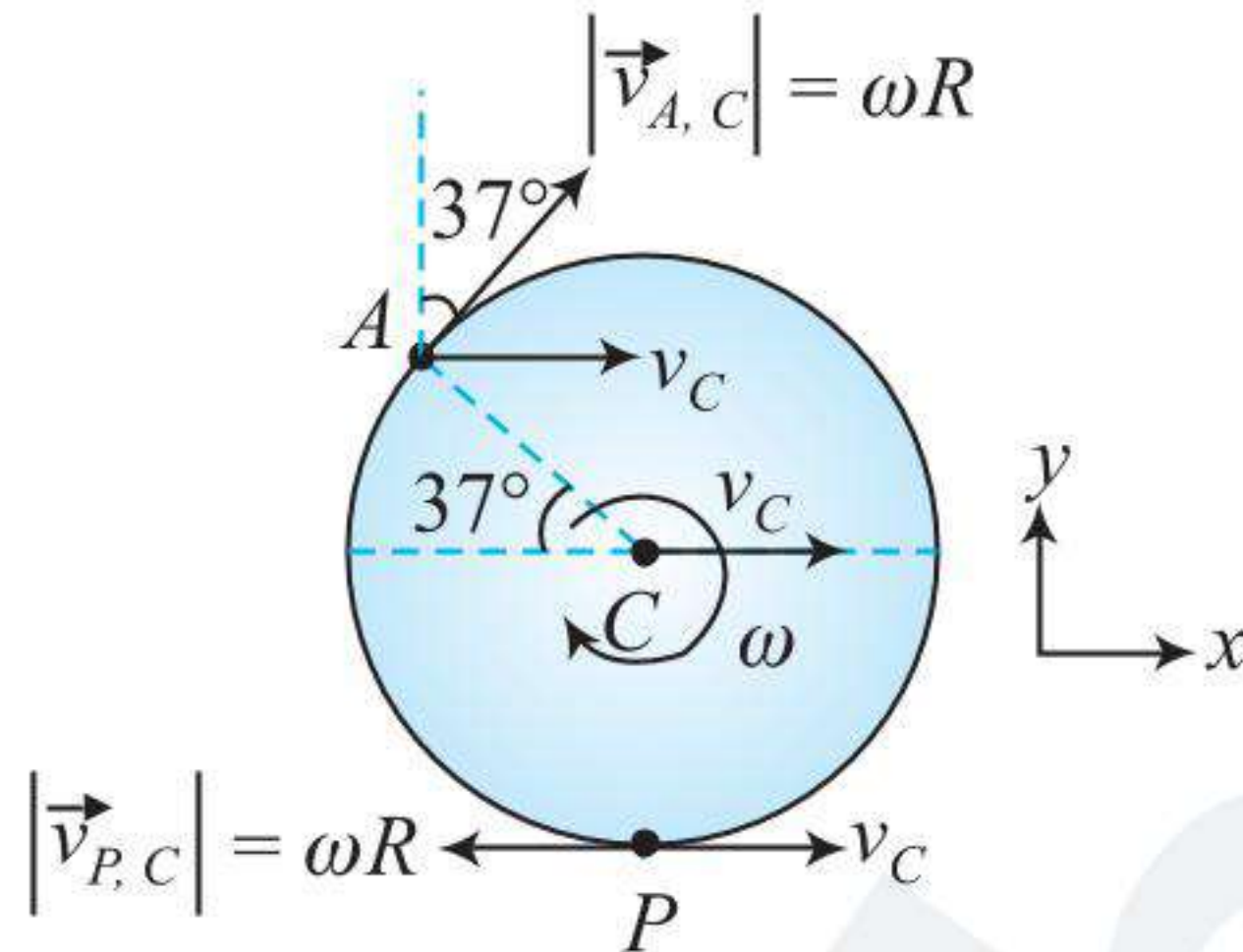
If the velocity of point P is zero, the rotation of the cylinder should be in clockwise sense.

Velocity of point A , $\vec{v}_A = \vec{v}_{A,C} + \vec{v}_C$

$$\Rightarrow \vec{v}_A = (\omega R \sin 37^\circ \hat{i} + \omega R \cos 37^\circ \hat{j}) + v_C \hat{i}$$

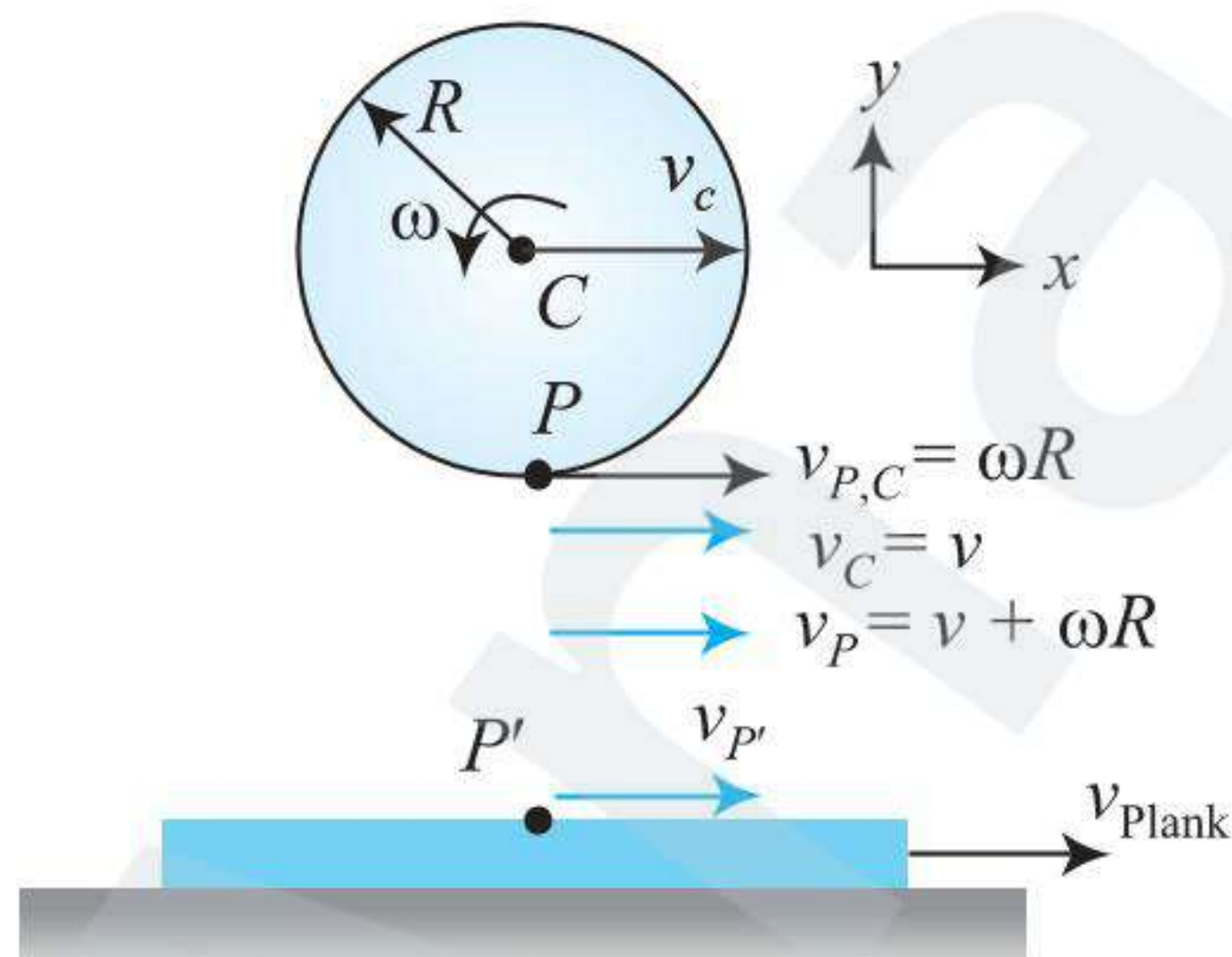
$$= (v_C + \omega R \sin 37^\circ) \hat{i} + \omega R \cos 37^\circ \hat{j}$$

$$= \left(25 + 5 \times 5 \times \frac{3}{5}\right) \hat{i} + 5 \times 5 \times \frac{4}{5} \hat{j} = (40\hat{i} + 20\hat{j}) \text{ m/s}$$



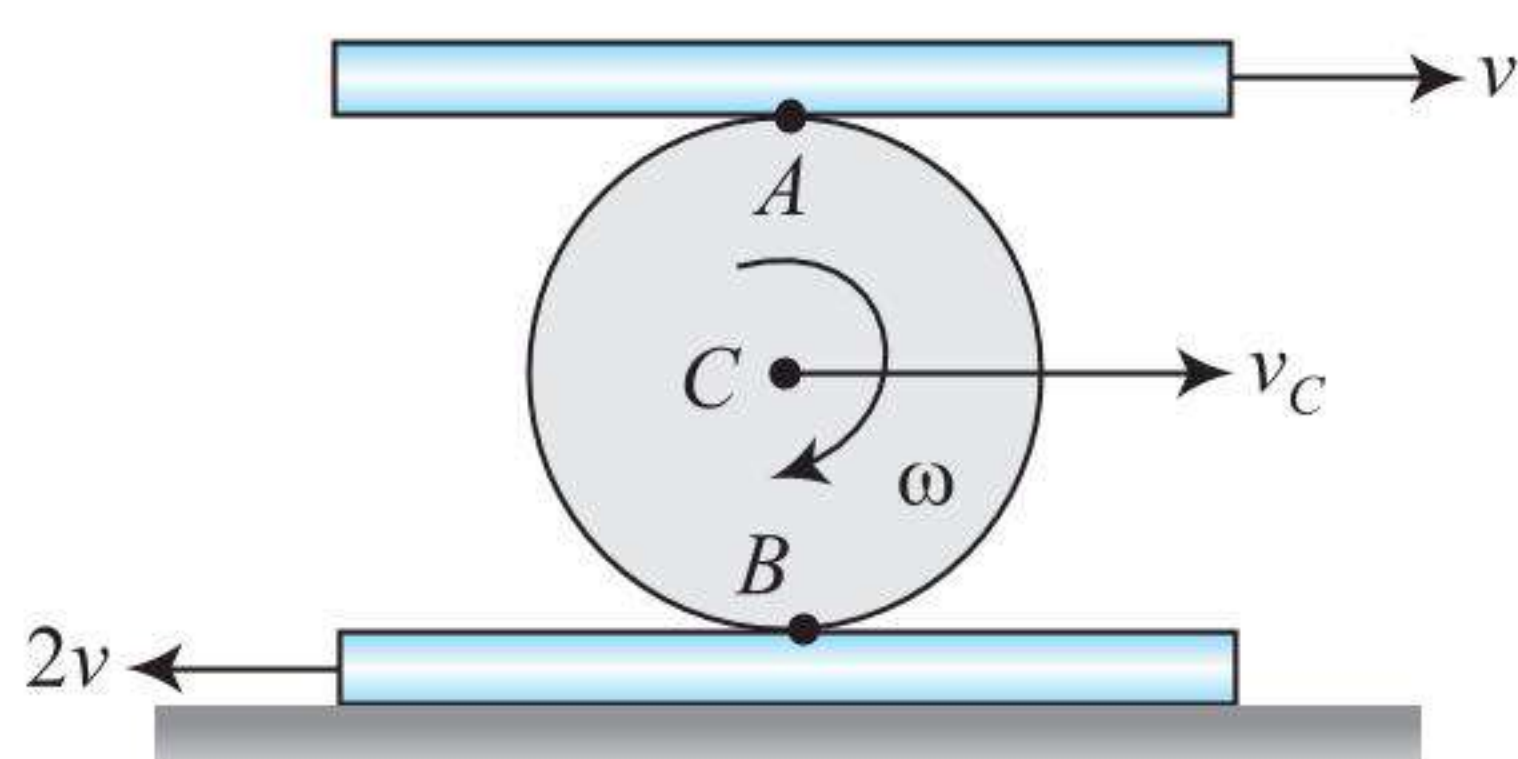
2. As disc is rolling, there should not be relative sliding between plank and disc at point of contact.

$\vec{v}_P = \vec{v}_{P'} =$ velocity of plank.



$$\vec{v}_P = \vec{v}_{P,C} + \vec{v}_C = \omega R + v_C = 2 \times \frac{1}{2} + 2 = 3 \text{ m/s}$$

3. As the disc is rolling points A and B which are on the disc and in contact with the planks will not slide. The velocities of points A and B should be the same as the planks. Let the disc rotate with an angular velocity ω and let the velocity of centre of the mass of the disc be v_C .



$$\text{Velocity of point } A, v_A, v_C + \omega R = v \quad \dots(i)$$

$$\text{Velocity of point } B, v_B, \omega R - v_C = 2v \quad \dots(ii)$$

From Eqs. (i) and (ii), $\omega = 3v/2R$ and $v_C = -v/2$

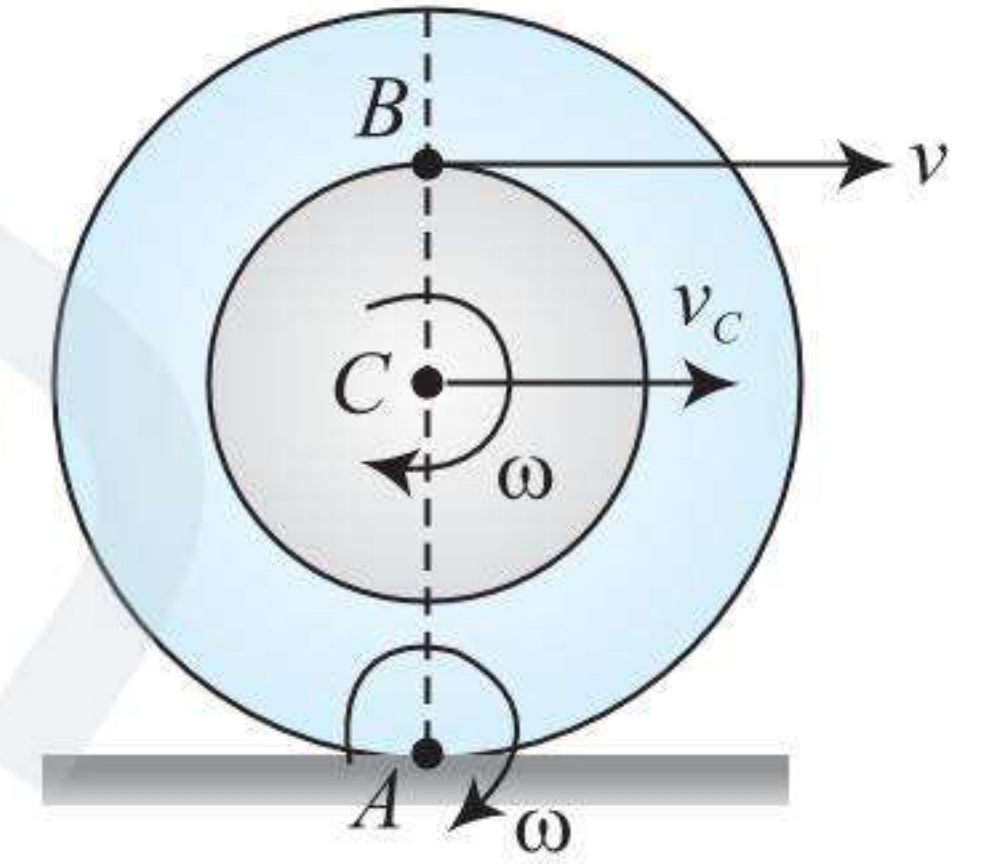
Hence, velocity of centre of disc will be $v/2$ towards the left and the angular velocity will be $3v/2R$ in the clockwise sense.

4. As the cotton reel is rolling, lowest point of the reel contact with ground should be ICR.

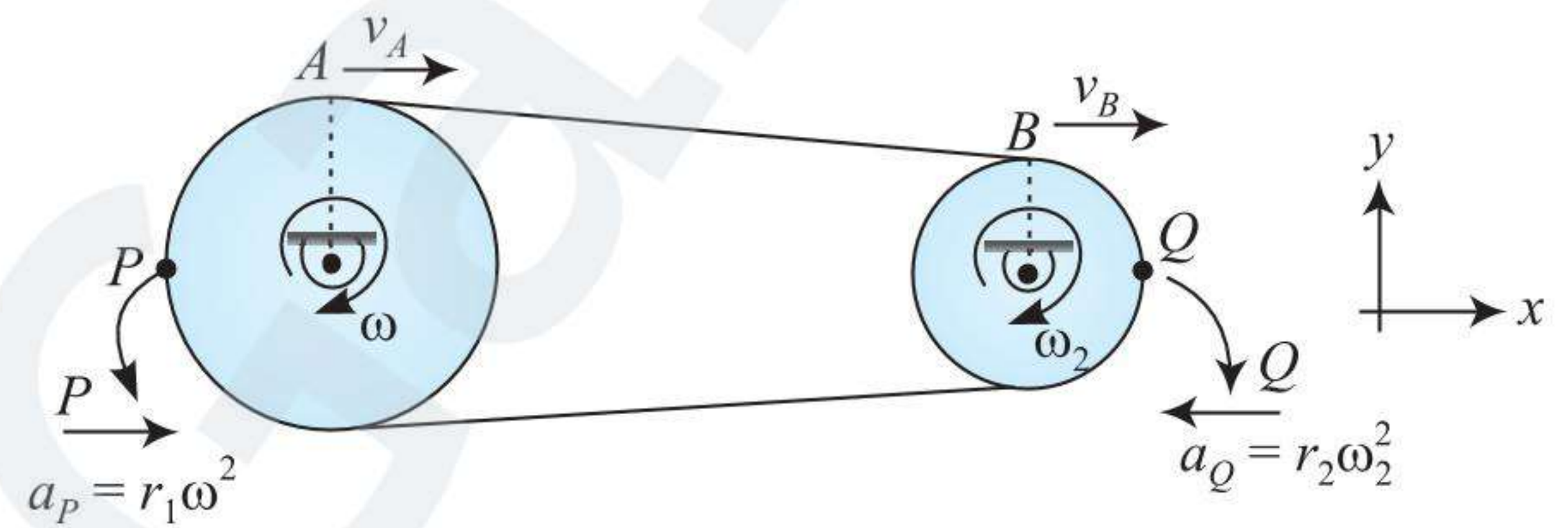
$$v_B = v_P = v = \omega(R+r)$$

$$\Rightarrow \omega = \frac{v}{(R+r)} \text{ (in clockwise sense)}$$

$$v_c = \omega R = \frac{vR}{(R+r)}$$



5. (i) As belt is not sliding $v_A = \omega r_1 = v_B$



$$(ii) v_B = \omega_2 r_2 \Rightarrow \omega_2 = \frac{\omega r_1}{r_2}$$

$$(iii) \vec{a}_P = r_1 \omega^2 (\hat{i}) \text{ and } \vec{a}_Q = r_2 \omega_2^2 (-\hat{i})$$

$$\Rightarrow \vec{a}_Q = r_2 \left(\frac{\omega r_1}{r_2} \right)^2 (-\hat{i}) = -\frac{\omega^2 r_1^2}{r_2} (\hat{i})$$

Hence, acceleration of point P w.r.t. point Q

$$\vec{a}_{P,Q} = \vec{a}_P - \vec{a}_Q = r_1 \omega^2 \left(1 + \frac{r_1}{r_2} \right) \hat{i}$$

6. The acceleration of P is $\vec{a}_P = \vec{a}_{PC} + \vec{a}_{CO} + \vec{a}_O \quad \dots(i)$

$$a_O = 0 \quad \dots(ii)$$

$$\vec{a}_P = \vec{a}_{PC} + \vec{a}_{CO}$$

$$\vec{a}_{PC} = r \omega_{PC}^2 \hat{j} + 0 = \frac{v^2}{r} \hat{j} \quad \dots(iii)$$

$$a_{CO} = r_{CO} \omega_{CO}^2 = \frac{v^2}{R-r} \hat{j} \quad \dots(iv)$$

$$\text{Using the above equations } a_P = v^2 \left[\frac{1}{r} + \frac{1}{R-r} \right] \hat{j}$$

7. Let P be a point passing through instantaneous axis of rotation. Therefore $v_P = 0$.

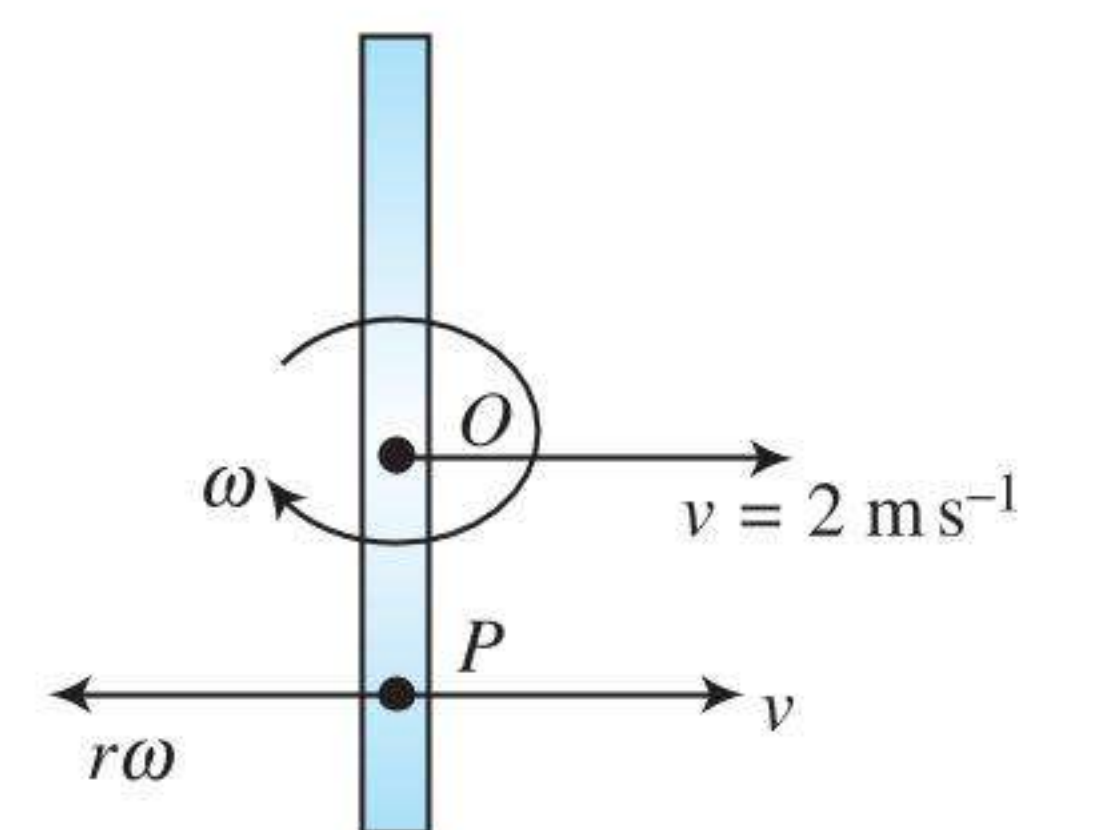
$$\Rightarrow |\vec{v}_{PO} + \vec{v}_O| = 0$$

$$\Rightarrow v_{PO} - v_O = 0 \Rightarrow v_{PO} = v_O$$

where $v_O = v$ and $v_{PO} = (OP)\omega$

$$\Rightarrow (OP)\omega = v$$

$$\Rightarrow OP = v/\omega = \frac{2}{2\pi} = \frac{1}{\pi} \text{ m}$$

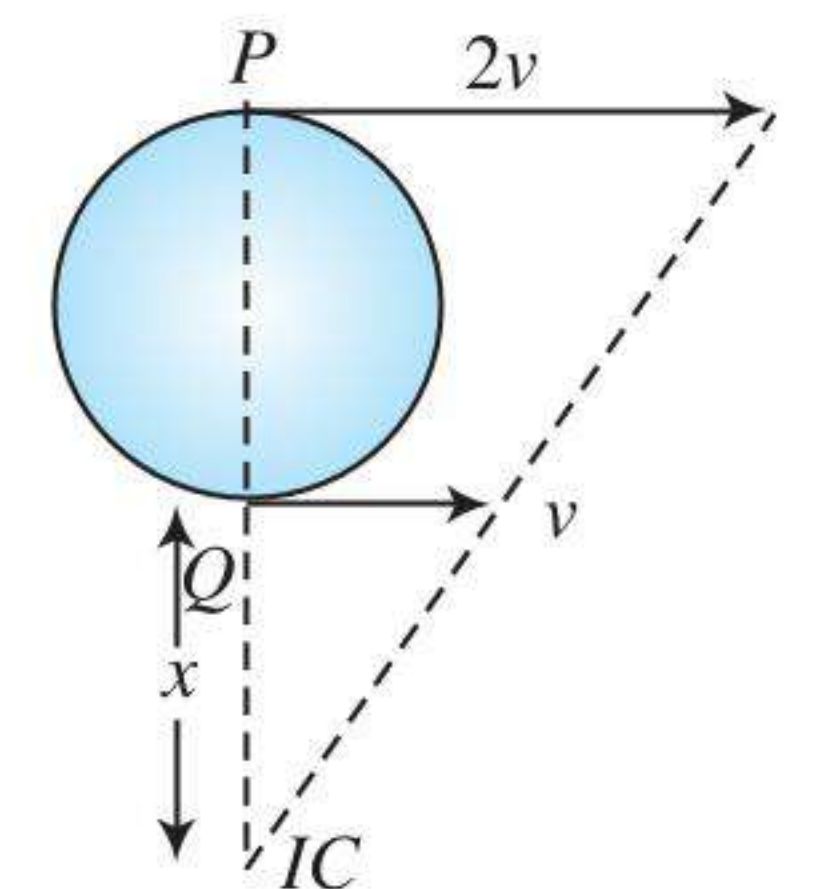


$$8. (a) \frac{v_P}{v_Q} = \frac{2R+x}{x}$$

$$\Rightarrow \frac{2v}{v} = \frac{2R+x}{x}$$

$$2x = 2R + x \therefore x = 2R$$

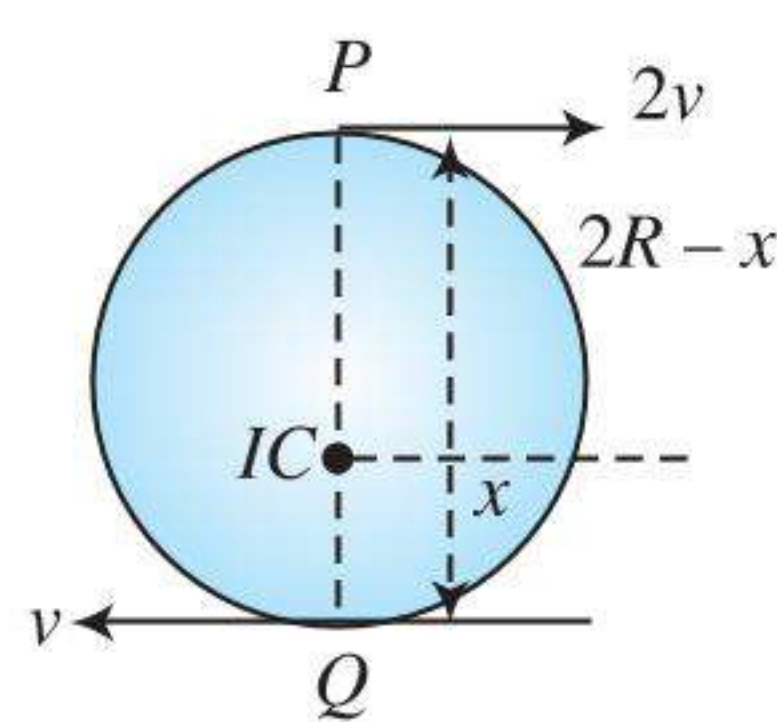
$$\therefore \omega = \frac{v_Q}{x} = \frac{v}{2R}$$



$$(b) \frac{v_P}{v_Q} = \frac{2R-x}{x}$$

$$\frac{2v}{v} = \frac{2R-x}{x}$$

$$\therefore x = 2R/3 \text{ and } \omega = \frac{v_Q}{x} = \frac{3v}{2R}$$

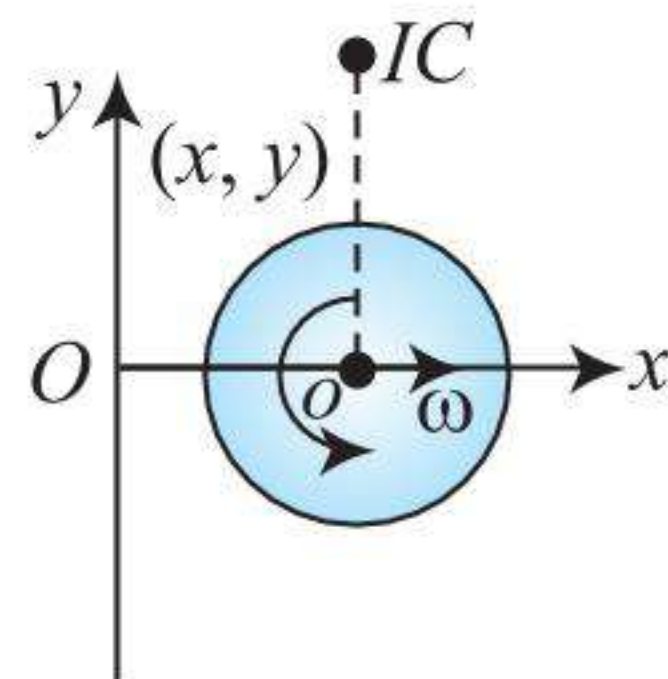


9. (a) Let the coordinates of instantaneous centre of rotation at any time t be (x, y) .

$$\text{Now, } v = \omega y$$

$$\sqrt{2ax} = \omega y$$

$$\therefore y^2 = \left(\frac{2a}{\omega^2}\right)x \text{ (parabola)}$$



- (b) $v = \omega y$

$$v = \frac{\alpha xy}{v}$$

$$\therefore xy = \frac{v^2}{\alpha} = \text{constant (rectangular hyperbola)}$$

Exercise 3.3

1. MR^2 (infact MI of any part of mass M of a ring of radius R about an axis passing through the geometrical centre and perpendicular to the plane of the ring is MR^2).

2. $\frac{MR^2}{2}$. These cases are same as circular disc. The distributions with respect to axis of rotation is same pattern as circular disc.

3. Moment of inertia of the plate about axis 1 (by taking rods perpendicular to axis 1)

$$I_1 = \frac{Mb^2}{3}$$

Moment of inertia of the plate about axis 2 (by taking rods perpendicular to axis 2)

$$I_2 = \frac{Ml^2}{12}$$

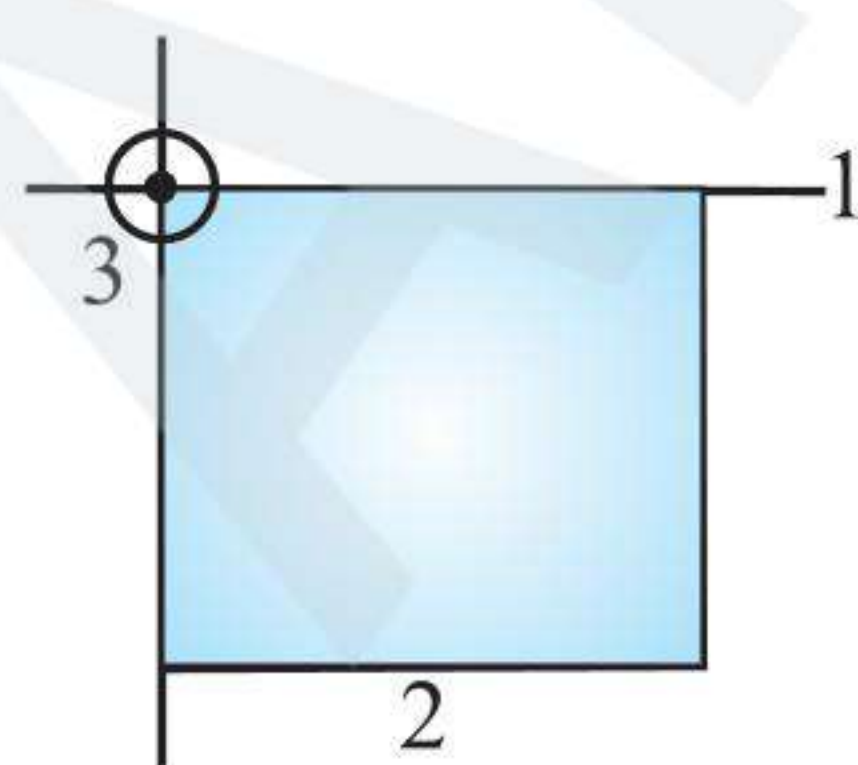
Moment of inertia of the plate about axis 3 (by taking rods perpendicular to axis 3)

$$I_3 = \frac{Mb^2}{12}$$

Moment of inertia of the plate about axis 4 (by taking rods perpendicular to axis 4)

$$I_4 = \frac{Ml^2}{3}$$

$$4. I_3 = I_1 + I_2 = \frac{Ml^2}{3} + \frac{Ml^2}{3} = \frac{2Ml^2}{3}$$



5. For calculation of the moment of inertia about centre, use parallel axis theorem

$$I = 2 \left[\frac{Ml^2}{12} + m \left(\frac{b}{2} \right)^2 \right] + 2 \left[\frac{mb^2}{12} + m \left(\frac{l}{2} \right)^2 \right]$$

$$I = \frac{2}{3} m(l^2 + b^2)$$

Hence, we can also use parallel axis theorem.

$$I_{PQ} = 2 \left(\frac{mb}{3} \right)^2 + mb^2 = \frac{5}{3} mb^2$$

$$6. I_P = I_{P_1} + I_{P_2}$$

$$I_P = \frac{Ml^2}{3} + \left(\frac{Ml^2}{12} + Mx^2 \right)$$

$$x^2 = l^2 + \left(\frac{l}{2} \right)^2 = \frac{5l^2}{4}$$

$$\Rightarrow I_P = \frac{Ml^2}{3} + \left[\frac{Ml^2}{12} + M \left(\frac{5l^2}{4} \right) \right] = \frac{5}{3} Ml^2$$

$$7. I = I_{\text{disc}} + I_{\text{ring}} + I_{\text{point masses}}$$

$$= \frac{mR^2}{2} + mR^2 + 4 \left[\frac{m}{4} \left(\frac{R}{2} \right)^2 \right] = \frac{7mR^2}{4}$$

8. In the system shown in the figure, the total moment of inertia can be given as the sum of moment of inertia of the rod and that of the spherical ball.

Moment of inertia of the rod about the axis passing through an end is $I_{\text{rod}} = \frac{Ml^2}{3}$.

Moment of inertia of the spherical ball about the axis passing through its geometrical centre is $I = \frac{2mr^2}{5}$.

Using parallel axis theorem, we get the moment of inertia about the axis passing through the end of rod as

$$I_{AA'} = \frac{2}{5} mr^2 + ml^2$$

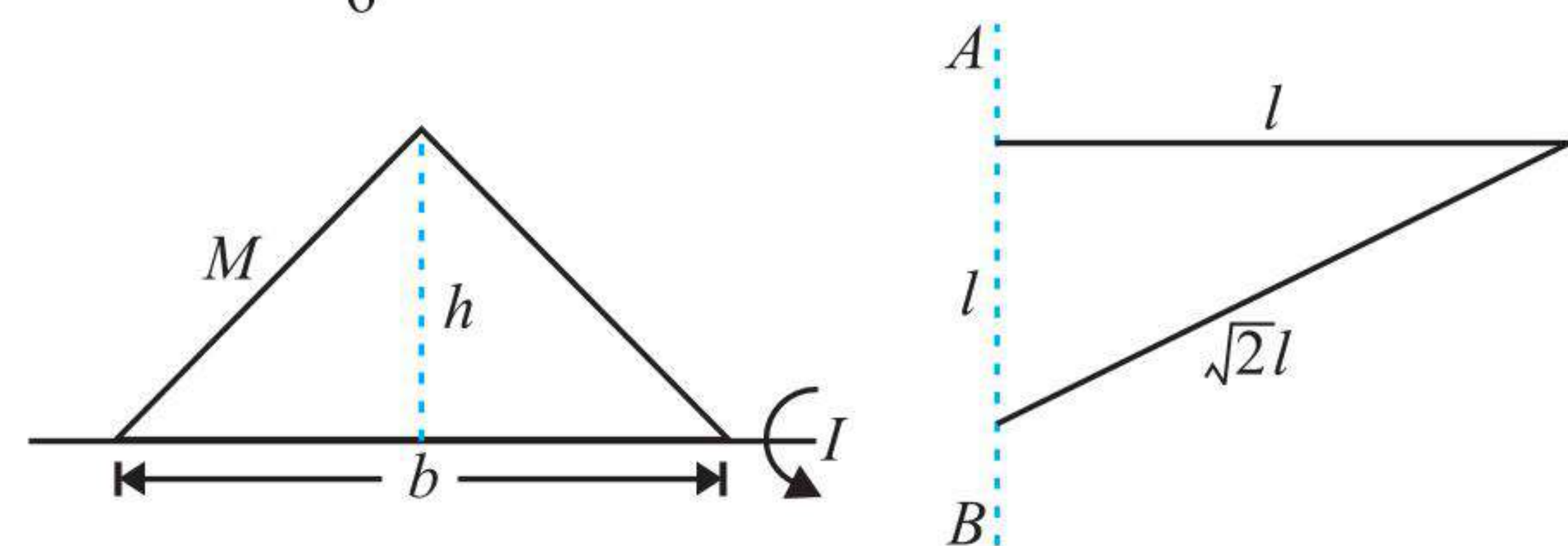
Thus total moment of inertia of the system can be given as

$$I_{\text{total}} = I_{\text{rod}} + I_{AA'}$$

$$= \frac{Ml^2}{3} + \left(\frac{2}{5} mr^2 + ml^2 \right)$$

9. We know moment of inertia of the triangular lamina about one of the side is

$$I = \frac{Mh^2}{6}$$



$$\text{Here in the problem } I = \frac{Ml^2}{6}$$

10. It is important to notice that the MI (Moment of inertia) of the remaining disc about the z -axis will be $I_{zz} = \frac{1}{2} MR^2$

$$\text{But } I_{xx} = I_{yy},$$

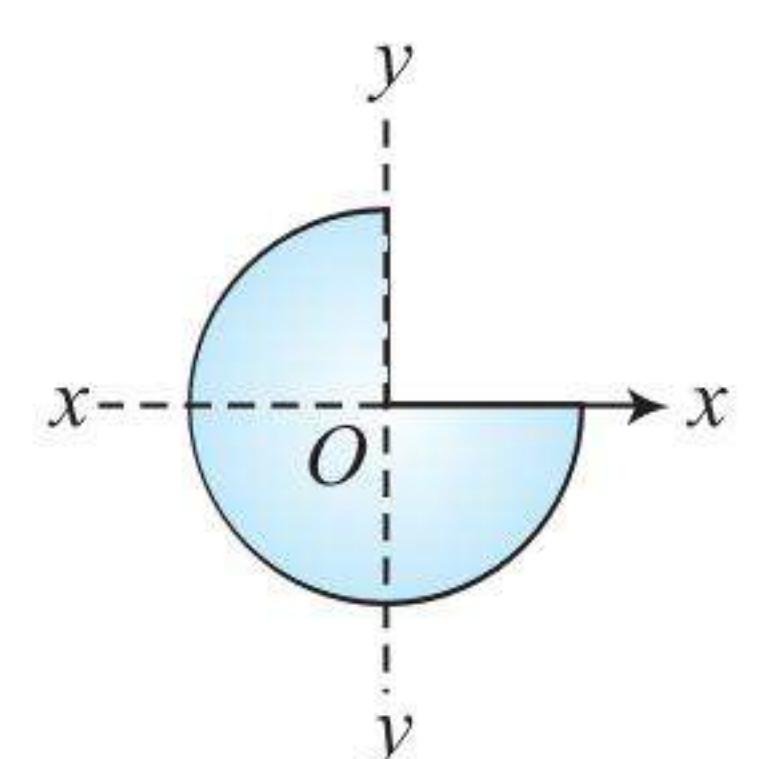
$$\therefore I_{xx} = \frac{1}{2} I_{zz} = \frac{1}{4} MR^2$$

$$11. I_0 = I_{CM} + Md^2$$

$$M(x^2 + y^2) = I_0 - I_{CM} \quad [= \text{a constant}]$$

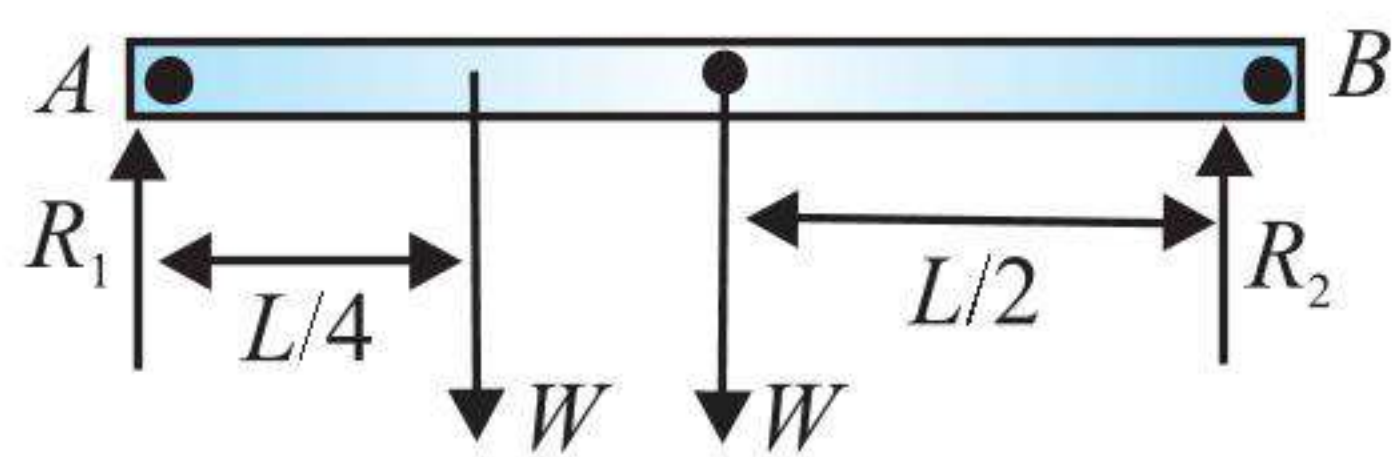
$$\Rightarrow M(x^2 + y^2) = M \left(l^2 + \frac{l^2}{4} \right)$$

$$\Rightarrow x^2 + y^2 = \frac{5l^2}{4}$$



Exercise 3.4

1. $R_1 + R_2 = 2W$... (i)

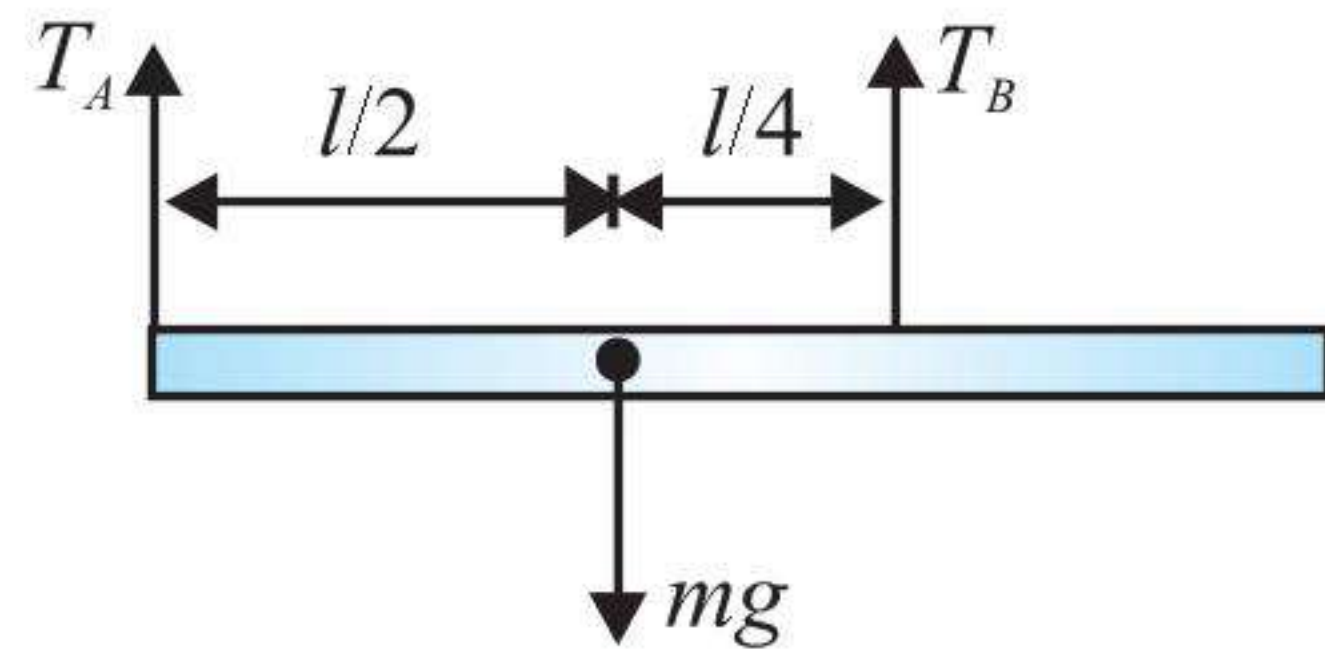


Balancing torque about A:

$$R_2 L = W \frac{L}{4} + W \frac{L}{2}$$

$$R_2 = \frac{3W}{4} \text{ and } R_1 = \frac{5W}{4}$$

2.



$$T_A + T_B = mg \quad \dots (i)$$

$$T_A \frac{l}{2} = T_B \frac{l}{4} \Rightarrow 2T_A = T_B \quad \dots (ii)$$

From Eqs. (i) and (ii),

$$T_A = \frac{mg}{3}, T_B = \frac{2mg}{3}$$

3. $f_1 = \mu_1 N_1, f_2 = \mu_2 N_2$

$$N_2 = f_1 = \mu_1 N_1 \quad \dots (i)$$

$$N_1 + f_2 = mg \quad \dots (ii)$$

In limiting case $f_1 = \mu_1 N_1$ and $f_2 = \mu_2 N_2$

Substituting the values of f_1 and f_2 in Eqs. (i) and (ii), we get

$$N_2 = \frac{\mu_1 mg}{1 + \mu_1 \mu_2}$$

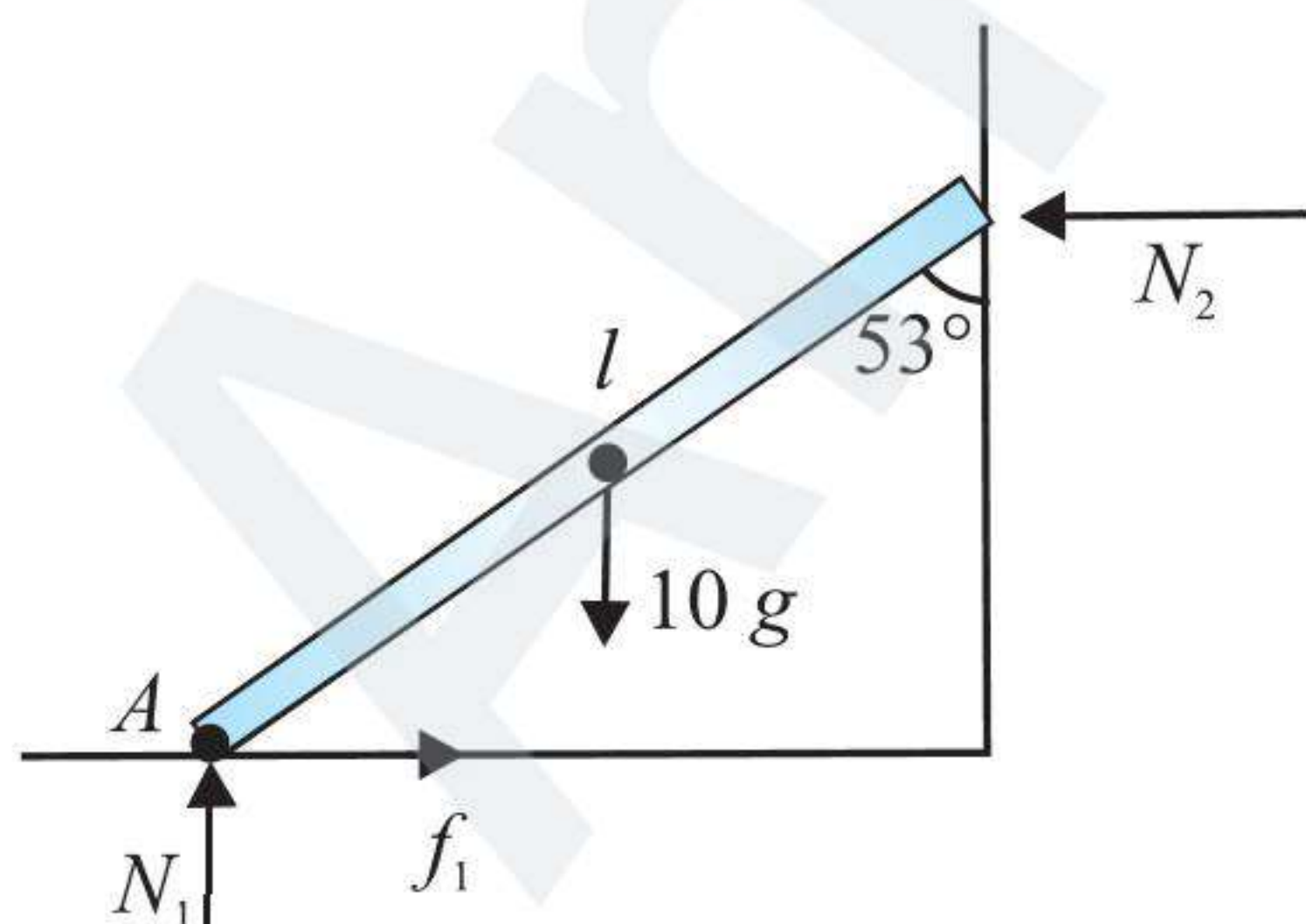
Taking torque about A and putting it to be zero,

$$mg \frac{l}{2} \cos \theta = f_2 l \cos \theta + N_2 l \sin \theta$$

$$mg \cos \theta = 2\mu_2 N_2 \cos \theta + 2N_2 \sin \theta$$

Put the value of N_2 and simplify to get $\tan \theta = \frac{1 - \mu_1 \mu_2}{2\mu_1}$

4. $N_1 = 10g = 100 \text{ N}$



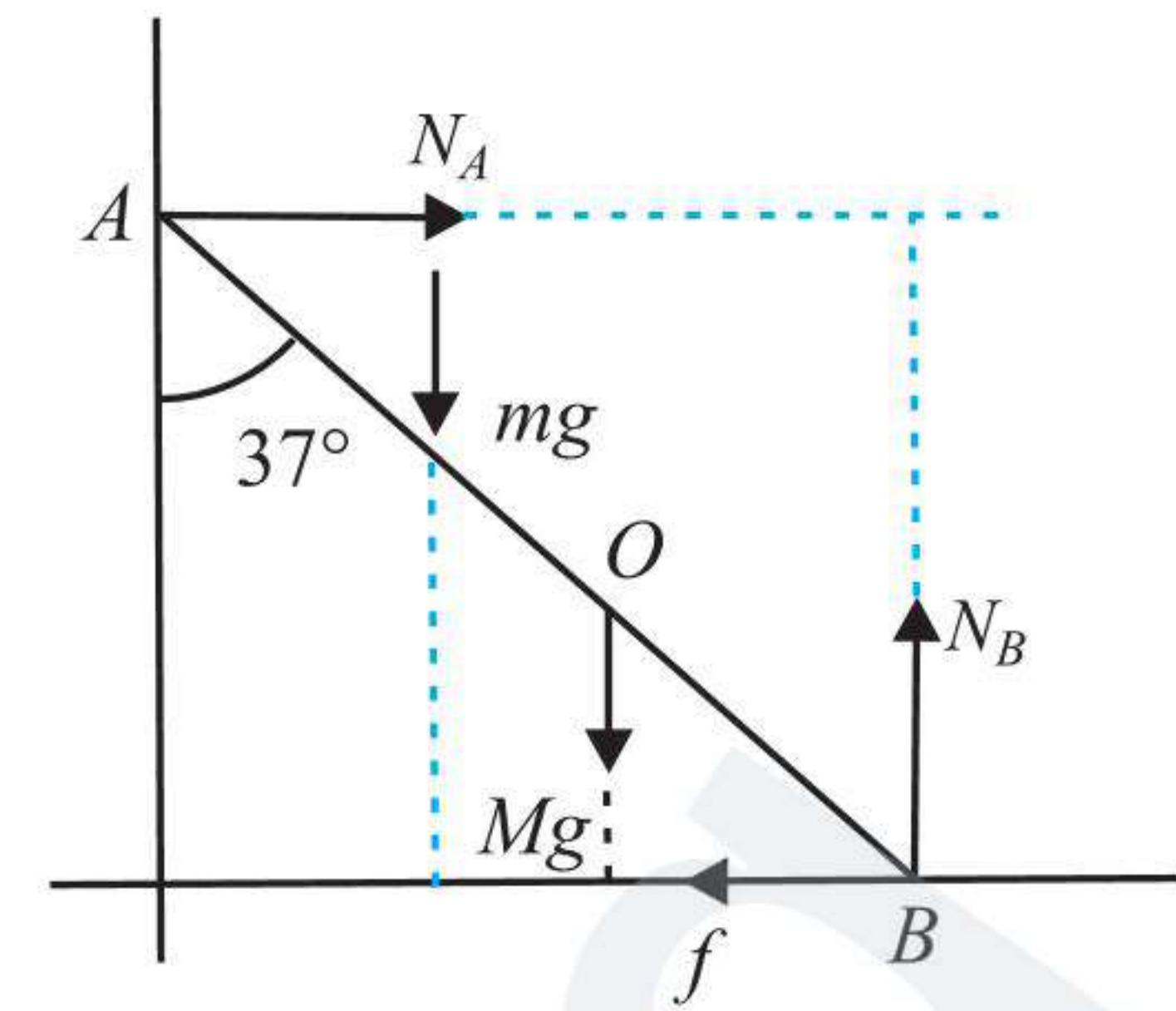
Balancing torque about A:

$$10g \frac{l}{2} \cos 37^\circ = N_2 l \cos 53^\circ$$

$$N_2 = \frac{200}{3} \text{ N}$$

Now $f_1 = N_2 = \frac{200}{3} \text{ N}$

5. If the electrician works safely, the ladder + man system should be in equilibrium



For equilibrium, $\sum F = 0$ and $\sum \tau = 0$

for $\sum F_x = 0, N_A = f$

for $\sum F_y = 0, N_B = mg + Mg = 60 \times 10 + 16 \times 10 = 760 \text{ N}$

Taking torque about B,

$$N_A 10 \cos 37^\circ = mg 8 \sin 37^\circ + Mg 5 \sin 37^\circ$$

$$N_A = \frac{\tan 37^\circ}{10} (65 \times 10 \times 8 + 16 \times 10 \times 5)$$

From (i), $f = N_A = 420 \text{ N}$

For maximum value of coefficient of friction,

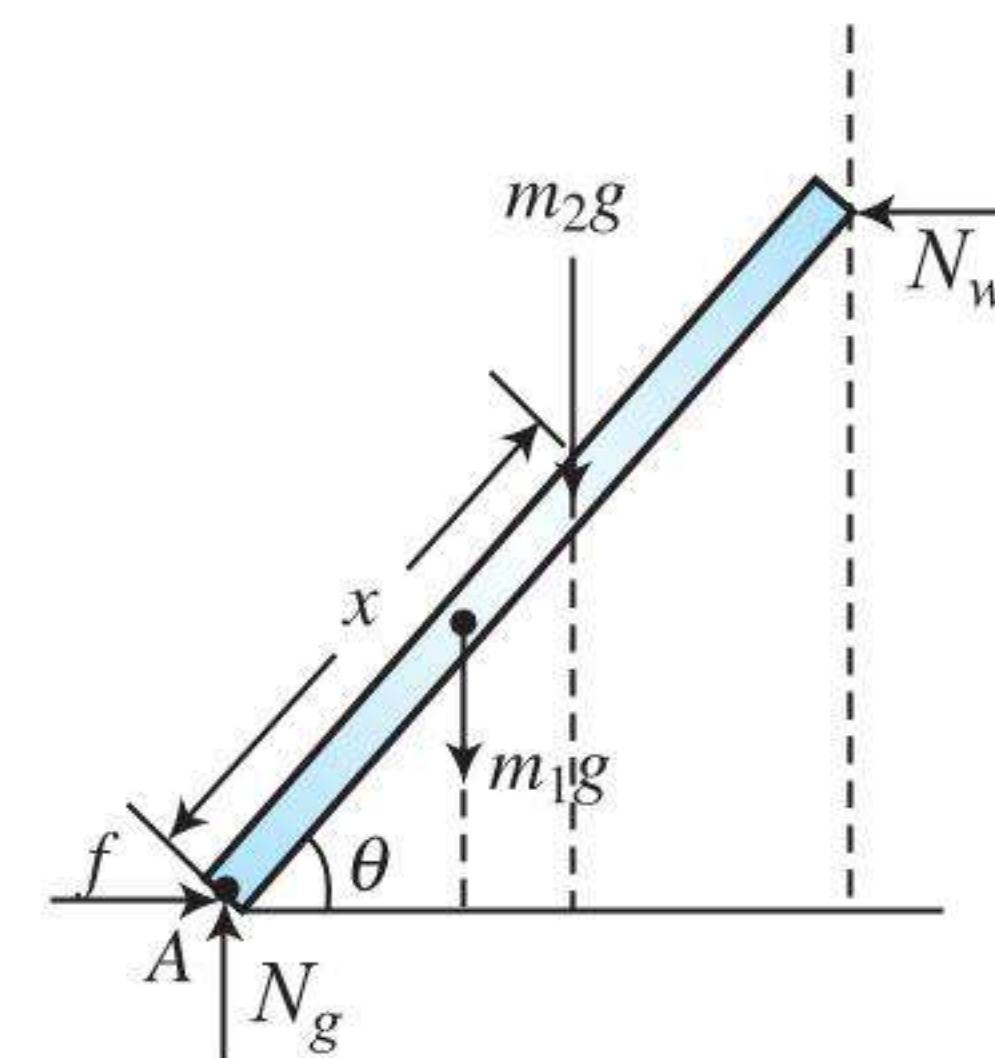
$$f \leq \mu N_B$$

$$\mu \geq \frac{f}{N_B} = \frac{420}{760} = 0.55$$

6. (a) $\sum F_x = f - N_w = 0 \quad \dots (i)$

$$\sum F_y = N_g - m_1 g - m_2 g = 0 \quad \dots (ii)$$

$$\sum \tau_A = -m_1 g \left(\frac{L}{2} \right) \cos \theta - m_2 g x \cos \theta + N_w L \sin \theta = 0$$



From the torque equation, $N_w = \left[\frac{1}{2} m_1 g + \left(\frac{x}{L} \right) m_2 g \right] \cot \theta$

Then, from Eq. (i), $f = N_w = \left[\frac{1}{2} m_1 g + \left(\frac{x}{L} \right) m_2 g \right] \cot \theta$

and from Eq. (ii), $N_g = (m_1 + m_2)g$

(b) If the ladder is on the verge of slipping when $x = d$, then

$$\mu = \frac{f_{x=d}}{N_g} = \frac{\left(\frac{m_1}{2} + \frac{m_2 d}{L} \right) \cot \theta}{m_1 + m_2}$$

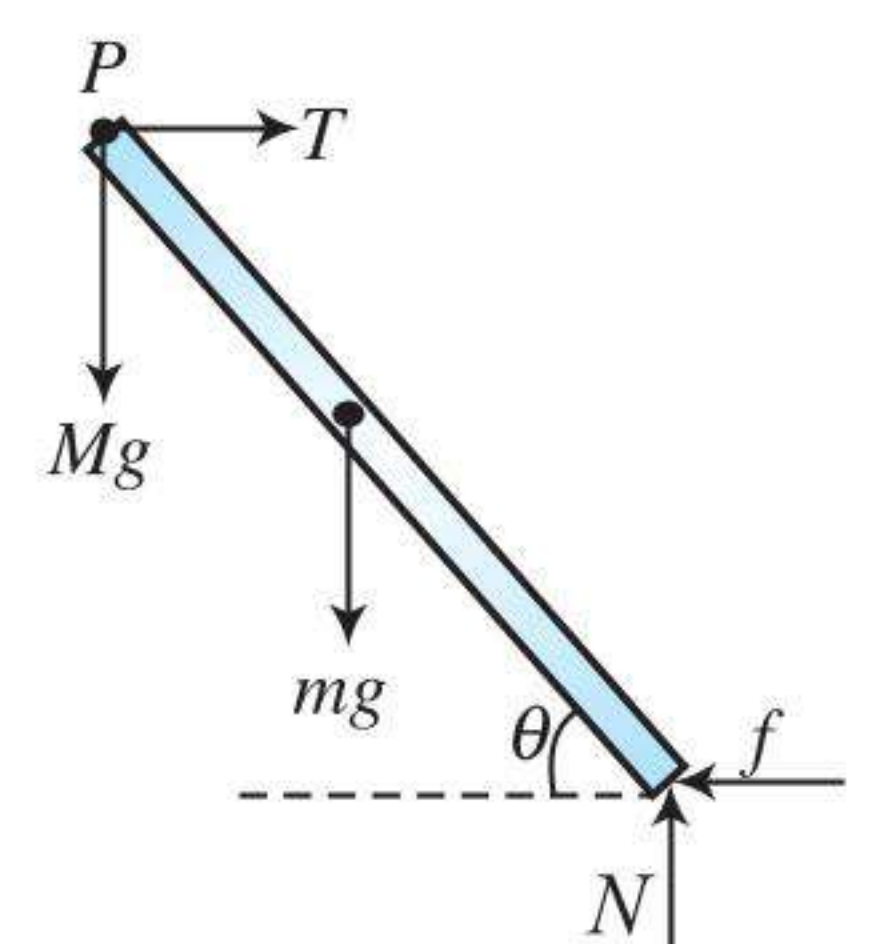
7. (a) We can use $\sum F_x = \sum F_y = 0$

and $\sum \tau = 0$ with pivot point at the contact on the floor.

Then $\sum F_x = T - \mu_s N = 0$

$$\sum F_y = N - Mg - mg = 0$$

$$\sum \tau = Mg(L \cos \theta) + mg \left(\frac{L}{2} \cos \theta \right) - T(L \sin \theta) = 0$$



Solving the above equations gives

$$M = \frac{m}{2} \left(\frac{2\mu_s \sin \theta - \cos \theta}{\cos \theta - \mu_s \sin \theta} \right)$$

This answer is the maximum value for M if $\mu_s < \cot \theta$. If $\mu_s \geq \cot \theta$, the mass M can increase without limit. It has no maximum value, and part (b) cannot be answered as stated either. In the case $\mu_s < \cot \theta$, we proceed.

(b) At the floor, we have the normal force in the y -direction and frictional force in the x -direction. The reaction force then is

$$R = \sqrt{N^2 + (\mu_s N)^2} = (M + m)g\sqrt{1 + \mu_s^2}$$

At point P , the force of the beam on the rope is

$$F = \sqrt{T^2 + (Mg)^2} = g\sqrt{M^2 + \mu_s^2(M + m)^2}$$

8. The forces acting on different parts are shown in figure. Consider the vertical equilibrium of 'the ladder plus the person' system. The forces acting on this system are its weight $(80 \text{ kg})g$ and the contact force $N + N = 2N$ due to the floor.

Thus, $2N = (80 \text{ kg})g$

or $N = (40 \text{ kg})(10 \text{ m/s}^2) = 400 \text{ N}$

Next consider the equilibrium of the left leg of the ladder. Taking torques of the forces acting on it about the upper end,

$$N(2 \text{ m}) \tan 30^\circ = T(1 \text{ m})$$

$$\text{or } T = N \frac{2}{\sqrt{3}} = (400 \text{ N}) \times \frac{2}{\sqrt{3}} = \frac{80}{\sqrt{3}} \text{ N}$$

9. When it is on the verge of slipping, the cylinder is in equilibrium.

$$\sum F_x = 0, \quad f_1 = N_2 = \mu_s N_1 \text{ and } f_2 = \mu_s N_2$$

$$\sum F_y = 0, \quad P + N_1 + f_2 = F_g$$

$$\sum \tau = 0, \quad P = f_1 + f_2$$

As P grows so do f_1 and f_2

Therefore, since $\mu_s = \frac{1}{2}$, $f_1 = \frac{N_1}{2}$ and $f_2 = \frac{N_2}{2} = \frac{N_1}{4}$

$$\text{then } P + N_1 + \frac{N_1}{4} = F_g$$

$$\text{and } P = \frac{N_1}{2} + \frac{N_1}{4} = \frac{3}{4}N_1$$

$$\text{so } P + \frac{5}{4}N_1 = F_g$$

$$\text{becomes } P + \frac{5}{4} \left(\frac{4}{3}P \right) = F_g \text{ or } \frac{8}{3}P = F_g$$

$$\text{Therefore } P = \frac{3}{8}F_g$$

10. (a) If the acceleration is a , we have $P_x = ma$ and $P_y + N - F_g = 0$. Taking the origin at the centre of gravity, the torque equation gives

$$P_y(L - d) + P_x h - Nd = 0$$

$$\text{Solving these equations, we find } P_y = \frac{F_g}{L} \left(d - \frac{ah}{g} \right)$$

(b) If $P_y = 0$, then

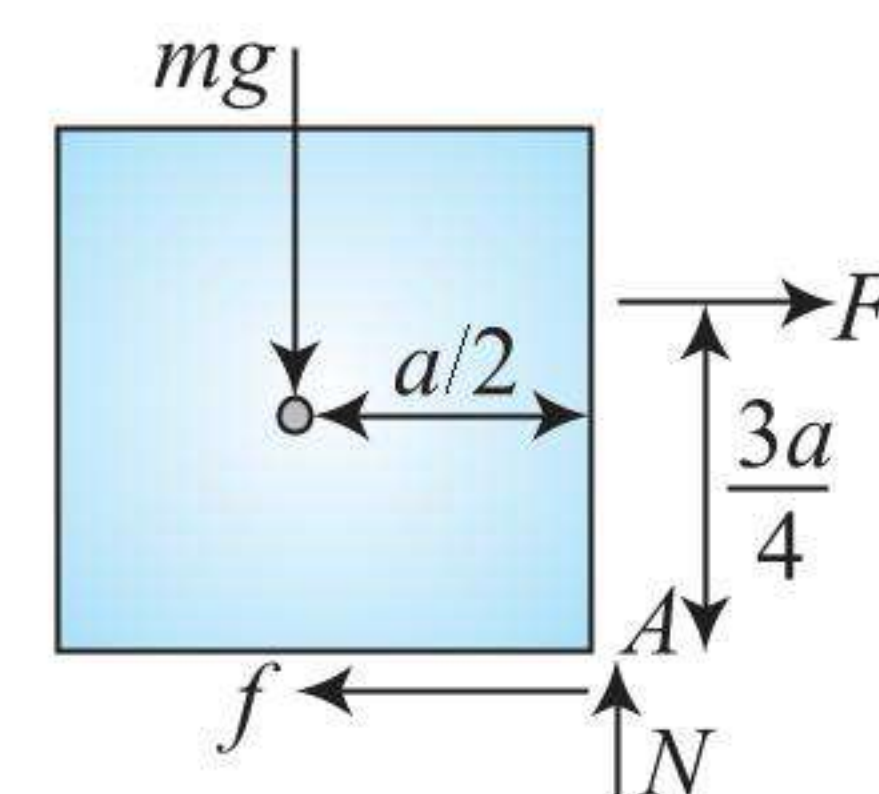
$$d = \frac{ah}{g} = \frac{(2.00 \text{ ms}^{-2})(1.50 \text{ m})}{10 \text{ ms}^{-2}} = 0.30 \text{ m}$$

Exercise 3.5

1. In the limiting case normal reaction will pass through A . The cube will tip about A if torque of F exceeds the torque of mg .

$$\text{Hence, } F \left(\frac{3a}{4} \right) > \left(\frac{a}{2} \right) mg \text{ or } F > \frac{2}{3} mg$$

Therefore, minimum value of F is $\frac{2}{3} mg$.



2. As there is no relative sliding between the blocks, the common acceleration of system $a = \frac{F}{2m} = \frac{kt}{2m}$

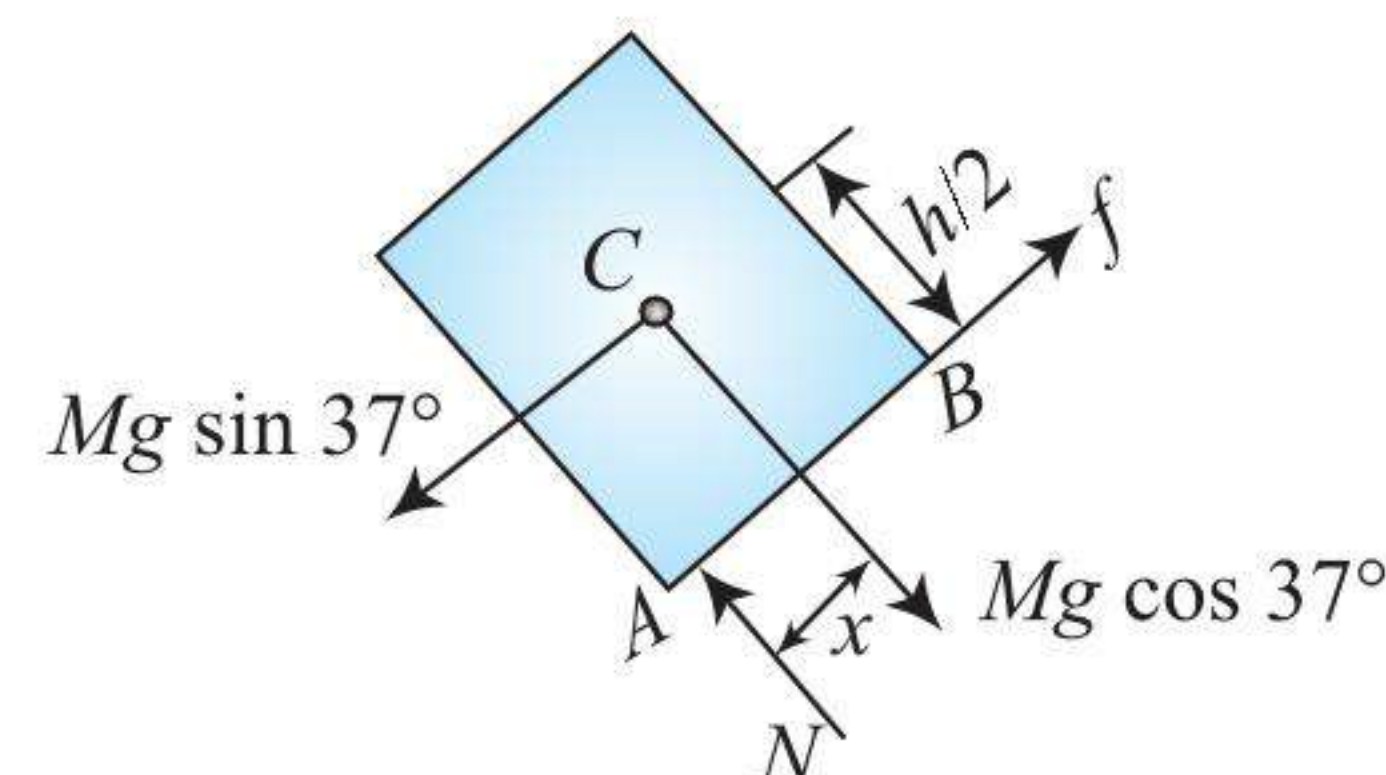
The force diagram of block B in reference frame attached to A is shown in figure. The normal force passes through left edge at the instant the block is about to topple.

The block 'A' will be on verge of toppling about point P torque of pseudo force equals torque due to its weight.

$$ma \frac{h}{2} = mg \frac{d}{2}$$

$$m \frac{kt}{2m} \frac{h}{2} = mg \frac{d}{2} \quad \therefore t = \frac{2dgm}{kh}$$

3. Here $\mu = 0.8$, $\frac{b}{h} = \frac{1}{3} = 0.33$; $\tan \theta = \frac{3}{4} = 0.75$



Since $\mu > b/h$ and $\mu > \tan \theta$, therefore the block has a greater tendency to topple over.

From the condition of translation equilibrium, we have

$$N = Mg \cos 37^\circ = (50)(10) \left(\frac{4}{5} \right) \text{ or } N = 400 \text{ N} < f_{\max}$$

$$f_{\max} = \mu N = 0.8(400) = 320 \text{ N}$$

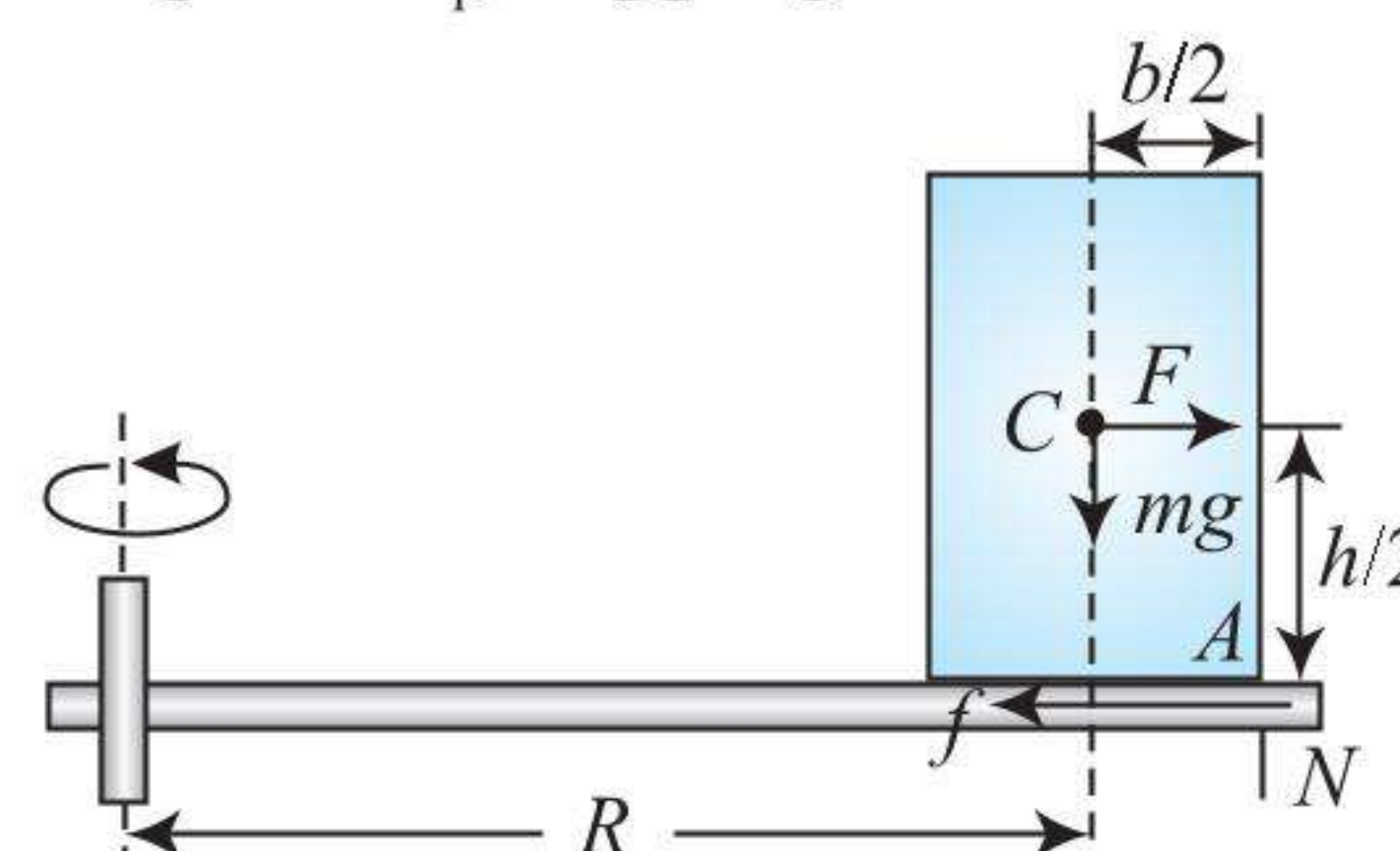
$$f = Mg \sin 37^\circ = (50)(10) \left(\frac{3}{5} \right) = 300 \text{ N}$$

For rotational equilibrium, $\sum \tau_c = 0 \Rightarrow Nx = f \frac{h}{2}$

$$\text{Thus, } x = \frac{f}{N} \frac{h}{2} = \left(\frac{300}{400} \right) \left(\frac{3}{2} \right) = \frac{9}{8} > \frac{b}{2}$$

Since the value of x is more than $b/2$ which is not possible, therefore the block topples over.

4. An observer on the disc, will observe a pseudo force F acting on the cylinder (the centrifugal force). The block can fall off either by slipping away or toppling about point P , depending on whichever takes place first. In any case friction force f acts opposite to F . The critical angular speed ω_1 , slipping occurs when $F = f$



$$\text{or } m\omega_1^2 R = \mu mg \Rightarrow \omega_1 = \sqrt{\frac{\mu g}{R}}$$

F tries to rotate the block about P , but the weight W opposes it. The rotation is possible if,

$$\tau_F \geq \tau_W$$

$$F \frac{h}{2} = mg \frac{b}{2} \Rightarrow F = mg \frac{b}{h}$$

$$m\omega_2^2 R = mg \frac{b}{h} \Rightarrow \omega_2 = \sqrt{\frac{bg}{hR}}$$

$$\therefore \mu > \frac{b}{h} \quad (\text{given})$$

$\therefore \omega_1 > \omega_2$ and the block falls off by rolling over at $\omega_1 = \omega_2$.

5. When the car is on the point of rolling over, the normal force on its inside wheels is zero.

$$\sum F_y = ma_y: N - mg = 0$$

$$\sum F_x = ma_x: f = \frac{mv^2}{R}$$

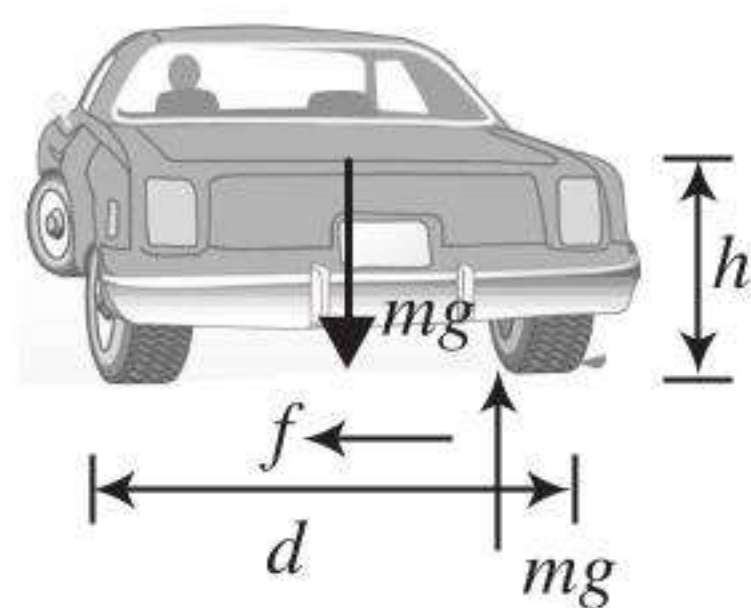
Take torque about the centre of mass:

$$fh - N \frac{d}{2} = 0$$

Then by substitution,

$$\frac{mv_{\max}^2}{R} h - \frac{mgd}{2} = 0 \Rightarrow v_{\max} = \sqrt{\frac{gdR}{2h}}$$

A wider wheelbase (larger d) and a lower centre of mass (smaller h) will reduce the risk of rollover.



Exercise 3.6

1. (a) $\tau_H = I_H \alpha$

$$mg \frac{l}{2} = \frac{ml^2}{3} \alpha \Rightarrow \alpha = \frac{3g}{2l}$$

$$(b) a_{tA} = \alpha l = \frac{3g}{2l} l = \frac{3g}{2}$$

$$a_{CA} = \omega^2 r = 0 \times l = 0$$

($\omega = 0$ just after release)

2. Free-body diagram of the rod

In the previous problem

$$a = \frac{3g}{2l}$$

$$\text{Hence, } a_{CM} = a_t = \frac{\alpha l}{2} = \frac{3g}{4}$$

From the free-body diagram in the vertical direction

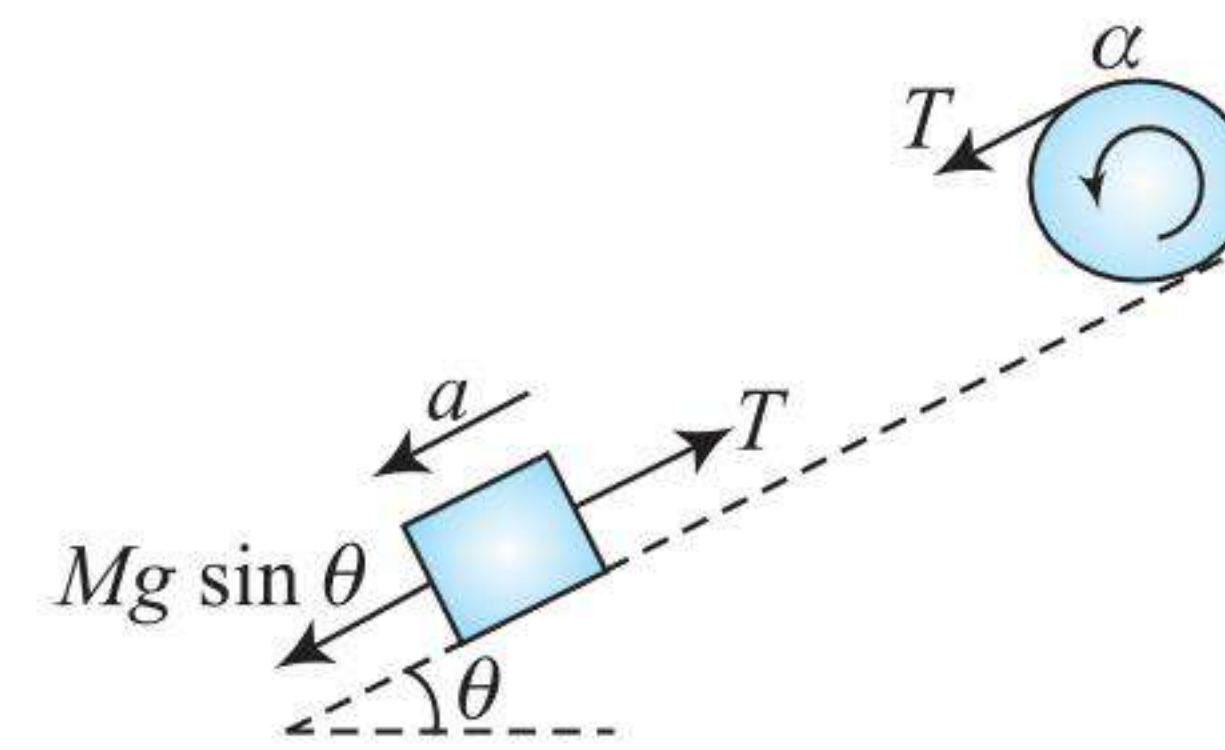
$$mg - N_1 = ma_{CM} = m \left(\frac{3g}{4} \right)$$

$$\Rightarrow N_1 = \frac{mg}{4}$$

In horizontal direction, $F_{\text{ext}} = ma_{CM} \Rightarrow N_2 = 0$

($\therefore a_{CM}$ in horizontal = 0 as $\omega = 0$ just after release).

3. Suppose the deceleration of the block is a . The linear deceleration of the rim of the wheel is also a . The angular deceleration of the wheel is $\alpha = a/r$. If the tension in the string is T , the equations of motion are as follows:



$$Mg \sin \theta - T = Ma \text{ and } Tr = I\alpha = \frac{Ia}{r}.$$

Eliminating T from these equations,

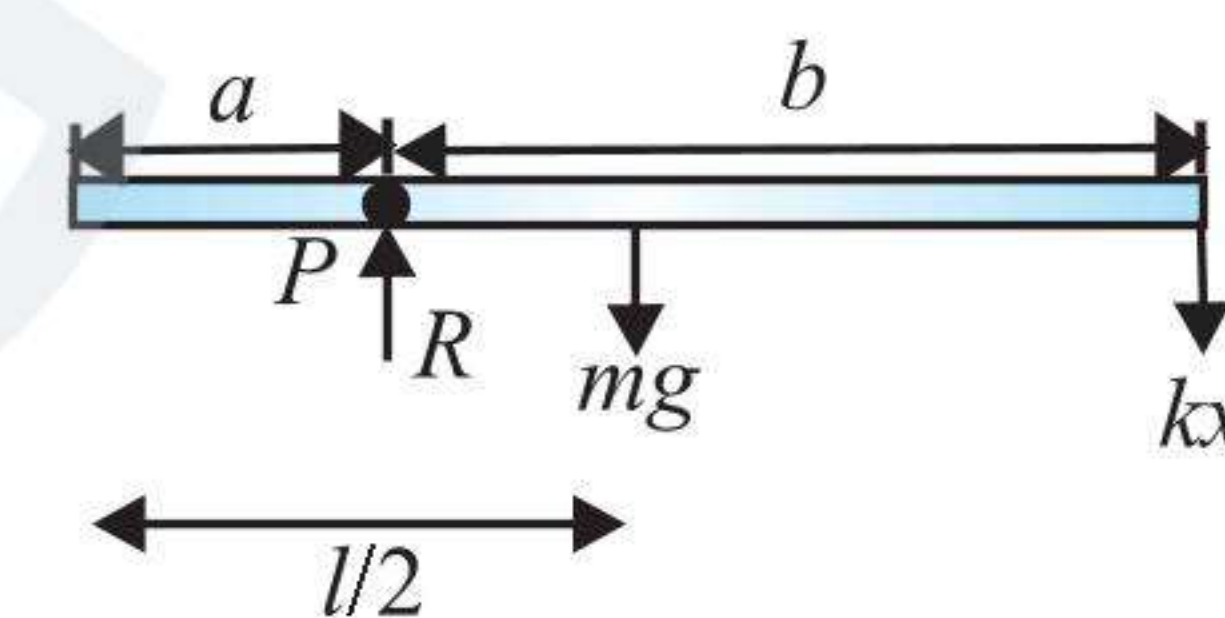
$$Mg \sin \theta - I \frac{a}{r^2} = Ma$$

$$\Rightarrow a = \frac{Mg r^2 \sin \theta}{(I + Mr^2)}$$

The initial velocity of the block up the incline is $v = \omega r$. Thus, the distance moved by the block before stopping is

$$x = \frac{v^2}{2a} = \frac{\omega^2 r^2 (I + Mr^2)}{2Mg r^2 \sin \theta} = \frac{(I + Mr^2) \omega^2}{2Mg \sin \theta}$$

4. Just after burning of a thread, torque about point P .



$$\tau_p = mg \left(\frac{l}{2} - a \right) + kxb$$

$$I_p \alpha = mg \left(\frac{l}{2} - a \right) + kxb$$

$$\text{where, } I_p = \frac{ml^2}{12} + m \left(\frac{l}{2} - a \right)^2 = \frac{56}{300} \text{ kg m}^2$$

$$\therefore \frac{56}{300} \alpha = \frac{28}{5} \Rightarrow \alpha = 30 \text{ rad/s}$$

Equation of motion at this instant

$$mg + kx - R = m\alpha_0 = m\alpha \left(\frac{l}{2} - a \right)$$

where a_0 is the acceleration.

$$20 + 6 - R = 2 \times 30 \times \frac{1}{10}$$

$$\Rightarrow R = 20 \text{ N}$$

5. The acceleration whole system, $a_1 = \frac{F}{m_1 + m_2}$

The acceleration of the point K w.r.t. the axis of the cylinder

$$a_2 = \alpha R$$

where α is given by

$$FR = I\alpha \text{ or } \alpha = \frac{FR}{m_1 R^2 / 2} = \frac{2F}{m_1 R}$$

$$\Rightarrow a_2 = \frac{2F}{m_1}$$

$\therefore$ The acceleration of the point K w.r.t. ground

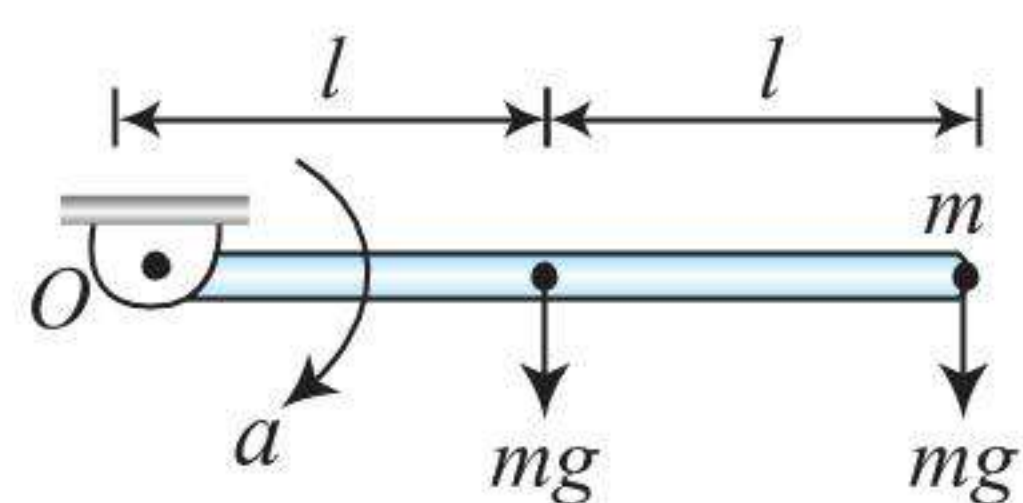
$$= a_1 + a_2 = \frac{F}{m_1 + m_2} + \frac{2F}{m_1} = F \frac{(3m_1 + 2m_2)}{m_1(m_1 + m_2)}$$

6. (i) $F_1 - F_2 = (m + M)a \Rightarrow a = \frac{F_1 - F_2}{m + M}$

$$(ii) F_1 r = \frac{mr^2}{2} \alpha \Rightarrow \alpha = \frac{2F_1}{mr}$$

7. (i) Torque equation: The internal forces and the reaction forces at O . Then the net torque about O is

$$\tau_0 = mgl + mg(2l) = 3mgl$$



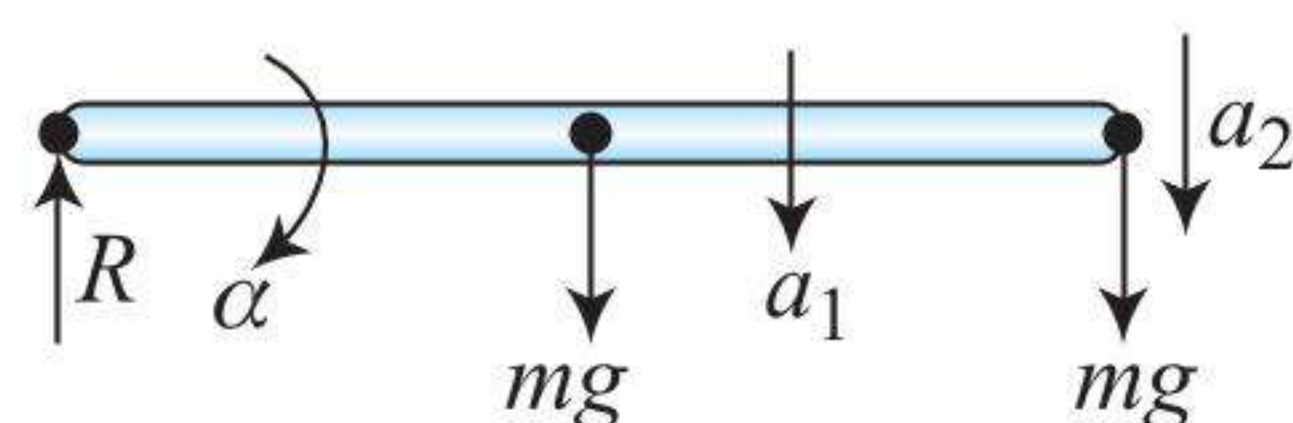
which produces an angular acceleration α . Using Newton's 2nd law of rotation $\tau = I\alpha$,

We have $3mgl = I_0\alpha$ where $I_0 = ml^2 + m(2l)^2 = 5ml^2$

This gives $\alpha = \frac{3g}{5l}$

- (ii) Force equation: Referring FBD, we have

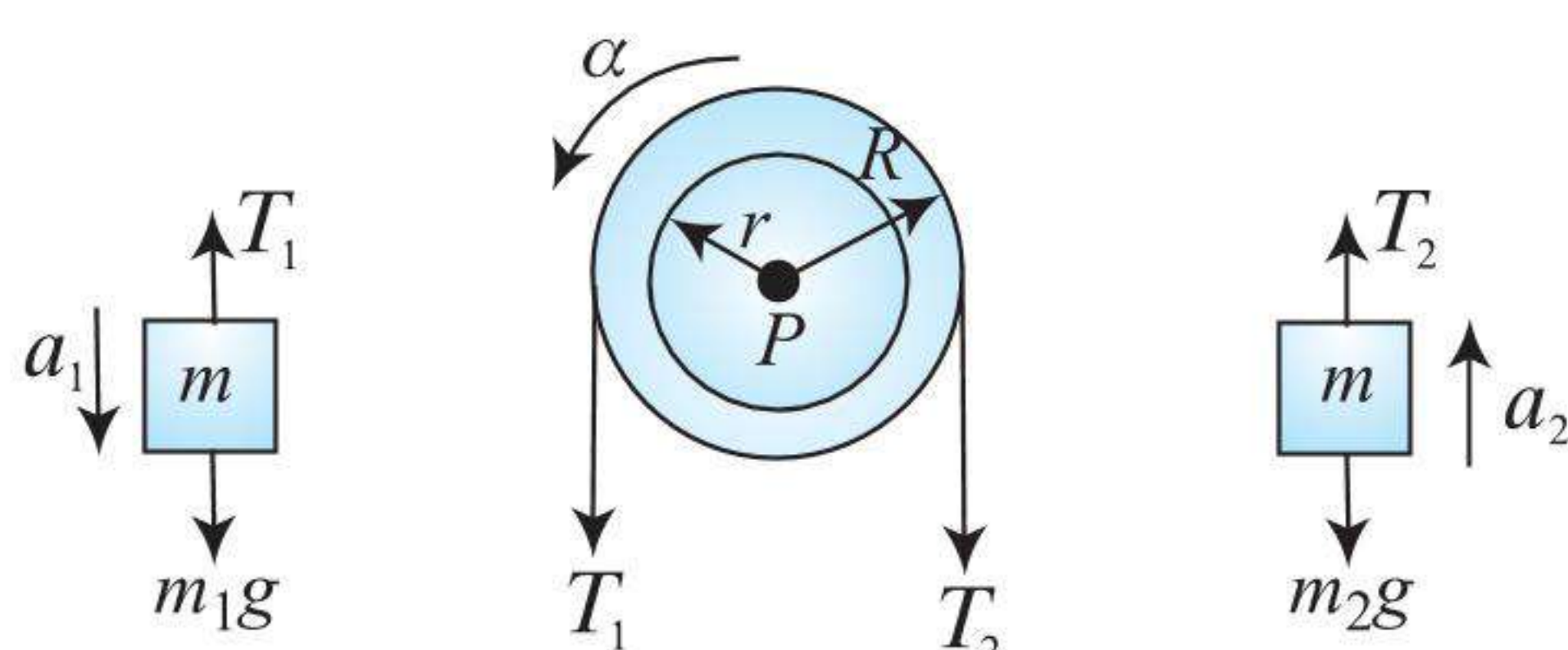
$$F_y = R - mg - mg = -ma_1 - ma_2$$



Substituting $a_1 = l\alpha$ and $a_2 = 2l\alpha$, we have $R = m(2g - 3l\alpha)$

Now substituting $\alpha = \frac{3g}{5l}$, we have $R = \frac{mg}{5}$

8. Torque equation: $T_1R - T_2r = I_c\alpha$... (i)



Writing force equation for block '1'

$$mg - T_1 = ma_1 = m(R\alpha) \Rightarrow T_1 = mg - m(R\alpha) \quad \dots (ii)$$

Writing force equation for block '2'

$$T_2 - mg = ma_2 \Rightarrow T_2 = mg + m(r\alpha) \quad \dots (iii)$$

From (i), (ii) and (iii)

$$(mg - mR\alpha)R - (mg + mr\alpha)r = I_c\alpha$$

$$mg(R - r) = [I_c + m(R^2 + r^2)]\alpha$$

$$\alpha = \frac{mg(R - r)}{I_c + m(R^2 + r^2)}$$

$$\text{But } I_c = mk^2 \Rightarrow \alpha = \frac{mg(R - r)}{mk^2 + m(R^2 + r^2)} = \frac{g(R - r)}{k^2 + (R^2 + r^2)}$$

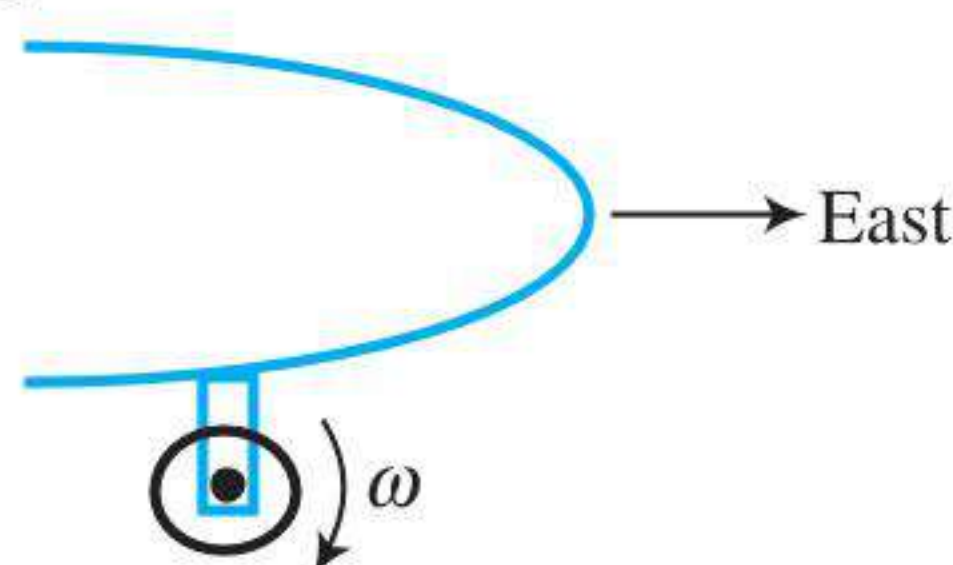
$$\text{Hence } a_1 = \alpha R = \frac{gR(R - r)}{k^2 + (R^2 + r^2)}$$

$$\text{and } a_2 = \alpha r = \frac{gr(R - r)}{k^2 + (R^2 + r^2)}$$

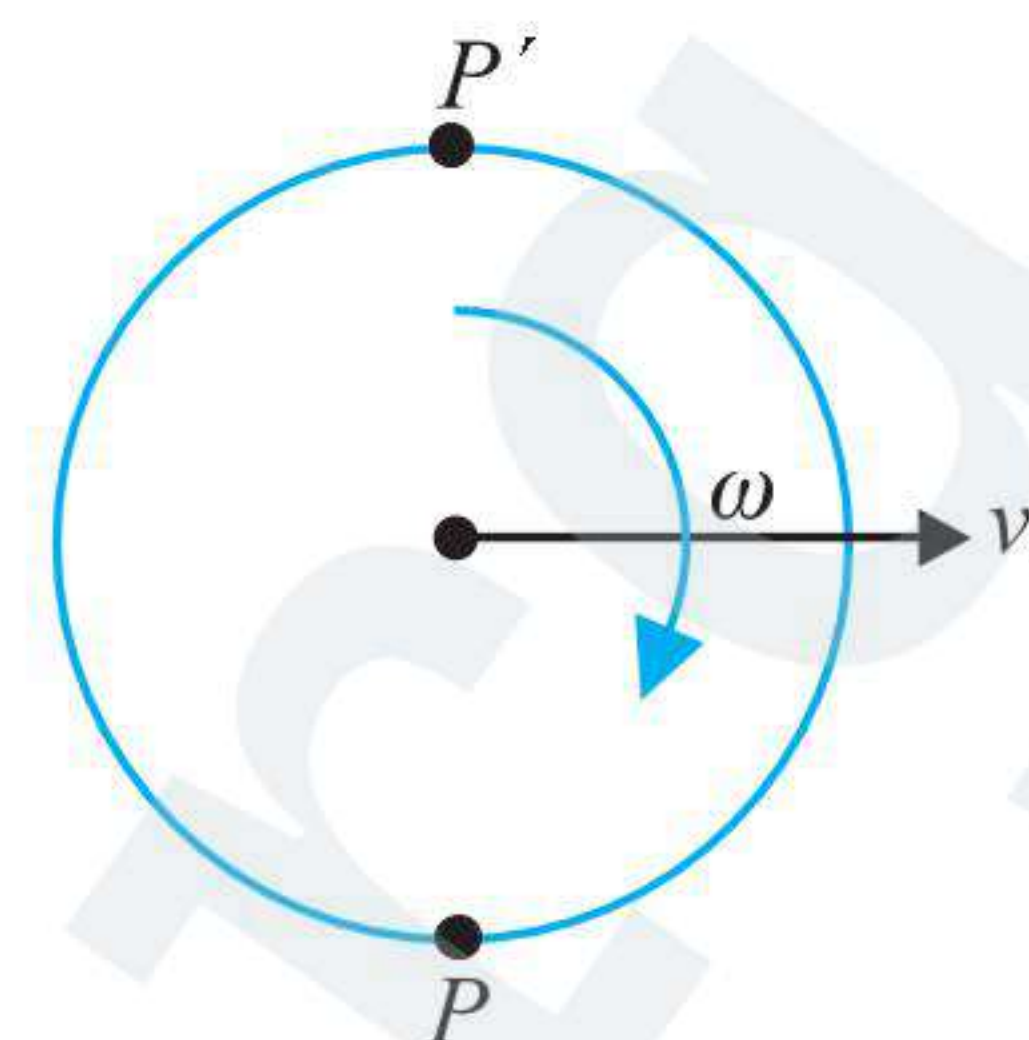
Exercises

Single Correct Answer Type

1. (4) The wheel should be rotated as shown. It is clockwise, so the direction of angular acceleration is inward or in north direction.



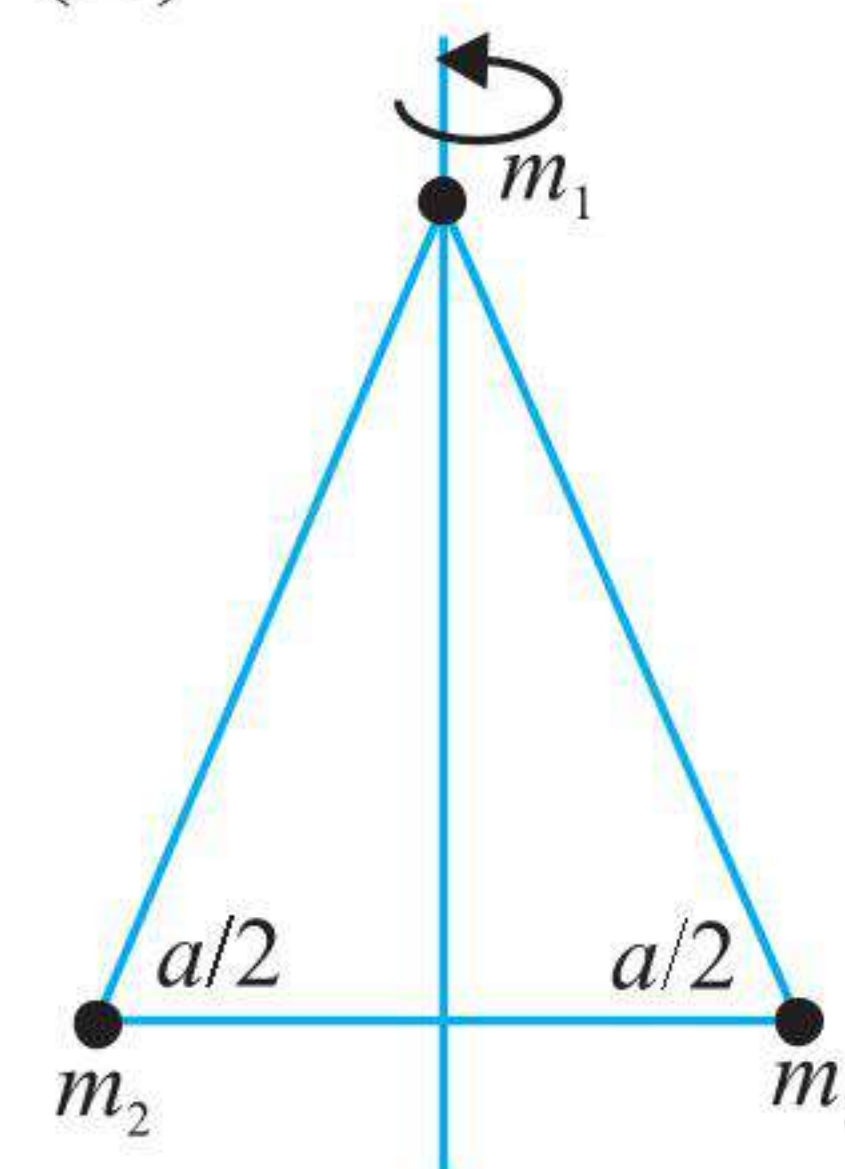
2. (2) Farther the mass from axis, greater will be the moment of inertia.
3. (4) The instantaneous axis of rotation will fall on the outside of the sphere. Since in case of pure rotation, the instantaneous axis of rotation will be on the centre. In case of pure rolling, the instantaneous axis of rotation will be on P .



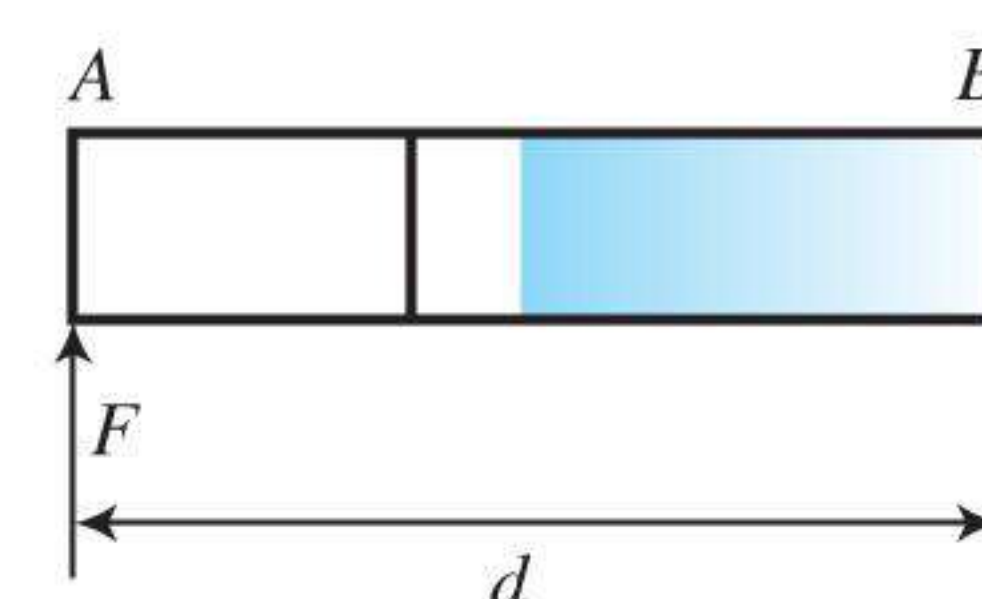
But if $v_c > \omega R$, then the instantaneous axis of rotation will fall out of the sphere.

4. (3) $I_B = I_A + Md^2$
5. (4) Distribution of mass is farthest.
6. (3) The distribution of mass is nearest to axis xx , hence moment of inertia is least about the xx -axis.
7. (2) MOI is $\sum m_i r_i^2$. About BC masses are spread far away than about any other axis.

$$8. (2) I = m_2 \left(\frac{a}{2} \right)^2 + m_3 \left(\frac{a}{2} \right)^2$$



9. (2) $\tau = I\alpha$, $\alpha = \tau/l$
- (1) $\tau = Fd$, I large
 α minimum
- (2) $\tau = Fd$, I minimum
 α maximum
- (3) $\tau = \frac{Fd}{2}$
- (4) $\tau = \frac{Fd}{2}$



10. (3) Because the planes of two rings are mutually perpendicular and centres are coincident, hence an axis, which is passing through the centre of one of the rings and to its plane, will be along the diameter of other ring. Hence, moment of inertia of the system

$$= I_{CM} + I_{diameter} = mr^2 + \frac{mr^2}{2} = \frac{3}{2} mr^2.$$

11. (2) According to the theorem of parallel axis,

$$I = I_{CG} + M(2R)^2$$

where I_{CG} = MI about an axis through the centre of gravity.

$$I = \frac{2}{5} MR^2 + 4MR^2 = \frac{22}{5} MR^2$$

$$\text{or } MK^2 = \frac{22}{5} MR^2 \Rightarrow K = \sqrt{\frac{22}{5}} R$$

12. (2) Given that $I_Q = 4I_P$

Here $I = \text{mass} \times (\text{radius})^2$. If σ is the linear mass density (mass/length) of the wire, then

$$\sigma \times 2\pi(nr) \times n^2 r^2 = 4(\sigma \times 2\pi r \times r^2)$$

$$\Rightarrow n^3 = 4 \Rightarrow n = (4)^{\frac{1}{3}}$$

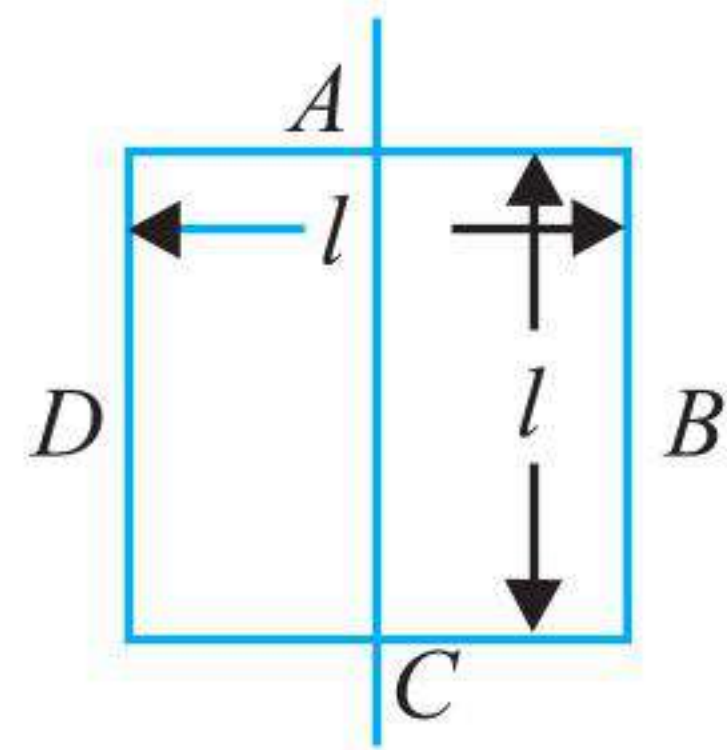
13. (3) Let M be the mass of each disc. Let R_A and R_B be the radii of discs A and B , respectively. Then $M = \pi R_A^2 t d_A = \pi R_B^2 t d_B$.

As $d_A > d_B$, so $R_A < R_B$. Now,

$$I_A = \frac{1}{2} MR_A^2, \quad I_B = \frac{1}{2} MR_B^2$$

$$\frac{I_A}{I_B} = \frac{R_A^2}{R_B^2} < 1, \text{ i.e., } I_A < I_B$$

14. (4) $I_{\text{median line}} = I_A + I_B + I_C + I_D$
- $$= 2 \times \frac{MF^2}{12} + 2M\left(\frac{l}{2}\right)^2$$
- $$= \frac{MF^2}{6} + \frac{MF^2}{2} = \frac{2}{3} MF^2$$



15. (1) Mass of each of the four parts = $\frac{M}{3}$

Mass of the plate including the cut piece = $\frac{4M}{3}$

Moment of inertia of the whole plate (including the cut piece) about the said axis = $\left(\frac{4M}{3}\right) \frac{l^2}{6}$

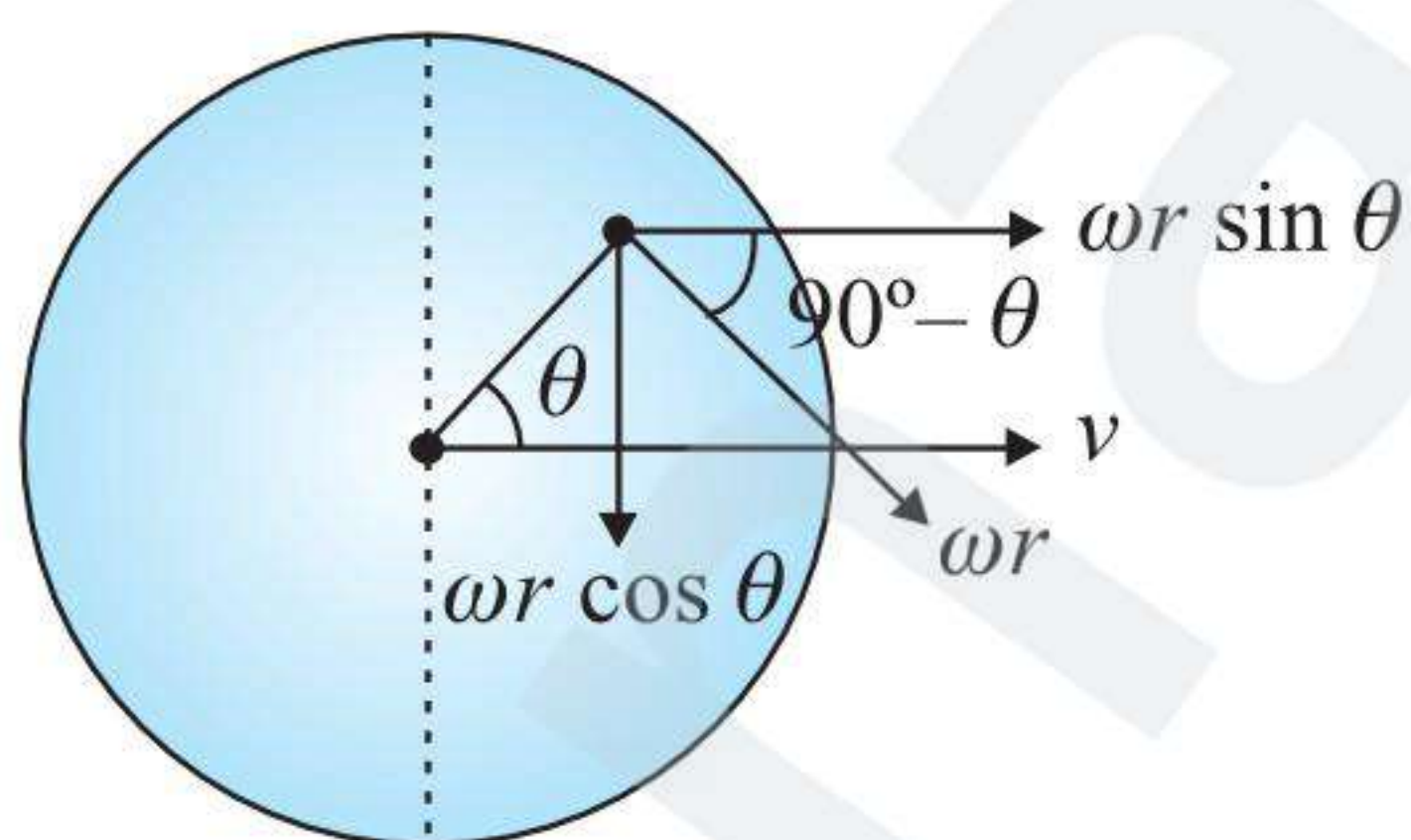
Now moment of inertia of the remaining portion should be $\frac{3}{4}$ of the above = $MF^2/6$.

16. (2) Here, $\omega = \frac{v}{R}$

$$V_{P_x} = \left(v + \frac{v}{R} r \sin \theta\right) \hat{i}$$

$$V_{P_y} = -\left(\frac{v}{R} r \cos \theta\right) \hat{j}$$

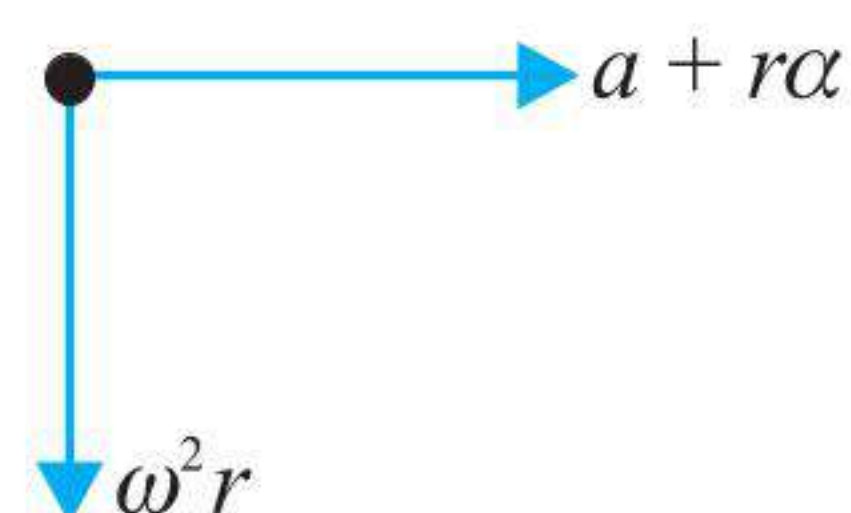
$$\therefore \vec{V}_P = \left(v + \frac{vr \sin \theta}{R}\right) \hat{i} - \frac{vr \cos \theta}{R} \hat{j}$$



17. (1) FBD of point P

In the earth frame, we have

$$\text{So, resultant} = \sqrt{(a + r\alpha)^2 + (\omega^2 r)^2}$$



18. (4) Moment of inertia of discs A and B about the axis through their centre of mass and perpendicular to the plane will be

$$I_{AA} = I_{BB} = \frac{1}{2} Mr^2$$

Now, moment of inertia of disc A about an axis through B , by theorem of parallel axis will be

$$I_{AA} = I_{BB} + M(2r)^2 = \frac{1}{2} Mr^2 + 4Mr^2 = \frac{9}{2} Mr^2$$

$$\text{So } I = I_{BB} + I_{AB} = \frac{1}{2} Mr^2 + \frac{9}{2} Mr^2 = 5Mr^2$$

19. (4) $\tau = (2mx^2)\alpha_1$

$$\tau = (2 \times m \times 4x^2)\alpha_2$$

$$\text{Equating } \tau, 8mx^2\alpha_1 = 2mx^2\alpha_2 \text{ or } \alpha_2 = \frac{\alpha_1}{4}$$

20. (2) Mass of incomplete ring = $m - \frac{M}{2\pi} \times \frac{\pi}{6} = M - \frac{M}{12} = \frac{11M}{12}$

$$\text{Moment of inertia of incomplete ring} = \left(\frac{11M}{12}\right) R^2 = \frac{11}{12} MR^2$$

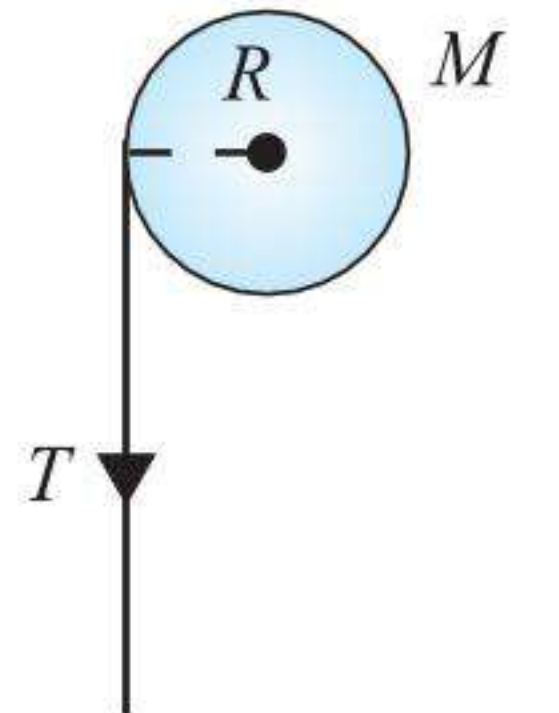
21. (3) Torque exerted on the disc

$$\tau = TR$$

$$\text{Now } \tau = I\alpha$$

$$\alpha = \frac{\tau}{I} = \frac{TR}{\frac{1}{2} MR^2}$$

$$= \frac{2TR}{MR^2} = \frac{2T}{MR}$$



22. (3) Tangential acceleration, $a = r\alpha = R\left(\frac{2T}{MR}\right) = \frac{2T}{M}$

23. (3) From the above problem, $T = \frac{Ma}{2}$.

$$\text{Now, } mg - T = ma$$

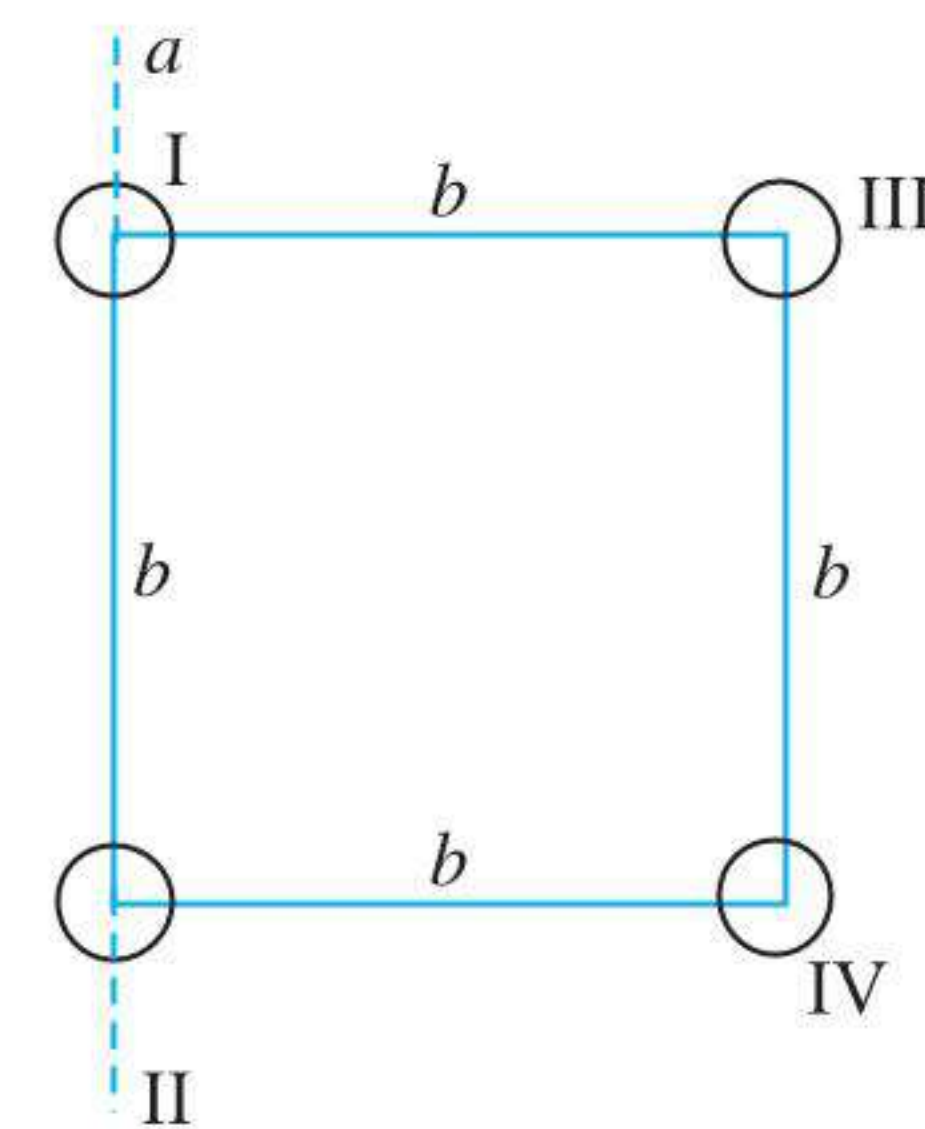
$$\text{or } mg - M\frac{a}{2} = ma$$

$$\text{or } mg = \left(m + \frac{M}{2}\right) a$$

$$a = \frac{2mg}{M + 2m}$$

24. (2) $T = \frac{Ma}{2} = \frac{Mmg}{M + 2m}$

25. (2) Combined moment of inertia of I and II = $2 \times \frac{2}{5} mr^2 = \frac{4}{5} mr^2$



Combined moment of inertia of III and IV

$$= 2 \left[\frac{2}{5} mr^2 + mb^2 \right] = \frac{4}{5} mr^2 + 2mb^2$$

Combined moment of inertia of the system

$$= \frac{4}{5} mr^2 + \frac{4}{5} mr^2 + 2mb^2 = \frac{8}{5} mr^2 + 2mb^2$$

26. (3) The moment of inertia of the uniform rod about an axis through one end and perpendicular to length is

$$I = \frac{ML^2}{3}$$

Torque ($\tau = I\alpha$) acting on the centre of gravity of the rod is given by

$$\tau = Mg \left[\frac{L}{2} \sin \theta \right]$$

$$\text{or } \frac{ML^2}{3} \alpha = Mg \frac{L}{2} \sin \theta$$

$$\alpha = \frac{3g}{2L} \sin \theta$$

27. (4) Taking torque about the attachment point for W , we get

$$-T_1(0.4L) + T_2(0.3L) + 500(0.2L) = 0$$

$$T = 1000 \text{ N, where } T_1 = T_2 = T$$

$$\Sigma F_y = 0 \Rightarrow 2T - W - 500 = 0$$

$$\Rightarrow W = 1500 \text{ N}$$

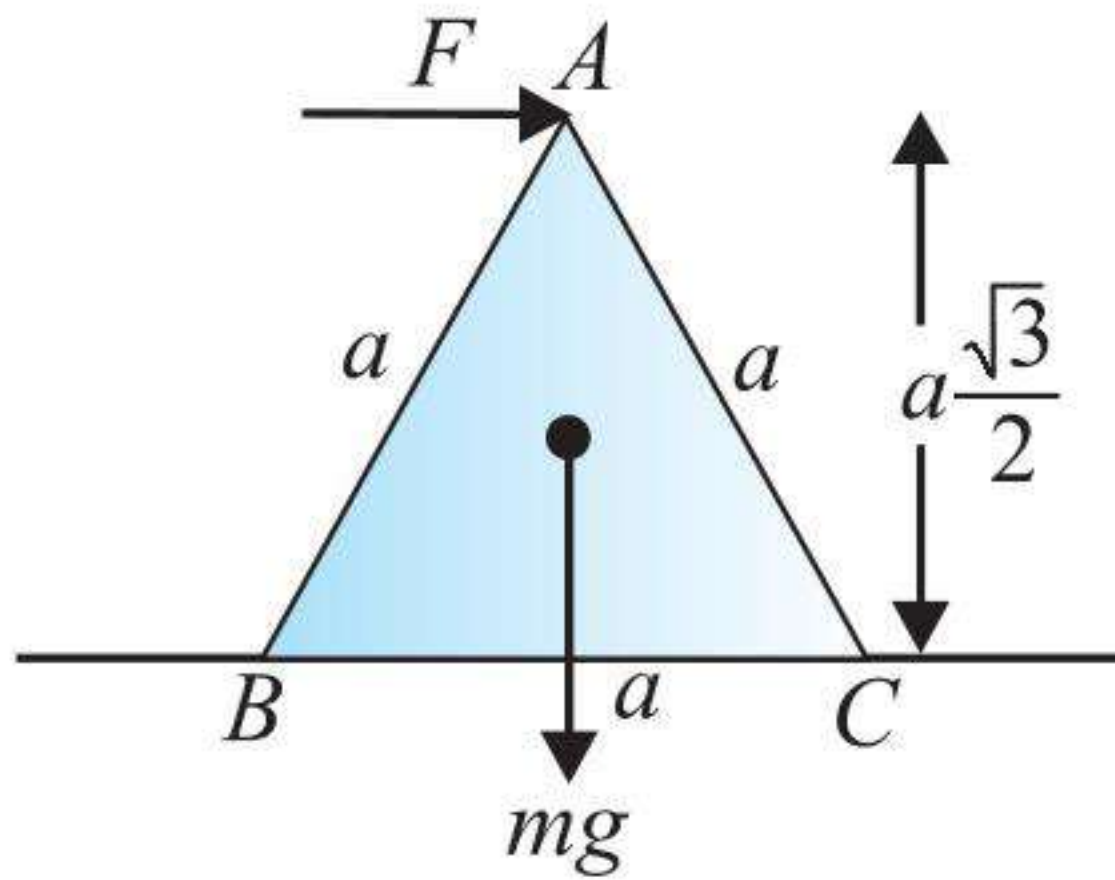
28. (3) Net torque about C is zero.

$$Fa + 100d = 0$$

$$F = -100 \frac{d}{a} = -150$$

$$\vec{F} = -150\hat{i}$$

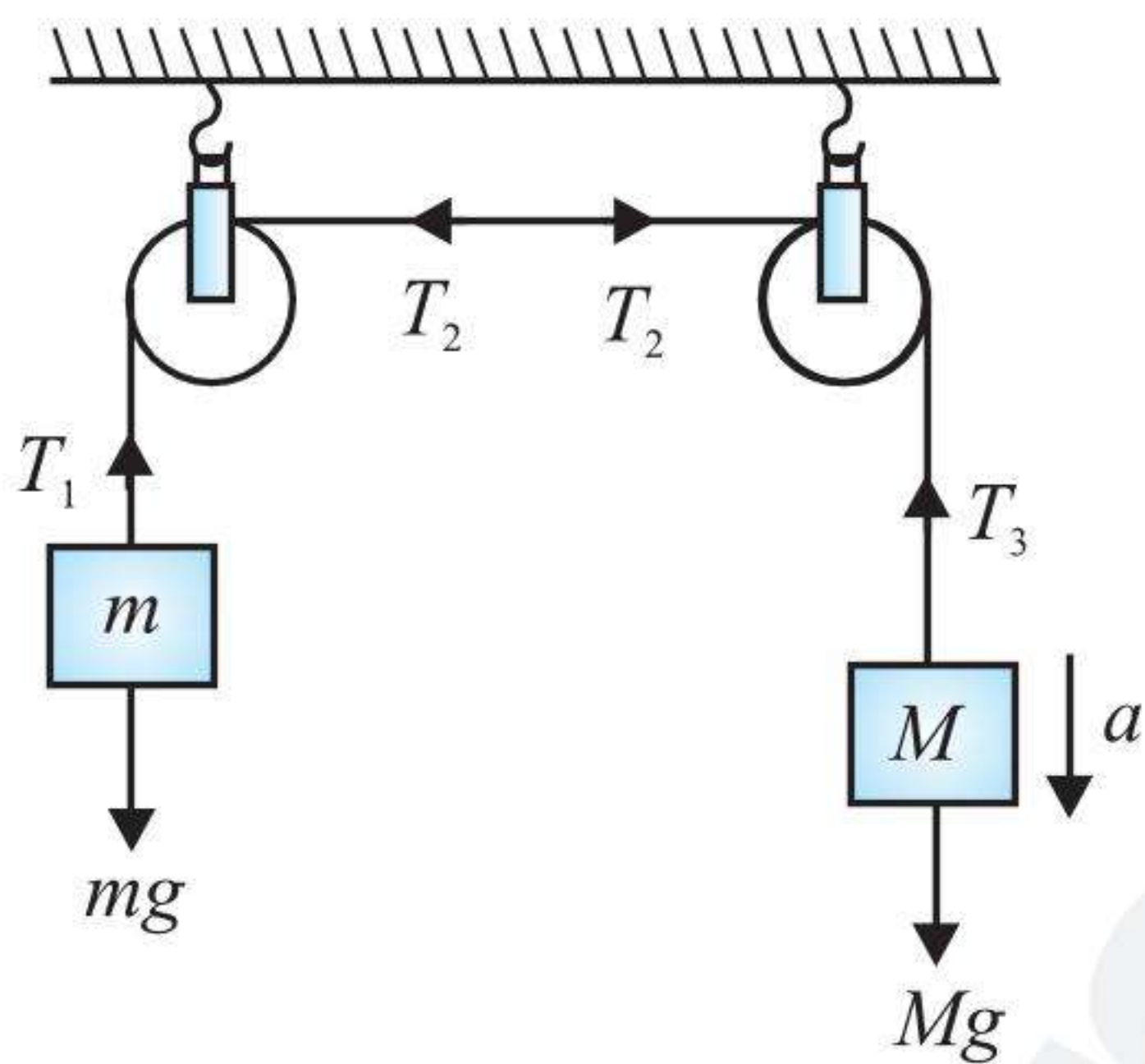
29. (1) The tendency of rotating will be about point C . For the minimum force, the torque of F about C has to be equal to the torque of mg about C .



$$F\left(a\frac{\sqrt{3}}{2}\right) = mg\left(\frac{a}{2}\right) \Rightarrow F = \frac{mg}{\sqrt{3}}$$

30. (1) $T_1 - mg = ma$... (i)

$$r(T_1 - T_2) = I\alpha$$
 ... (ii)



$$Mg - T_3 = Ma$$
 ... (iii)

$$r(T_3 - T_2) = I\alpha$$
 ... (iv)

$$\text{and } a = R\alpha$$

From Eqs. (ii) and (iv), we get

$$T_3 - T_1 = \frac{2Ia}{R^2}$$

From Eqs. (i) and (iii), we get

$$(M - m)g = (M + m)a + T_3 - T_1$$

$$(M - m)g = (M - m)a + \frac{2Ia}{r^2}$$

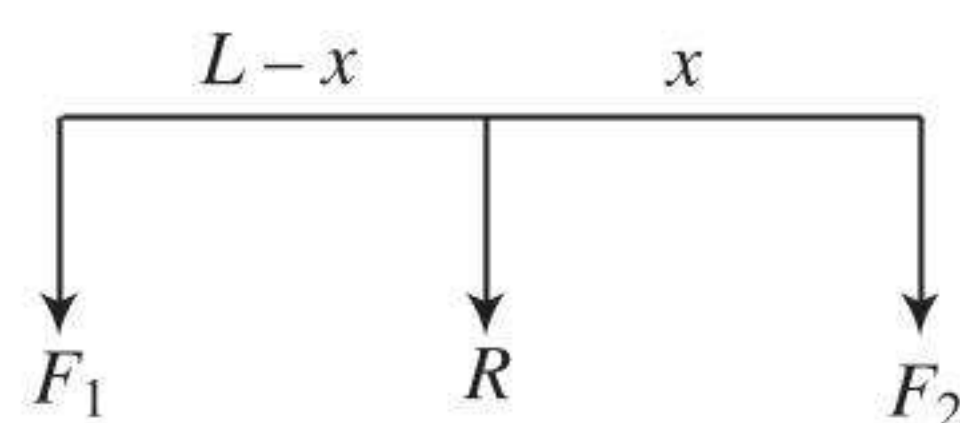
$$\Rightarrow a = \frac{(M - m)g}{\left(M + m + \frac{2I}{r^2}\right)}$$

31. (4) For $\tau = 0$

$$F_1(L - x) = F_2x$$

$$F_2\left(\frac{3L}{4} - x\right) = F_1\left(x + \frac{L}{4}\right)$$

$$\Rightarrow \frac{L - x}{x + (L/4)} = \frac{x}{(3L/4) - x}$$



$$\frac{3L}{4} = x + \frac{3x}{4} + \frac{x}{4} = 2x \Rightarrow 3L = 8x$$

So putting the value of x

$$F_1\frac{5L}{8} = F_2\frac{3L}{8} \Rightarrow F_1 : F_2 = 3 : 5$$

32. (3) The force should pass through the centre of mass of the assembly.

33. (4) Given $\alpha_A = 2\alpha = 5 \text{ m/s}^2 \Rightarrow \alpha = \frac{5}{2} \text{ m/s}^2$

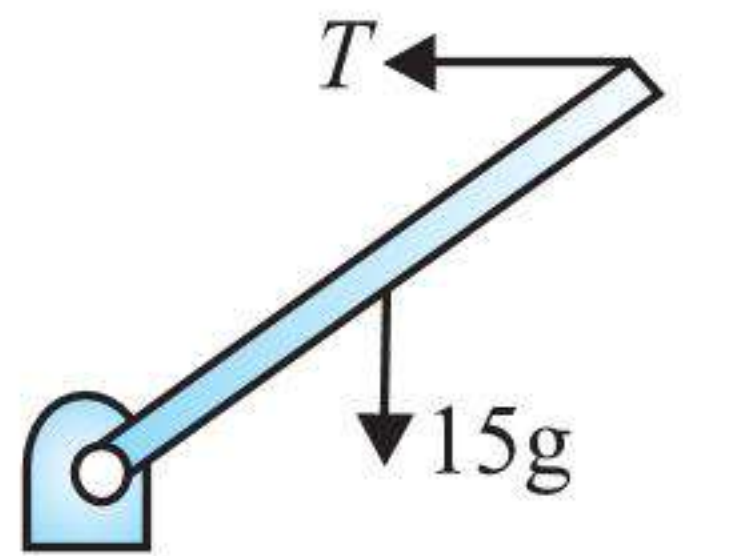
$$\text{Hence, acceleration of } B, a_B = 1(\alpha) = \frac{5}{2} \text{ m/s}^2$$

34. (3) The free-body diagram of the rod is shown in the figure. (Force exerted by the hinge on the rod are not shown.)

Take torque about the hinge,

$$T \times 3 - 15g \times 2 = 0$$

$$\Rightarrow T = 100 \text{ N}$$



35. (3) Perpendicular distance of F_3 is greatest from O , hence torque of F_3 is greatest.

36. (3) Mass of cut disc: $m_1 = M/4$

Moment of Inertia of original disc about axis 1:

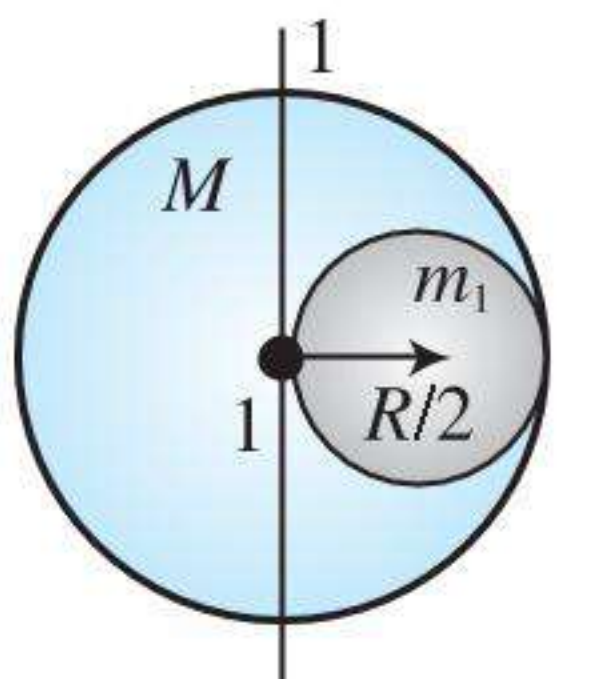
$$I = \frac{1}{4}MR^2$$

Moment of Inertia of small disc about axis 1:

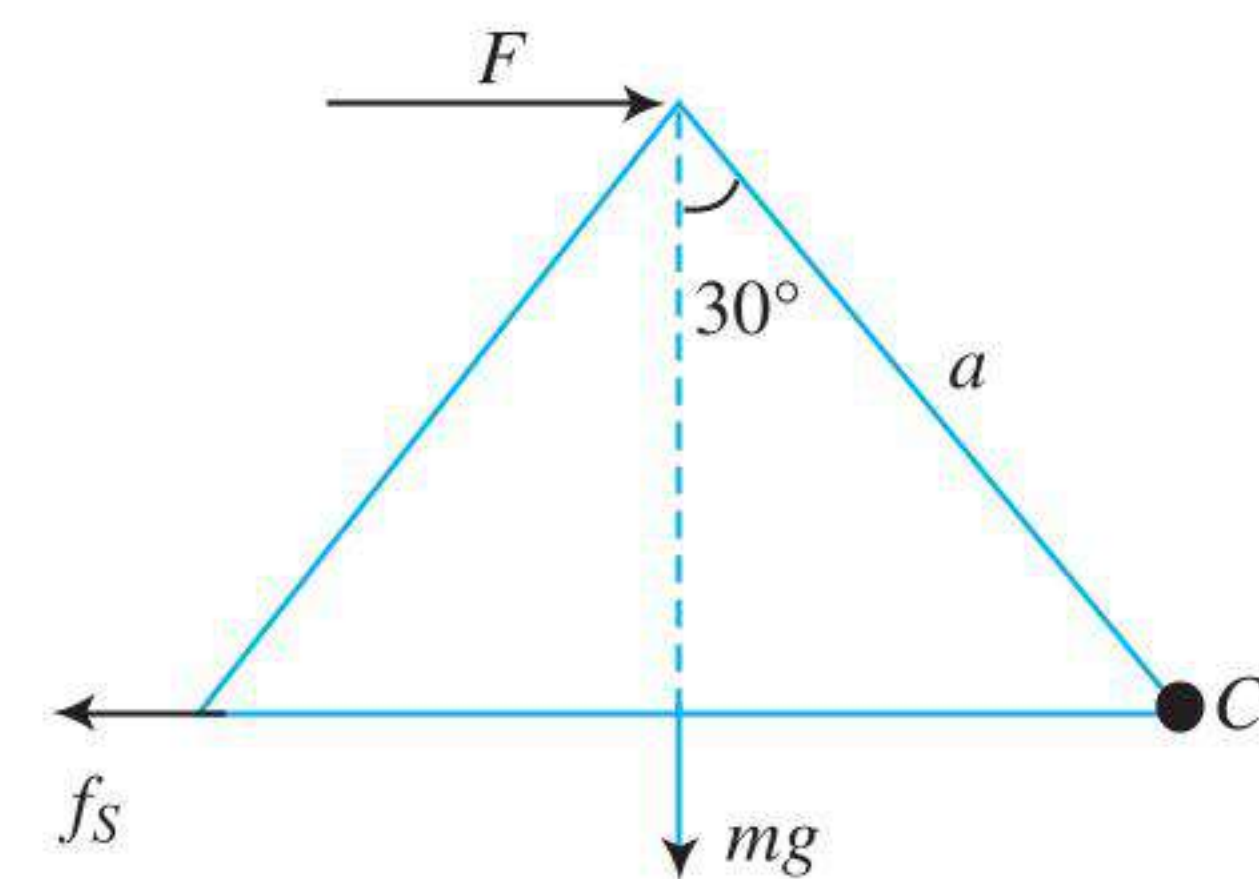
$$\begin{aligned} I' &= \frac{1}{4}m_1\left(\frac{R}{2}\right)^2 + m_1\left(\frac{R}{2}\right)^2 \\ &= \frac{5}{16}m_1R^2 = \frac{5}{64}MR^2 \end{aligned}$$

Required moment of inertia:

$$I - I' = \frac{1}{4}MR^2 - \frac{5}{64}MR^2 = \frac{11}{64}MR^2$$



37. (3) About C , for toppling $\tau_F > \tau_{mg}$



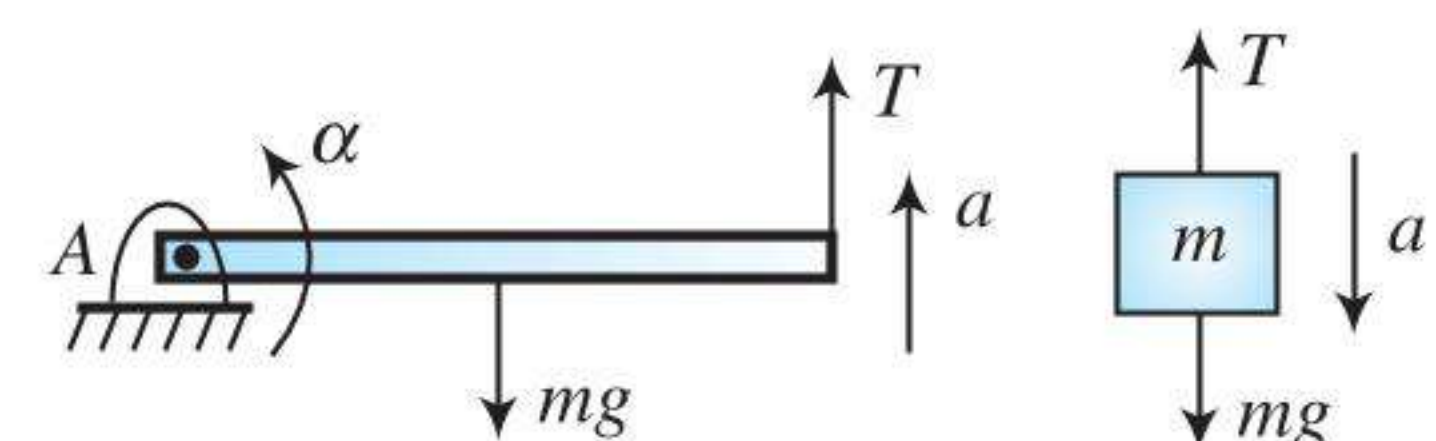
$$Fa \cos 30^\circ > mg \frac{a}{2} \Rightarrow F > \frac{mg}{\sqrt{3}}$$

$$\text{If no translation, then } f_s = F > \frac{mg}{\sqrt{3}}$$

$$\text{but } f_s \leq \mu mg$$

$$\Rightarrow \frac{mg}{\sqrt{3}} \leq \mu mg \Rightarrow \mu \geq \frac{1}{\sqrt{3}}$$

38. (3) $\alpha = \frac{a}{l}$... (i)



$$mg - T = ma$$

$$\dots (ii)$$

$$Tl - mg \frac{l}{2} = \frac{ml^2}{3} \alpha \quad \dots(iii)$$

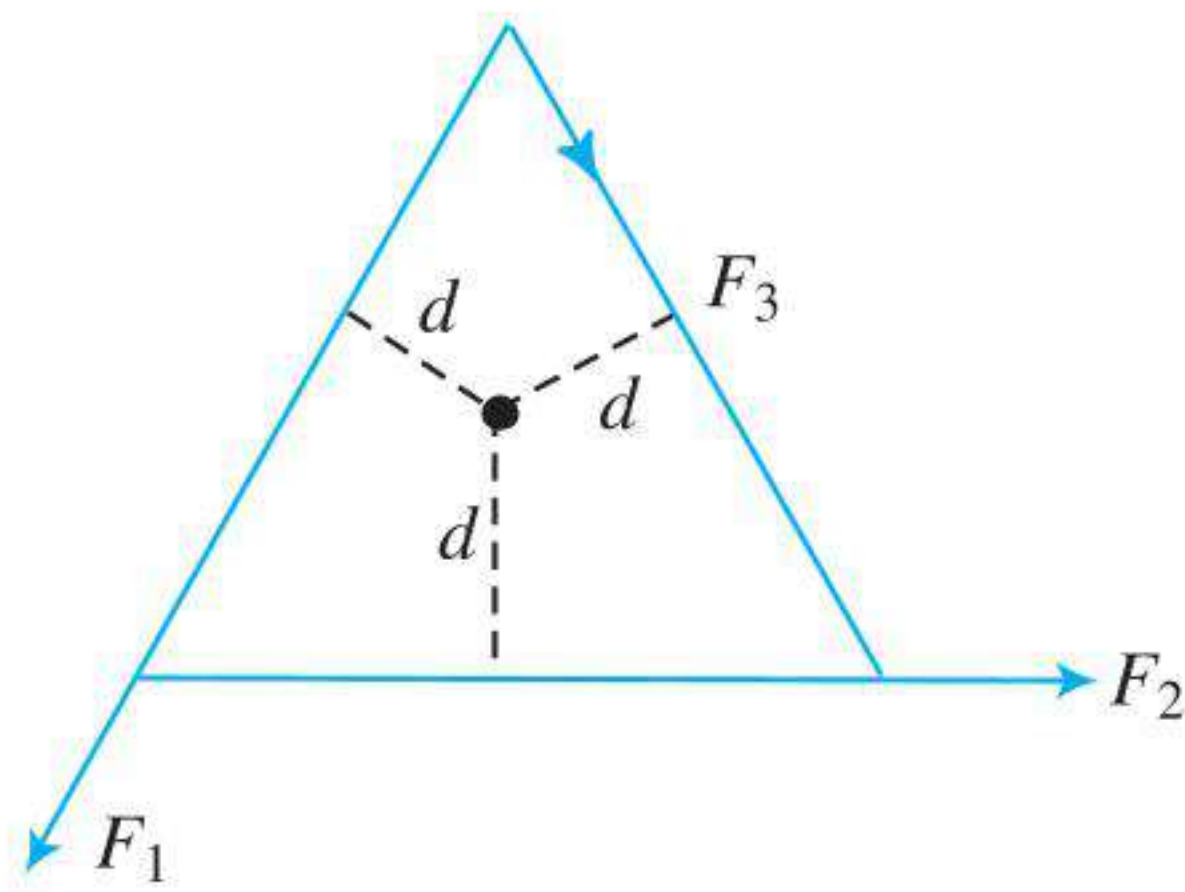
Solving the above equations: $a = \frac{3g}{8}$

39. (3) Clearly, the block shall topple about its edge through O . The torque FL of the applied force is clockwise. The torque $MgL/2$ of the weight is anticlockwise.

Applying condition for rotational equilibrium.

$$-FL + mgL/2 = 0 \text{ or } F = \frac{mg}{2}$$

40. (4) The torques F_1d and F_2d of F_1 and F_2 respectively are counterclockwise. The torque F_3d is clockwise. Applying condition for rotational equilibrium,



$$F_1d + F_2d - F_3d = 0 \text{ or } F_3 = F_1 + F_2$$

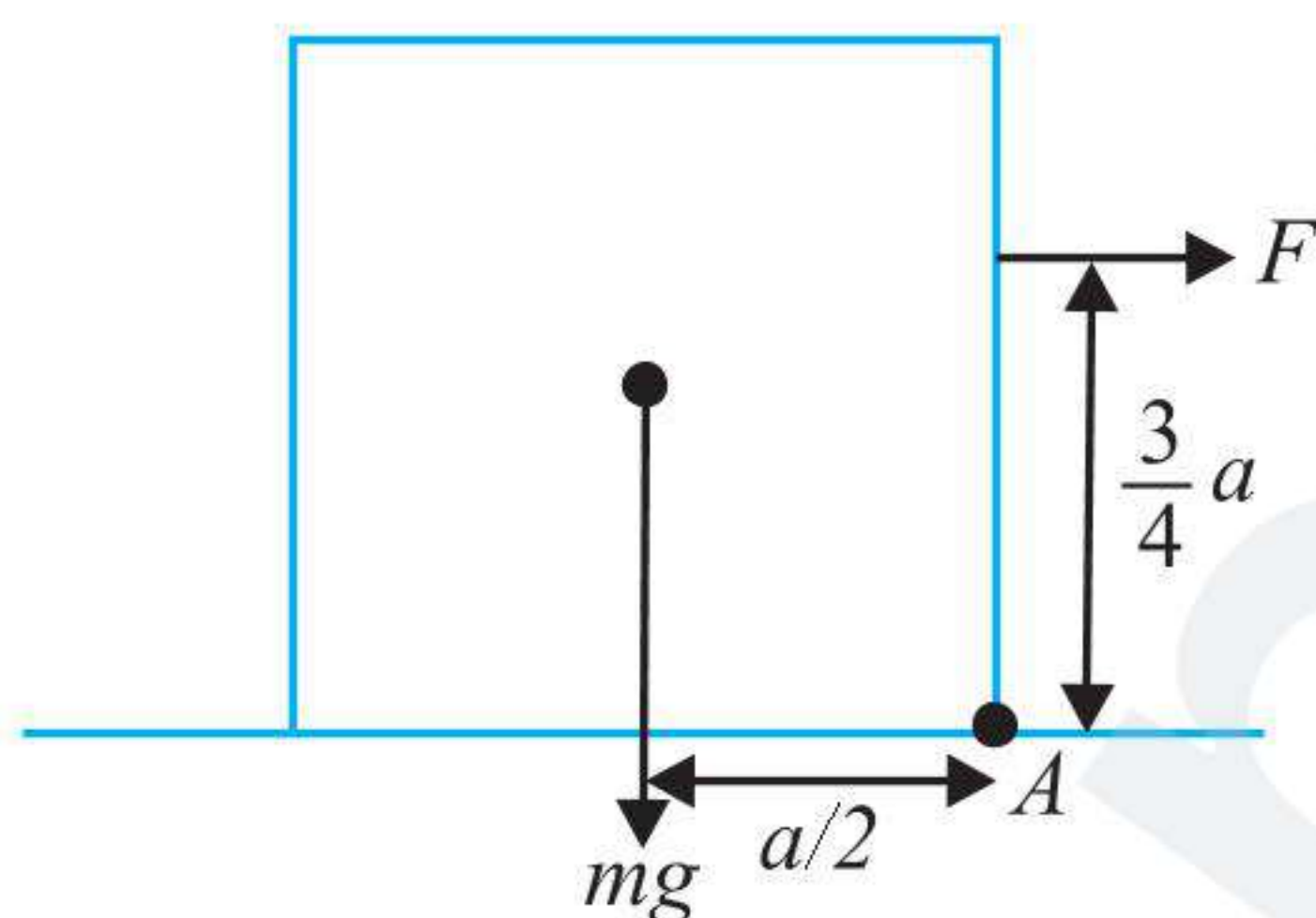
41. (3) Balancing the torque:

$$\text{for the first case: } 16l_1 = ml_2$$

$$\text{for the second case: } ml_1 = 4l_2$$

$$\text{Divide them to get } m = 8 \text{ kg}$$

42. (2) See the figure. For tilting about A the clockwise torque (due to F) should be greater than the anticlockwise torque about A .



$$mg \times \frac{a}{2} = \frac{F3a}{4}; F = \frac{2mg}{3}$$

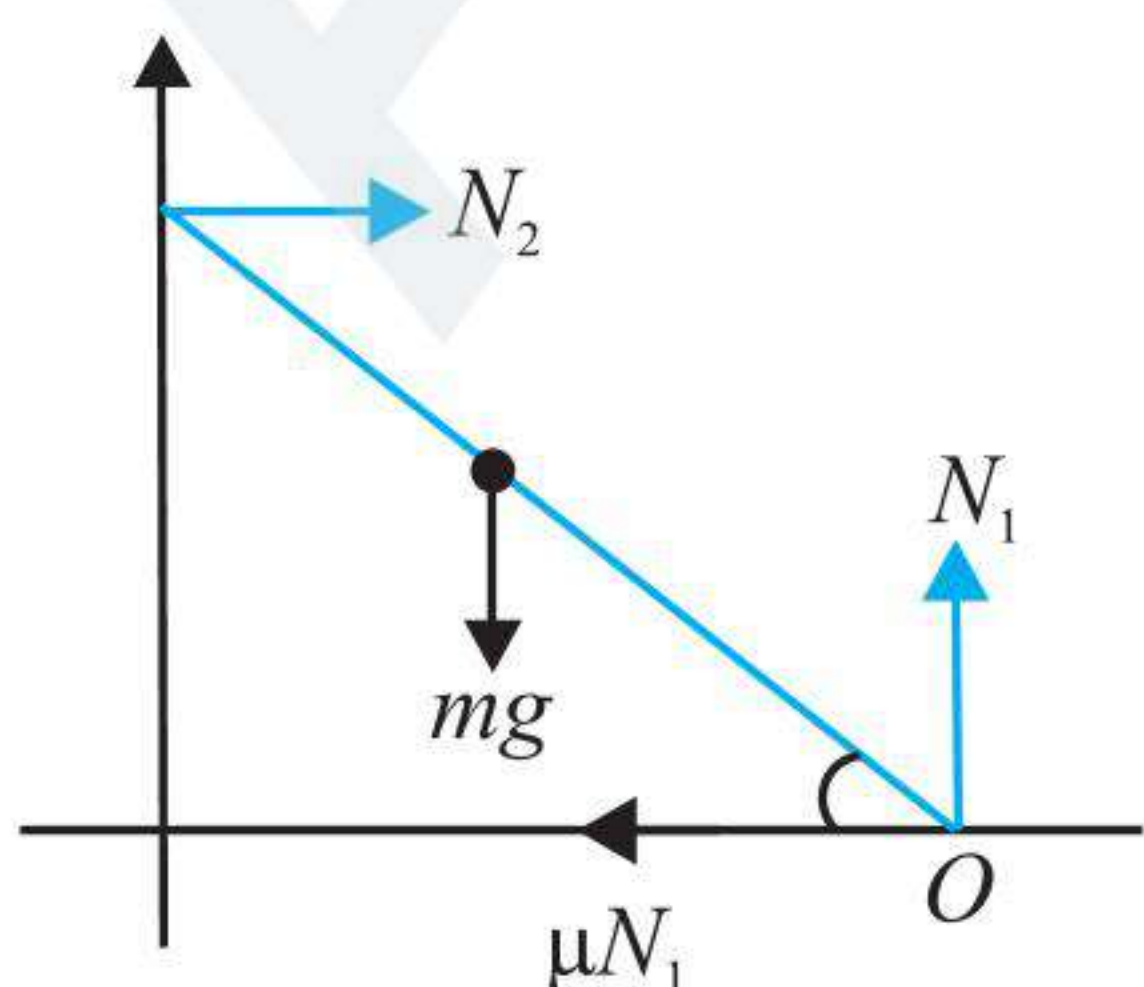
43. (4) $mg = N_1$

$$\mu N_1 = N_2 \text{ or } N_2 = \mu mg$$

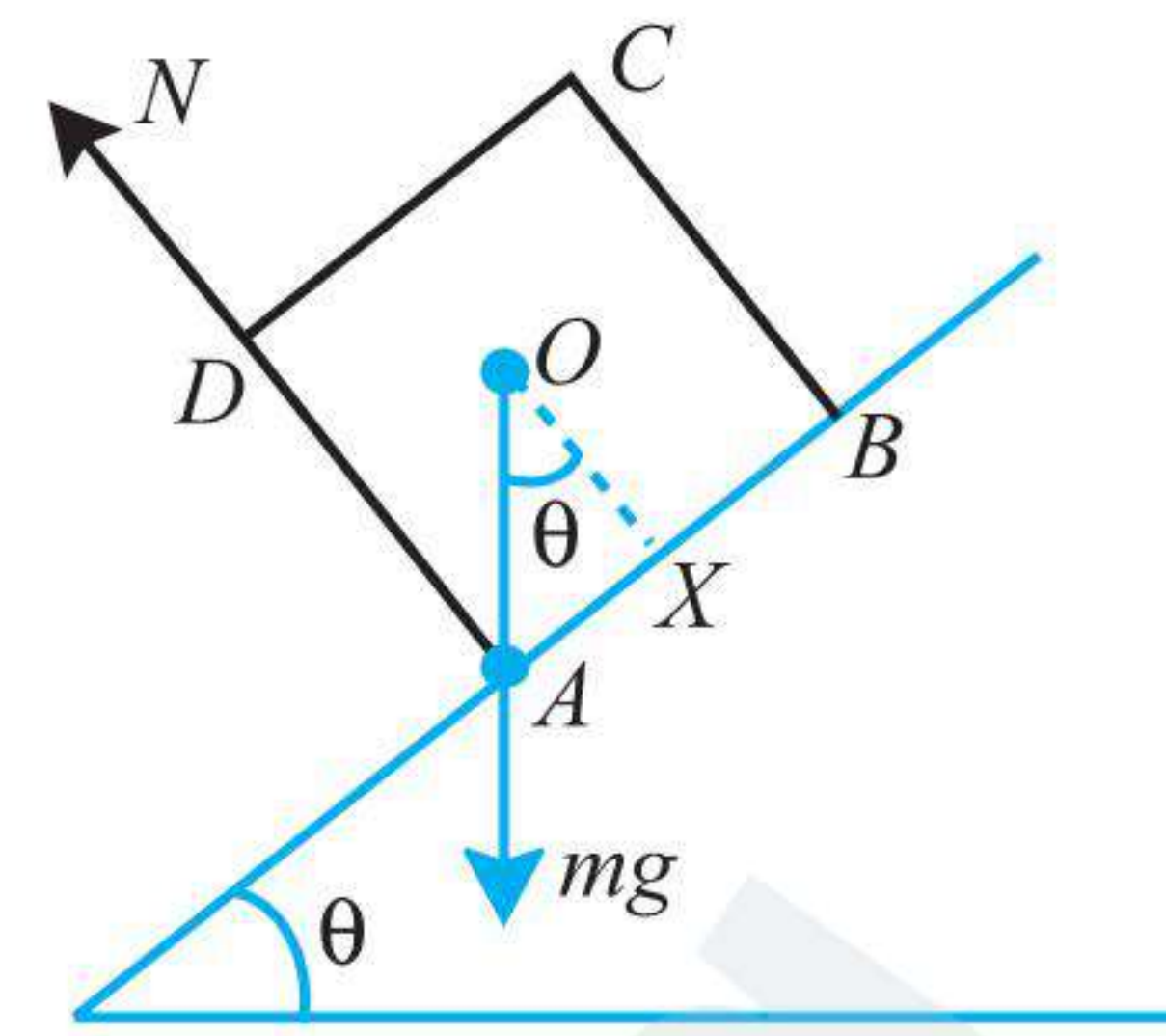
Taking moment about O , we get

$$\mu mgl \sin \theta = mg \frac{l}{2} \cos \theta$$

$$\tan \theta = \frac{1}{2\mu} \text{ or } \theta = \tan^{-1}\left(\frac{1}{2\mu}\right)$$



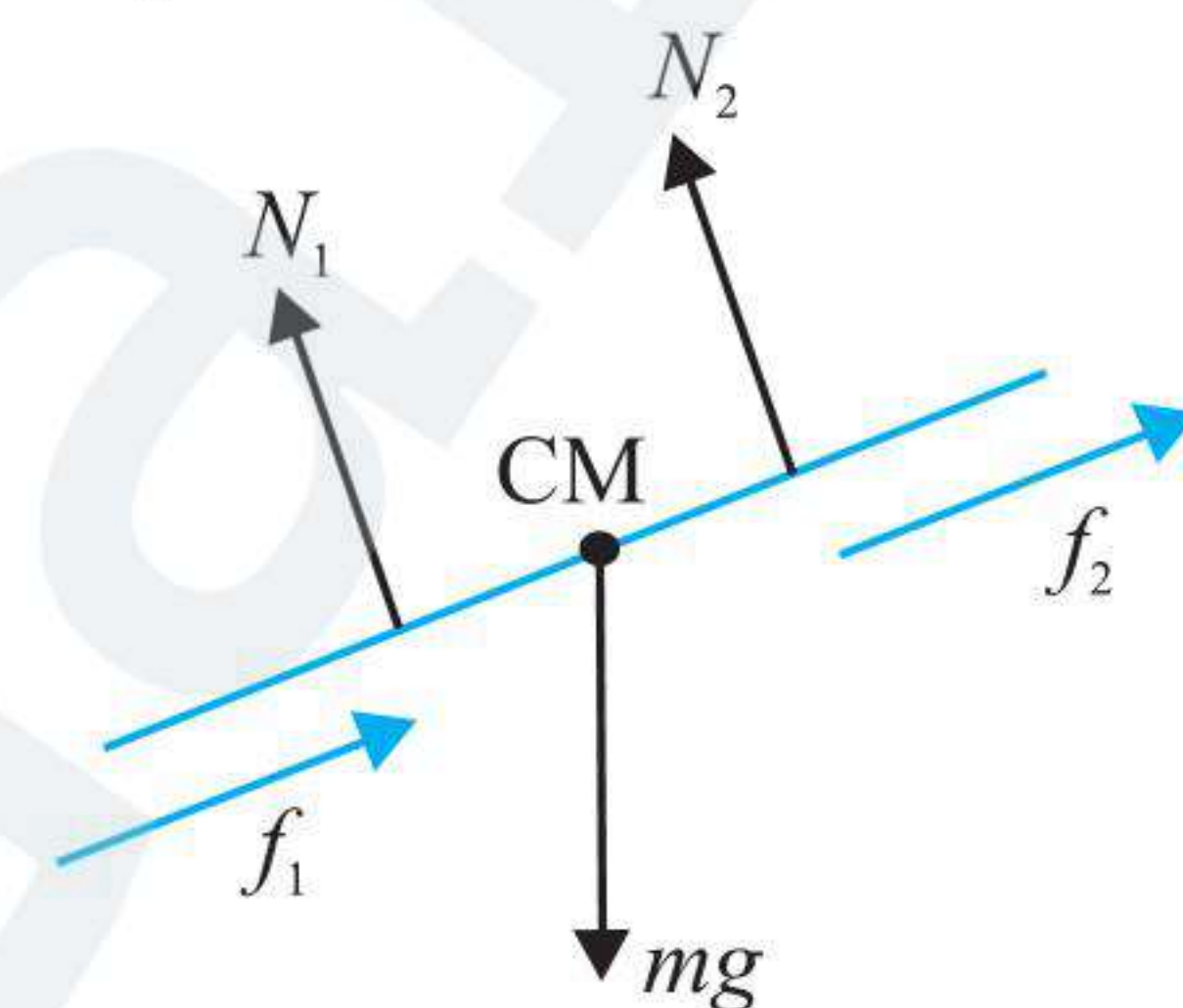
44. (3) The cube will not topple until the line of action of CM passes through the cube.



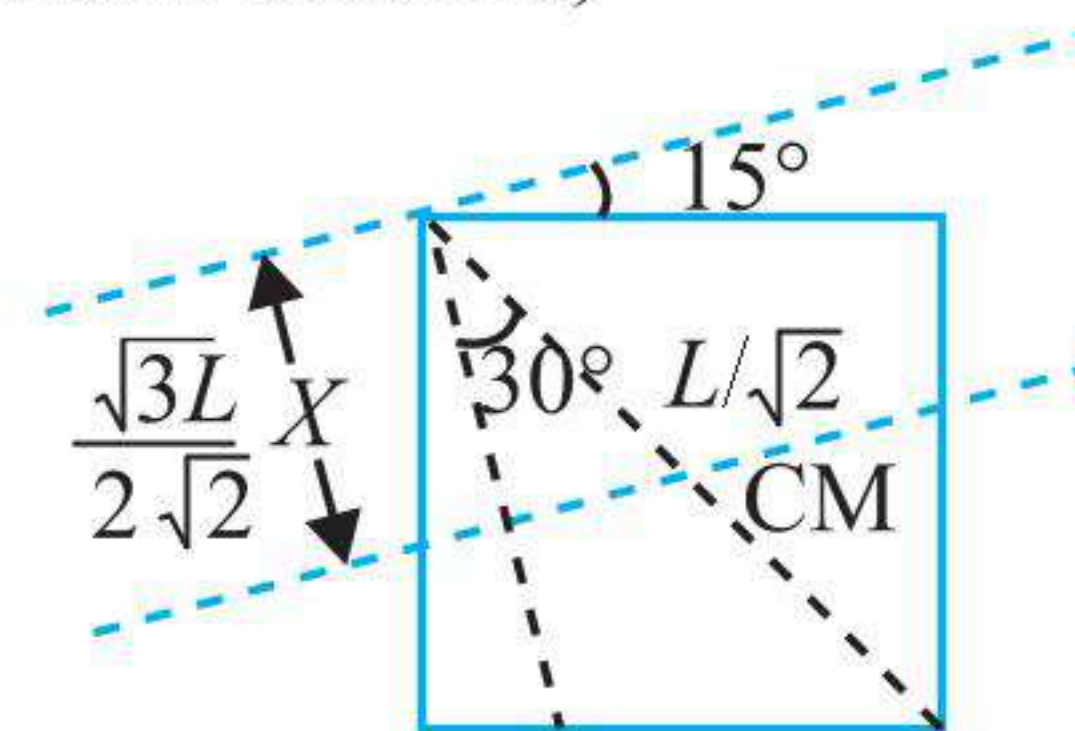
$$\text{That is, } \tan \theta = \frac{AX}{OX} = \frac{a/2}{a/2} \text{ or } \theta = 45^\circ$$

45. (3) As rod is in equilibrium. The torque of all the forces about the CM is zero.

$$N_1 \frac{l}{4} = N_2 \frac{l}{4} \Rightarrow N_1 = N_2$$



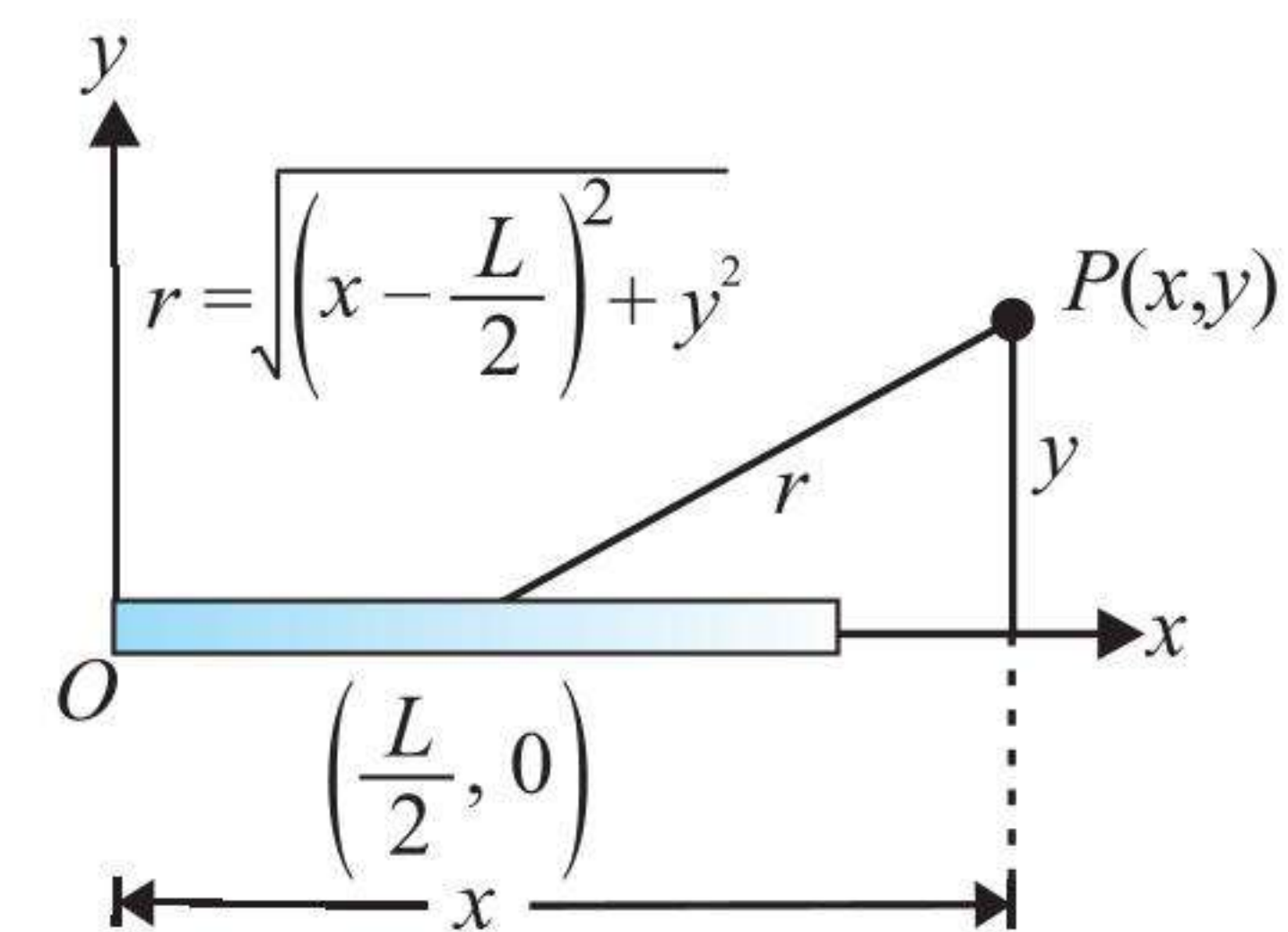
46. (2) From parallel axis theorem,



$$I_1 = \frac{ML^2}{12}, I_2 = I_1 + Mx^2 = \frac{ML^2}{12} + M\left(\frac{\sqrt{3}L}{2\sqrt{2}}\right)^2 = \frac{11ML^2}{24}$$

47. (2) Let us take any point P on x - y plane. The MI of the rod about O is

$$I_P = I_{CM} + Mr^2$$

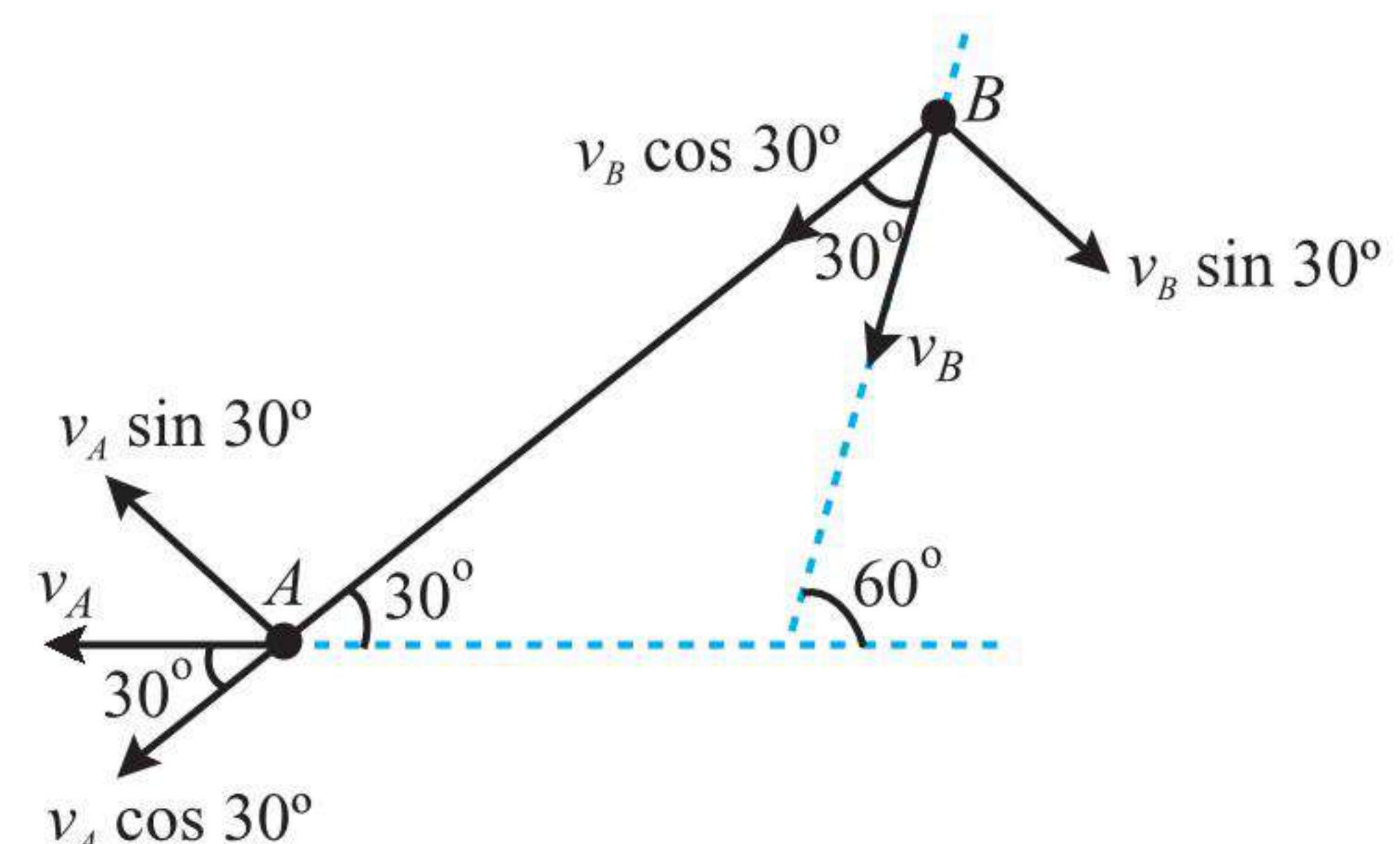


$$\text{Hence, } I_P = \frac{ML^2}{12} + M\left[\left(x - \frac{L}{2}\right)^2 + y^2\right]$$

$$\text{MI of the rod about } O, I_0 = \frac{ML^2}{3}$$

Applying $I_P = I_0$, we get an equation of a circle.

48. (2)



The velocities of the ends A and B along the length of rod should be the same.

$$\text{Hence, } v_A \cos 30^\circ = v_B \cos 30^\circ \Rightarrow v_A = v_B = v$$

Hence, the angular velocity of the rod is

$$\omega = \frac{(v_{AB})_{\perp}}{L} = \frac{2v \sin 30^\circ}{L} \Rightarrow \omega = \frac{v}{L}$$

$$49. (1) (MI)_{CG} = M \left(\frac{a^2 + b^2}{12} \right)$$

According to the theorem of parallel axis,

$$\begin{aligned} (MI)_{\text{required axis}} &= (MI)_{CG} + M(OA)^2 \\ &= M \left(\frac{a^2 + b^2}{12} \right) + M \left(\frac{a^2 + b^2}{4} \right) = M \left(\frac{a^2 + b^2}{3} \right) \end{aligned}$$

$$\begin{aligned} 50. (2) \text{ Total MI} &= I_1 + I_2 + I_3 + I_4 \\ &= 2I_1 + 2I_2 \quad [I_1 = I_3; I_1 = I_4] \\ &= 2(I_1 + I_2) \end{aligned}$$

$$\text{Now, } I_2 = I_3 = \frac{ML^2}{3}$$

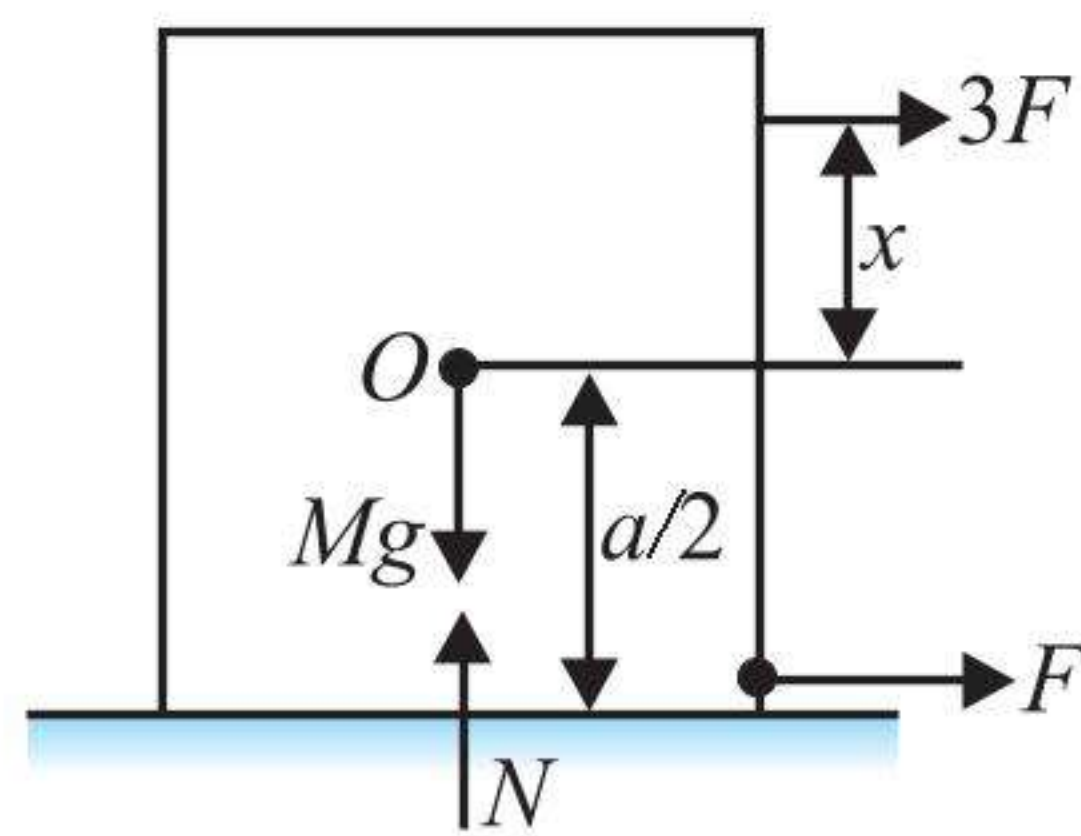
Using parallel axis theorem, we have

$$I = I_{CM} + Mx^2 \text{ and } x = \sqrt{l^2 + \left(\frac{l}{2}\right)^2}$$

$$I_1 = I_4 = \frac{ML^2}{12} + M \left[\sqrt{l^2 + \left(\frac{l}{2}\right)^2} \right]^2$$

$$\text{Putting all values, we get } I = \frac{10 ML^2}{3}$$

51. (2) The free-body diagram of the block can be drawn as shown. As the body has to move in pure translational motion, the torque about the centre of gravity must be zero.



$$3F \times x = F \times \frac{a}{2} \Rightarrow x = \frac{a}{6}$$

$$52. (2) I_1 = \frac{ml^2}{12}$$

$$I_2 = m \left[\frac{l}{2\pi} \right]^2 = \frac{ml^2}{4\pi^2}$$

$$\frac{I_1}{I_2} = \frac{ml^2}{12} \times \frac{4\pi^2}{ml^2} = \frac{\pi^2}{3}$$

53. (2) The two forces along the y -direction balance each other.

Hence, the resultant force is $2F$ along the x -direction.

Let the point of application of the force be at $(0, y)$.

(By symmetry, x -coordinate will be zero).

For rotational equilibrium:

$$F(a) + F(a) + F(a+y) - F(a-y) = 0$$

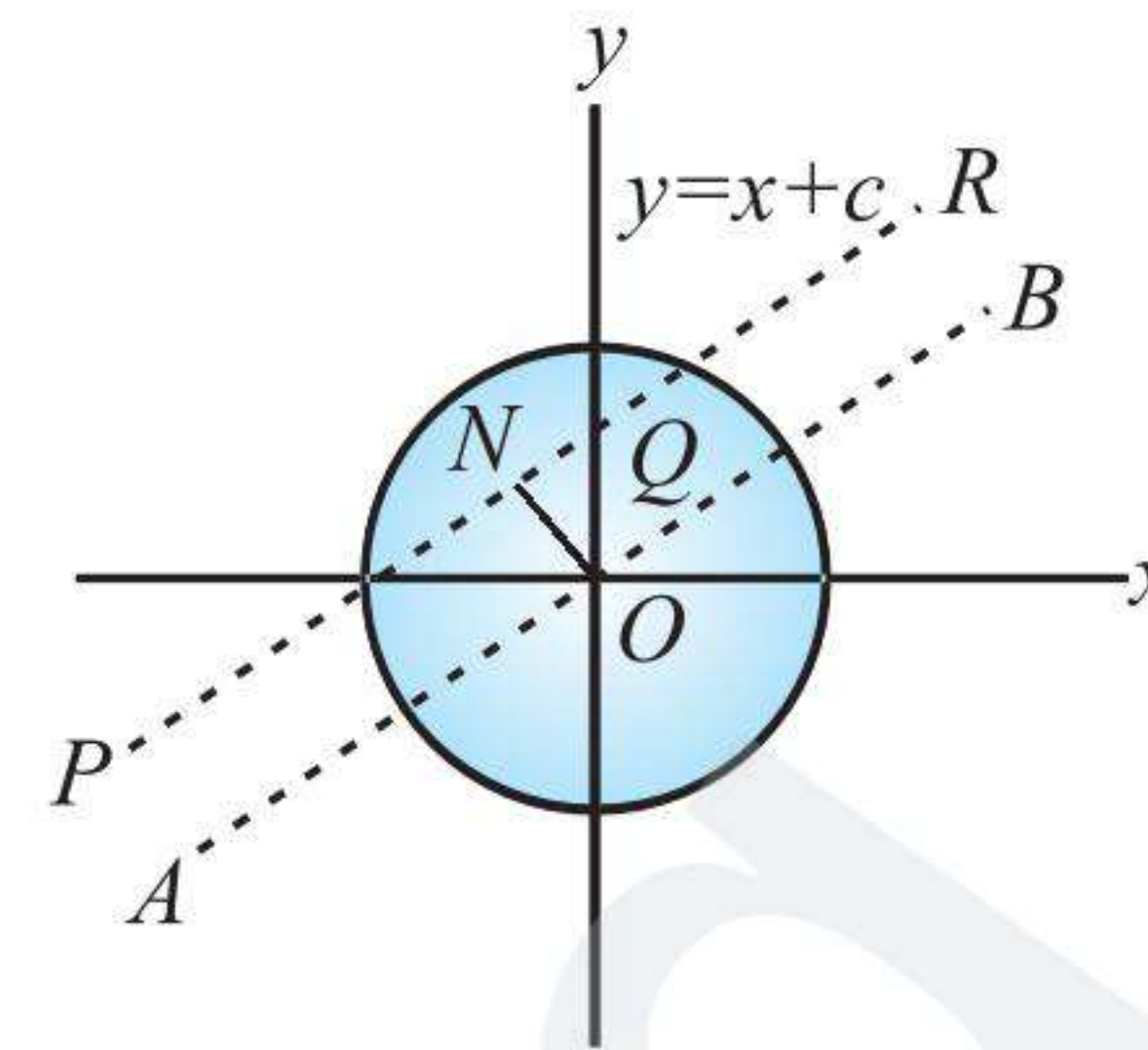
$$\Rightarrow y = -a$$

Alternative method: Torque will only be produced by the two forces acting along y -direction in anticlockwise direction. To balance this torque, we should apply a force $2F$ in order to produce a torque in the clockwise direction, which is only possible if we apply a force at a point below the x -axis.

$$\text{Then } \tau = F(a) + F(a) - 2F \times y = 0$$

$$\Rightarrow y = a$$

54. (2)



$$I_{PQR} = I_{AOB} + M(ON)^2 = \frac{1}{4} MR^2 + M \left(\frac{c}{\sqrt{2}} \right)^2$$

$$\text{But } I_z = \frac{1}{2} MR^2$$

$$\therefore c = \pm \frac{R}{\sqrt{2}}$$

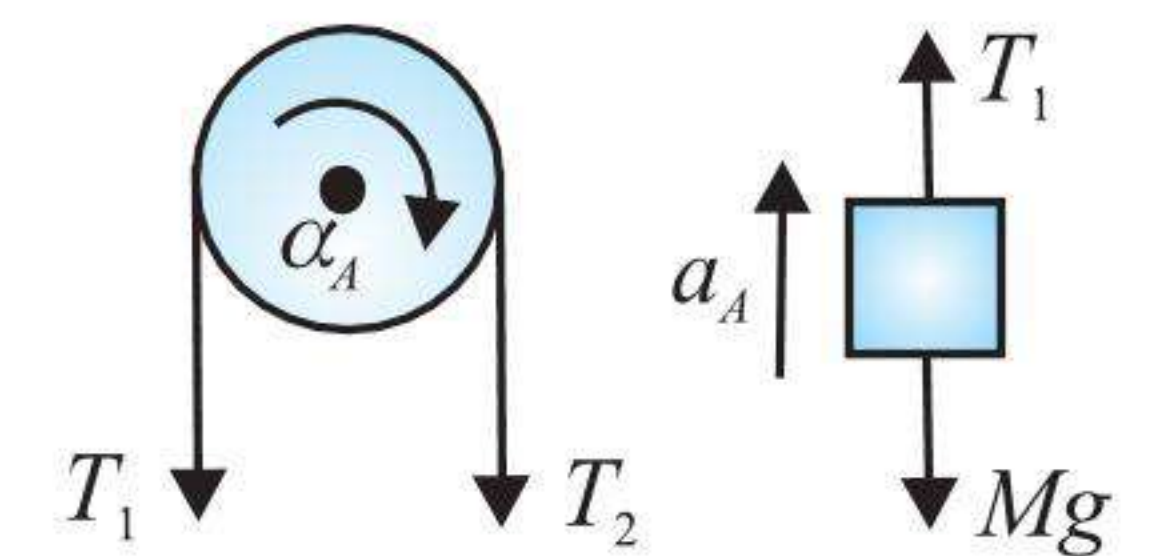
55. (2) In figure, $(T_2 - T_1)R = \frac{MR^2}{2} \alpha_A$

$$T_1 - Mg = Ma_A$$

$$T_2 = 2Mg$$

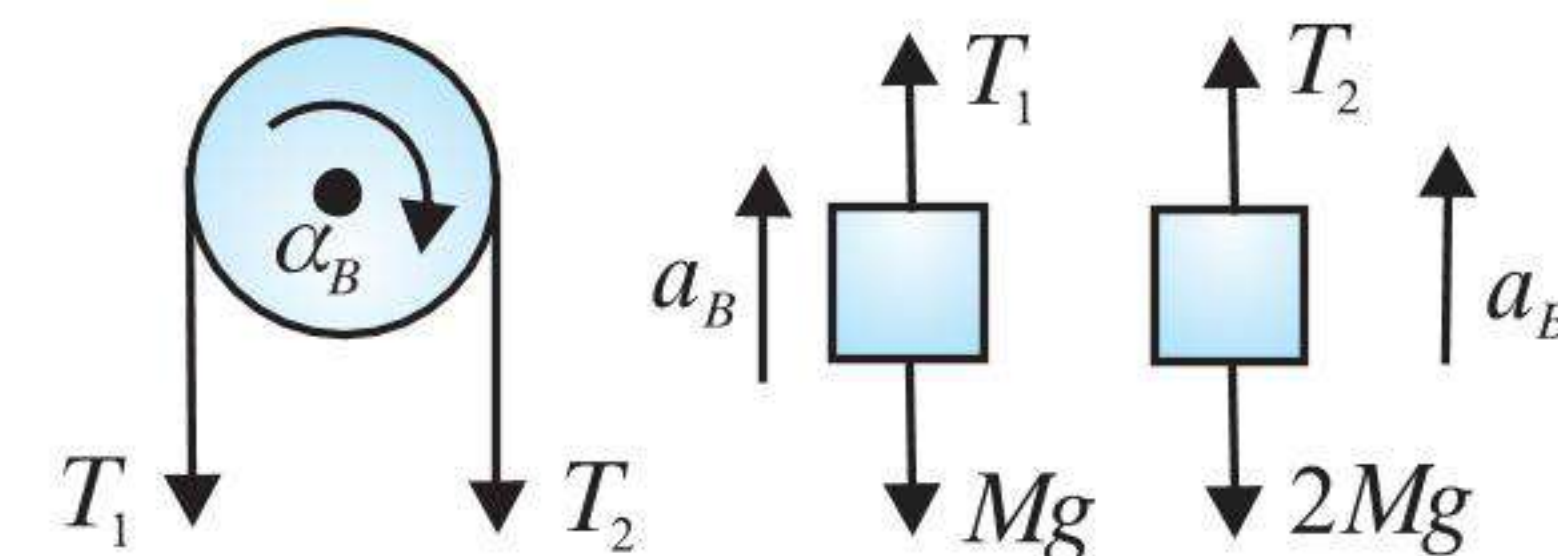
$$a_A = R\alpha_A$$

$$\alpha_A = \frac{2g}{3R}$$



From figure,

$$(T_2 - T_1) \times R = \frac{MR^2}{2} \alpha_B$$



$$T_1 - Mg = Ma_B$$

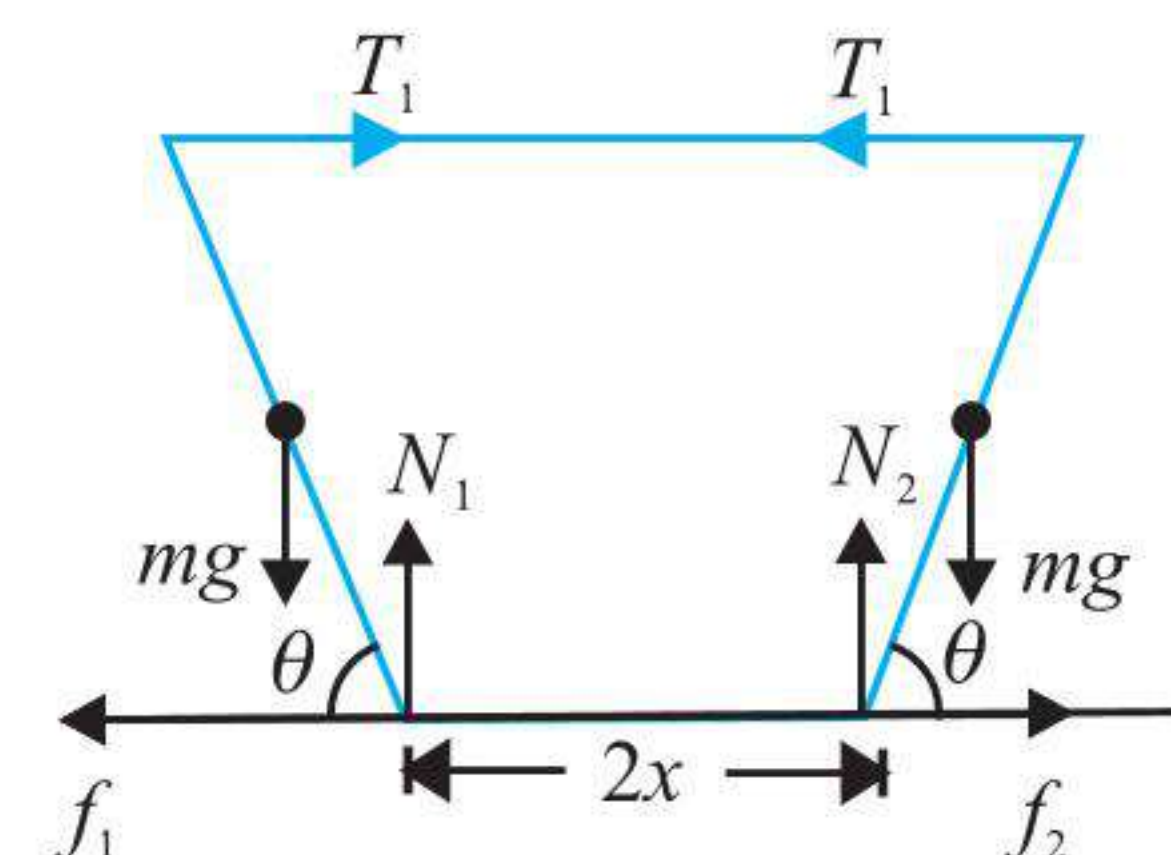
$$2Mg - T_2 = 2Ma_B$$

$$a_B = R\alpha_B$$

$$\alpha_B = \frac{2g}{7R}$$

So, $\alpha_A > \alpha_B$

56. (2) For the equilibrium of the entire structure, $N_1 + N_2 = 2Mg$



$$Mg \left(\frac{l}{2} \cos \theta \right) + N_2 \times 2x - Mg \left(\frac{l}{2} \cos \theta + 2x \right) = 0$$

$$\text{Given } N_2 = Mg, N_1 = Mg$$

$$\text{For individual boards, } T_1 = f_1, N_1 = Mg$$

$$\text{and } T_1 \times l \sin \theta = Mg \frac{l}{2} \cos \theta$$

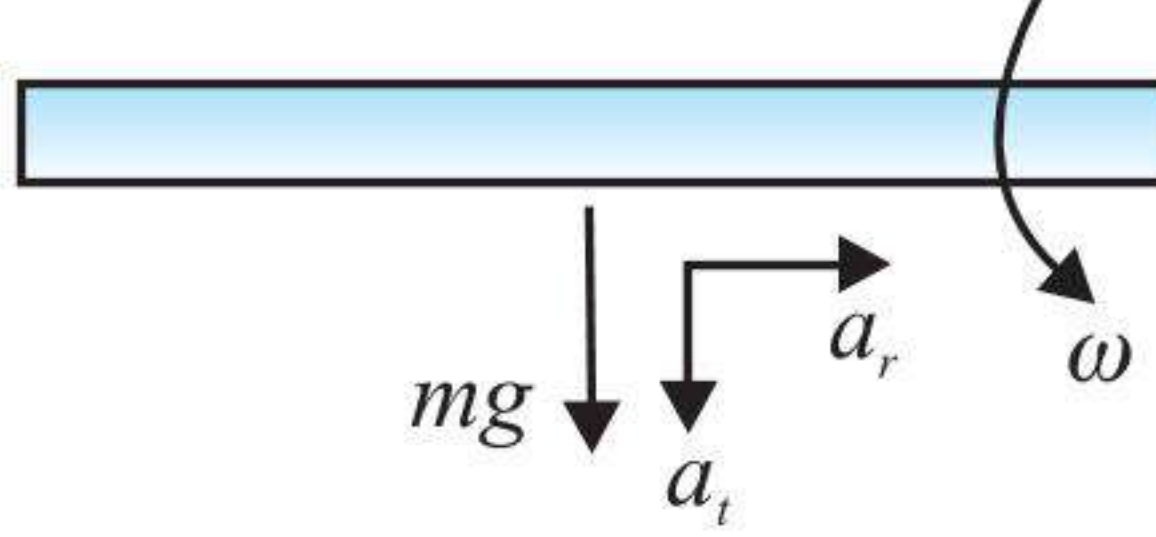
$$f_1 = T_1 = \frac{Mg}{2 \tan \theta}$$

$$\text{For safe equilibrium, } f_1 < f_L = \mu_s Mg$$

$$\frac{Mg}{2 \tan \theta} = \mu_s Mg \Rightarrow \tan \theta > \frac{1}{2\mu_s} \Rightarrow \theta = 45^\circ$$

So the minimum value of θ for this type of arrangement is 45° .

57. (3) The angular velocity of the rod about the pivot when it passes through the horizontal position is given by

$$mg \times \frac{L}{2} \sin 30^\circ = \frac{mL^2}{3} \times \frac{\omega^2}{2}$$


$$\omega = \sqrt{\frac{3g}{2L}}$$

Radial acceleration of the centre of mass (as centre of mass is moving in a circle of radius $L/2$) is given by

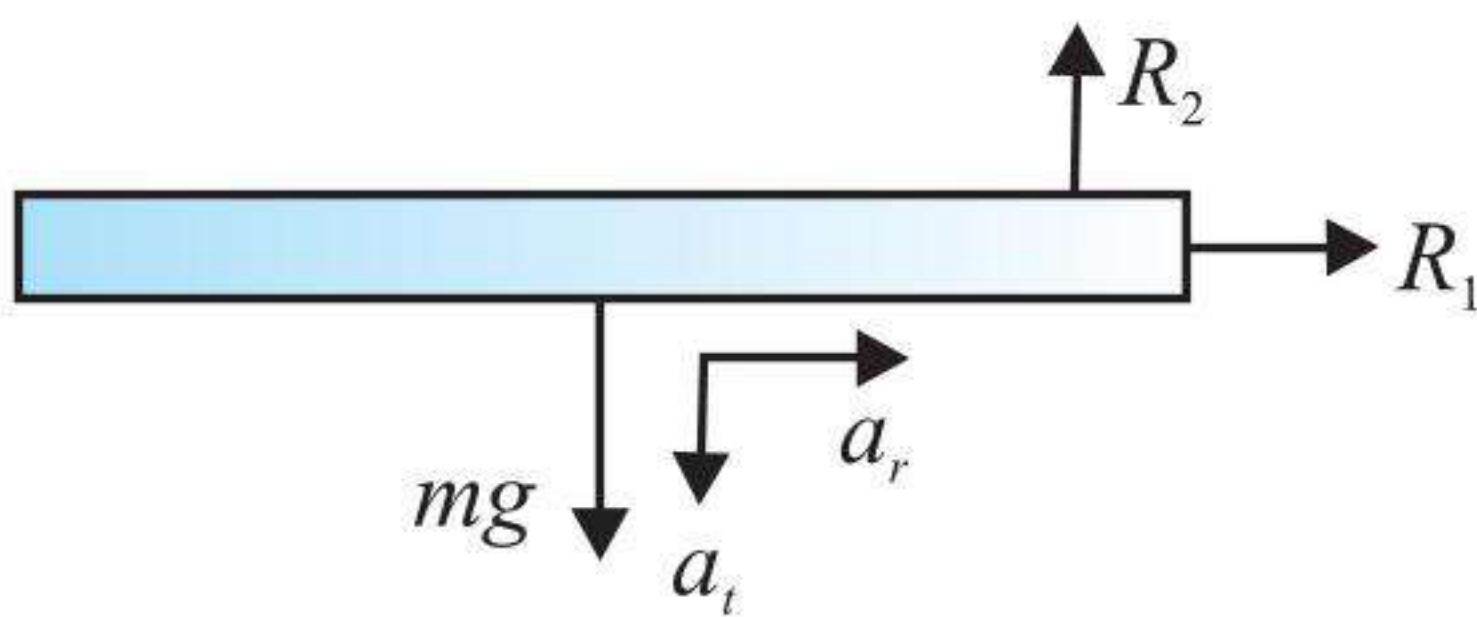
$$a_r = \omega^2 \frac{L}{2} = \frac{3g}{4}$$

Torque about pivot, in the horizontal position, is $\tau = mg \frac{L}{2} = I\alpha$

$$\alpha = \frac{mgL/2}{mL^2/3} = \frac{3g}{2L}$$

Tangential acceleration of the centre of mass, $a_t = \frac{L}{2} \alpha = \frac{3g}{4}$

Draw the FBD of the rod at an instant when it passes through the horizontal position. Use Newton's second law of equation.



$$R_1 = ma_r = \frac{3mg}{4}$$

$$mg - R_2 = m \times a_t = \frac{3mg}{4}$$

$$R_2 = \frac{mg}{4}$$

So, reaction force by the pivot on the rod, $R = \sqrt{R_1^2 + R_2^2} = \sqrt{10} mg/4$ at an angle of $\tan^{-1}(R_2/R_1) [= \tan^{-1}(1/3)]$ with the horizontal.

58. (2) $mg \frac{L}{2} = \frac{1}{3} mL^2 \alpha$

$$\alpha = \frac{3g}{2L}$$

$$\text{and } \alpha x = g \Rightarrow \frac{3g}{2L} x = g$$

$$x = \frac{2L}{3}$$

$$\text{distance from B: } L - x = \frac{L}{3}$$

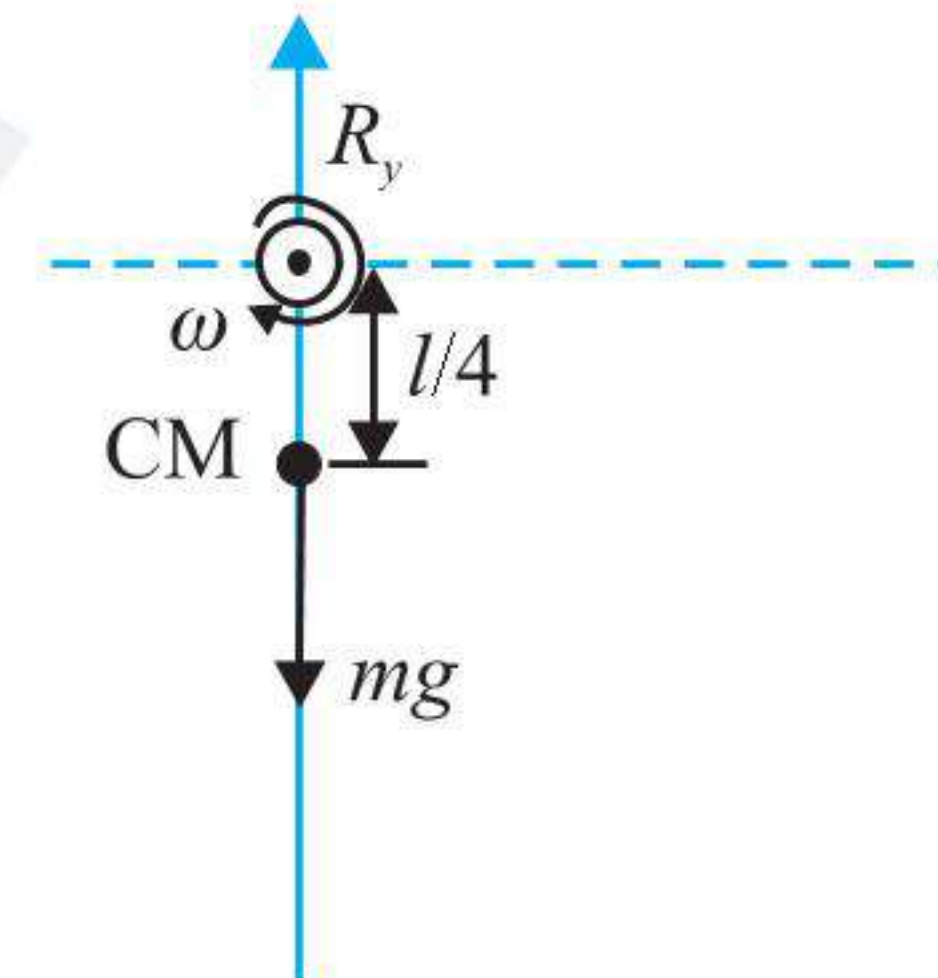
59. (3) KE of the rod is maximum when it is in the vertical position. From conservation of energy, we get

$$\frac{mgl}{4} = \frac{1}{2} \left[\frac{mL^2}{12} + \frac{mL^2}{16} \right] \omega^2$$

$$\frac{\omega^2 l}{4} = \frac{6g}{7}$$

$$\text{and } R_y - mg = m(l/4) \omega^2$$

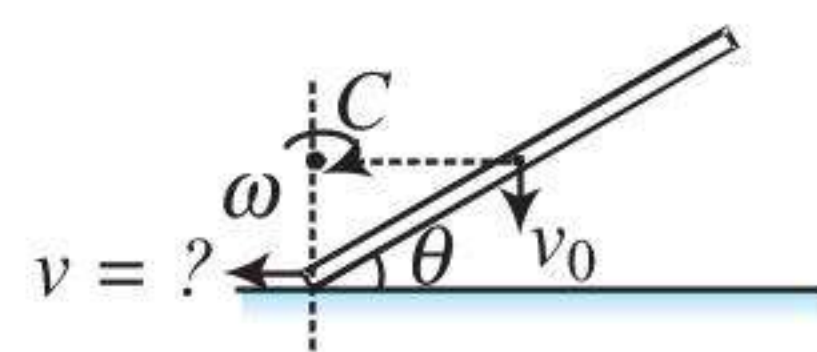
$$R_y = mg + m \left(\frac{6g}{7} \right) = \frac{13mg}{7}$$



60. (4) C is instantaneous centre of rotation

$$\omega \frac{L}{2} \cos \theta = v_0$$

$$\omega \frac{L}{2} \sin \theta = v \Rightarrow v = v_0 \tan \theta$$



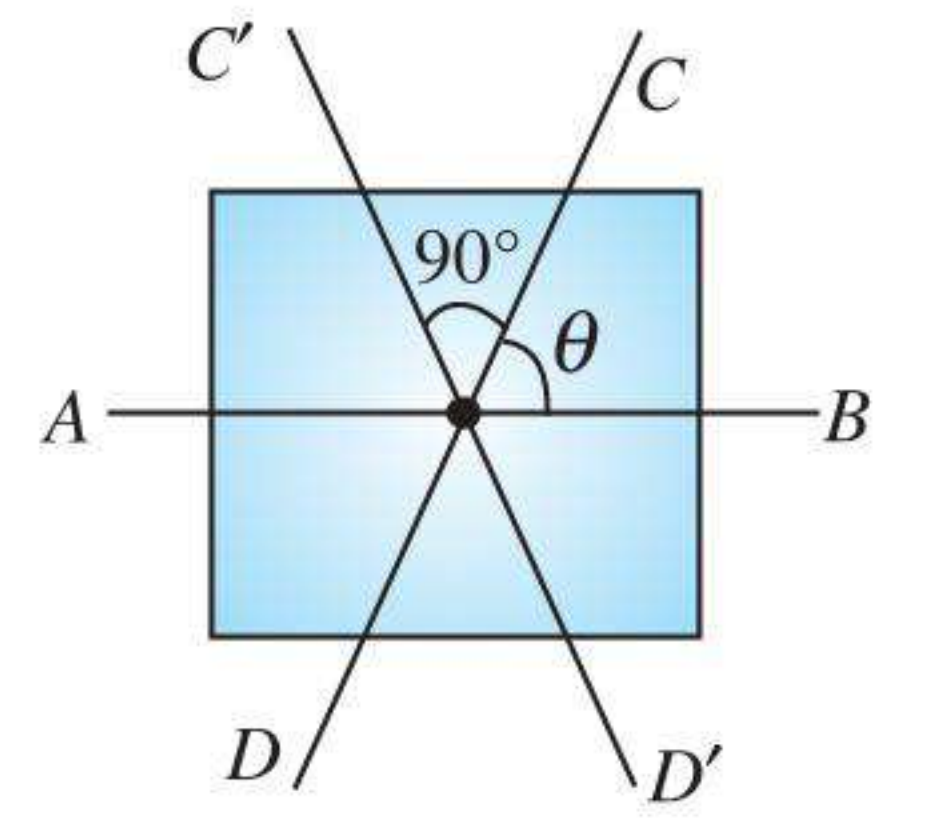
61. (1) As shown in the figure

$$I_{CD} = I_{C'D'}$$

$$\text{So, } 2I_{CD} = I \Rightarrow I_{CD} = \frac{I}{2}$$

$$\text{Also, } 2I_{AB} = I \Rightarrow I_{AB} = \frac{I}{2}$$

$$\text{So } I_{CD} = I_{AB}$$



62. (2) Moment of inertia of semicircular portions about x and y axes are same. But moment of inertia of straight portions about x-axis is zero.

$$\therefore I_x < I_y \text{ or } \frac{I_x}{I_y} < 1$$

63. (1) $m_2 g \cdot 1 = m_1 g \cdot 3 \Rightarrow m_2 = 3m_1$

$$4m_1 g \cdot 3 = m_3 g \Rightarrow m_3 = 12m_1$$

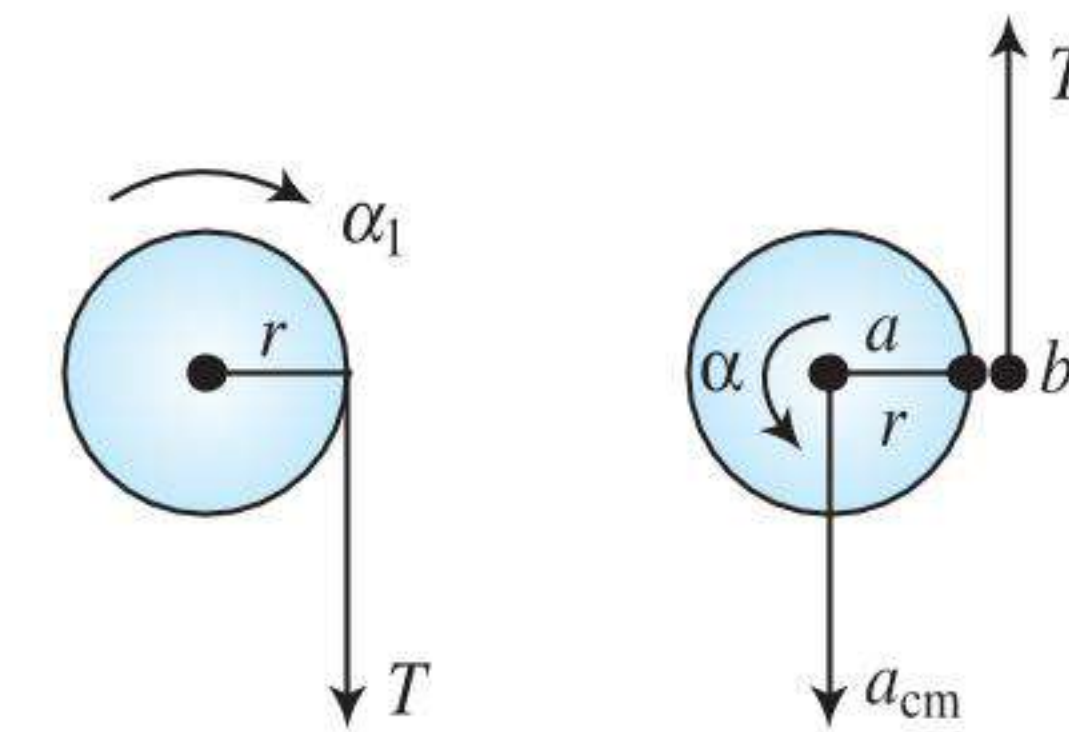
$$16m_1 g \cdot 3 = m_4 g \Rightarrow m_1 = \frac{m_4}{48} = \frac{48}{48} = 1 \text{ kg}$$

64. (2) $T_r = \frac{mr^2}{2} \alpha_1$... (i)

$$T_r = \frac{mr^2}{2} \alpha$$
 ... (ii)

$$\alpha_1 = \alpha$$
 ... (iii)

From (i) and (ii)



Acceleration of point b = acceleration of point a

$$r\alpha_1 = a_{cm} - r\alpha$$

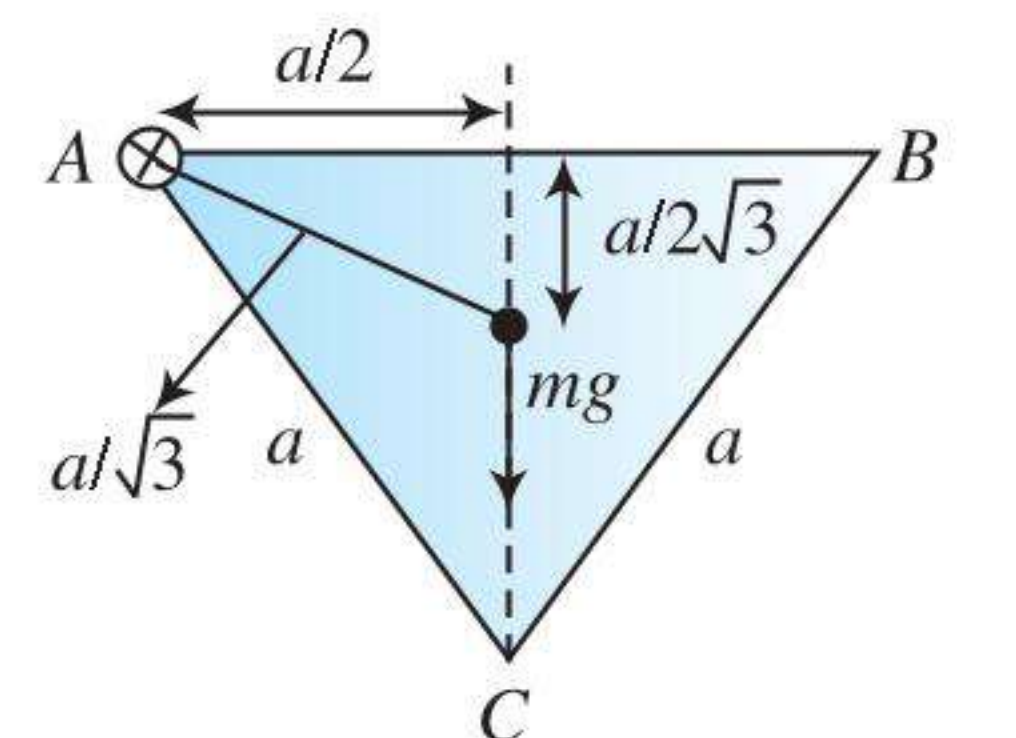
$$\text{Hence, } a_{cm} = 2r\alpha$$

... (iv)

65. (3) Torque about: $mg \frac{a}{2} = I\alpha$

$$\Rightarrow \alpha = \frac{mga}{2I}$$

$$\Rightarrow \text{Acceleration} = \frac{a}{\sqrt{3}} \alpha = \frac{mga^2}{2\sqrt{3}I}$$



66. (2) Net torque, $\tau = 30g \times 2 - 20g \times 2 = 20g$

$$I = 30 \times 2^2 + 20 \times 2^2 = 120 + 80 = 200 \text{ kg m}^2$$

$$\alpha = \frac{\tau}{I} = \frac{20g}{200} = \frac{g}{10} = \frac{10}{10} = 1.0 \text{ rad s}^{-2}$$

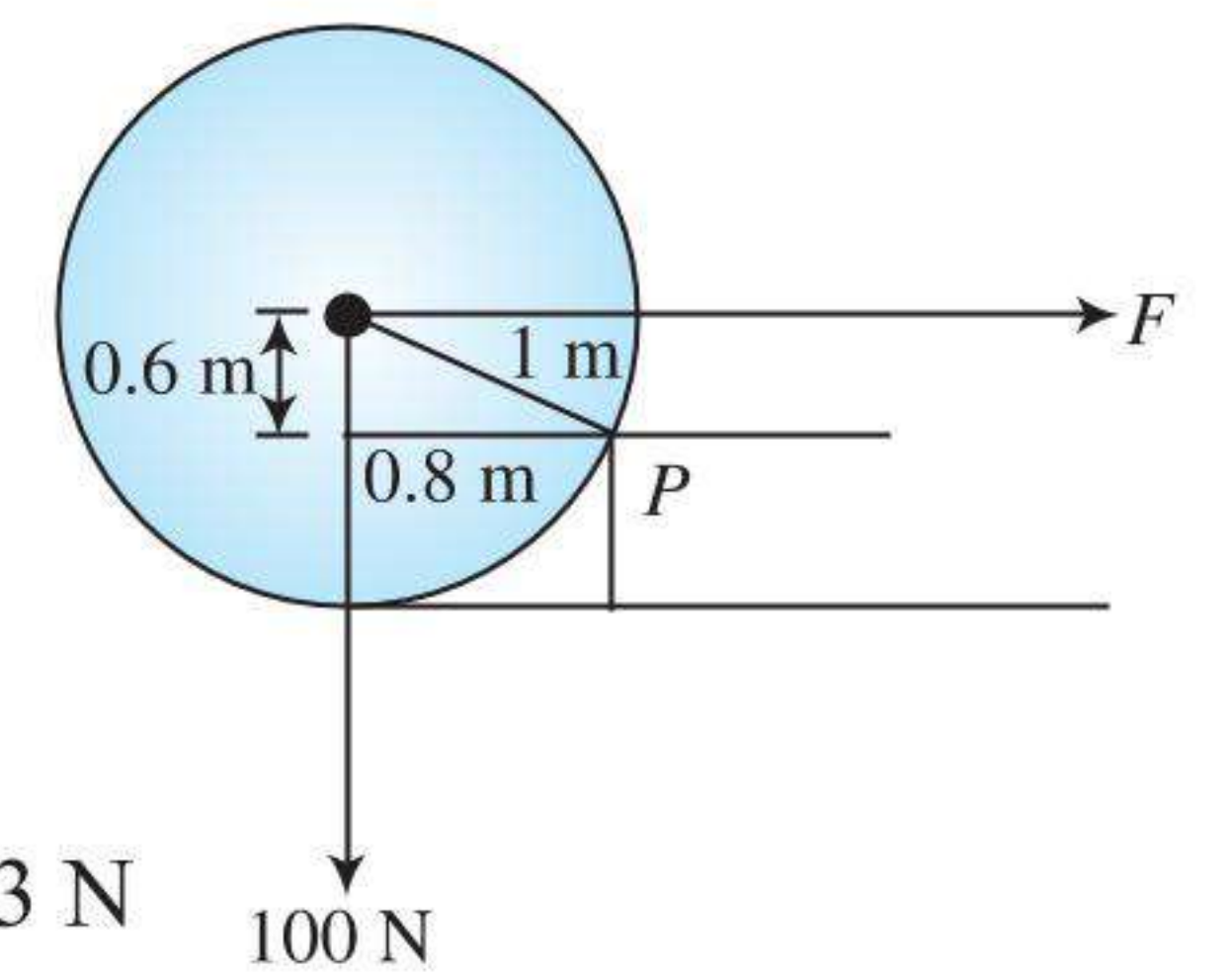
67. (3) $F' \times \frac{R}{5} = FR$ or $F' = 5F$

68. (4) Taking moments of forces about point P, we get

$$F \times 0.6 = 100 \times 0.8$$

$$\text{or } F = \frac{100 \times 0.8}{0.6} \text{ N}$$

$$= \frac{800}{6} \text{ N} = \frac{400}{3} \text{ N} = 133.3 \text{ N}$$



69. (3) Equation of motion is

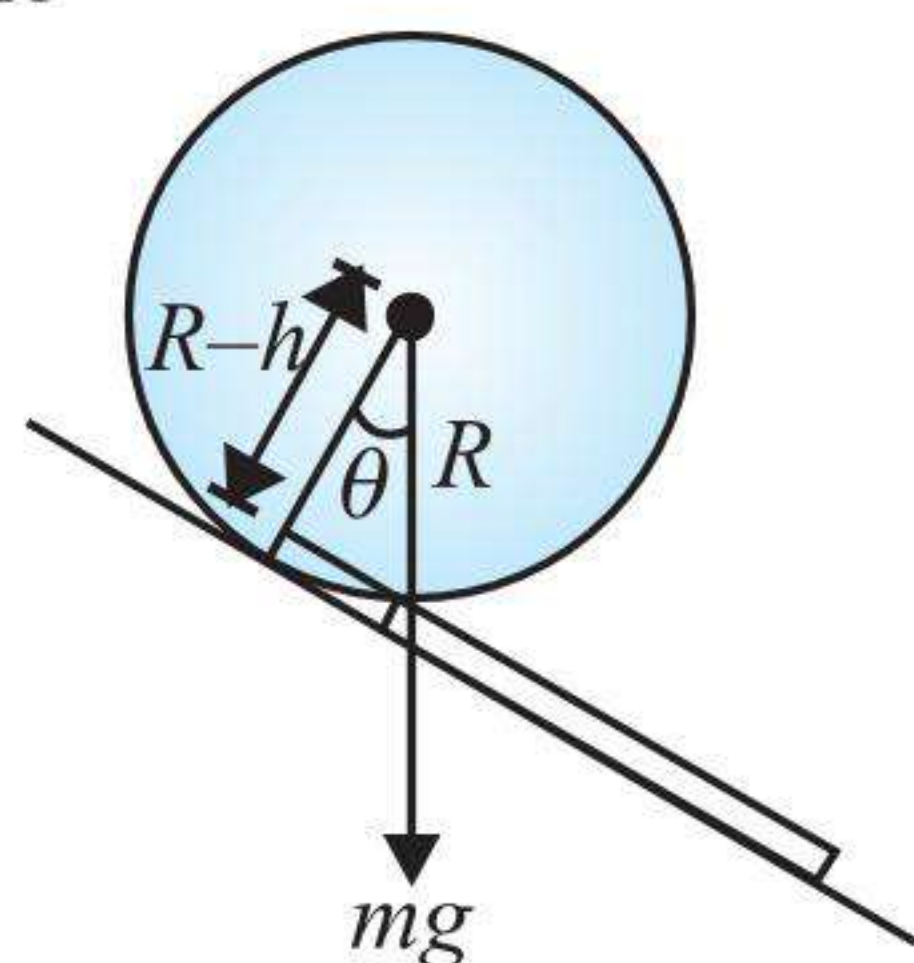
$$T = Mg - Ma = Mg - \frac{2}{3} Mg = \frac{1}{3} Mg$$

70. (1) The horizontal component of the force exerted on the bar by the hinge must balance the applied force $\vec{F}$, and so has magnitude 120.0 N and is to the left. Taking torques about point A, $(120.0 \text{ N})(4.00 \text{ m}) + F_v(3.00 \text{ m})$ so the vertical component is -160 N , with the minus sign indicating a downward component, exerting a torque in a direction opposite that of the horizontal component. The force exerted by the bar on the hinge is equal in magnitude and opposite in direction to the force exerted by the hinge on the bar.

$$\text{Net force} = \sqrt{[(160)^2 + (120)^2]} = 200 \text{ N}$$

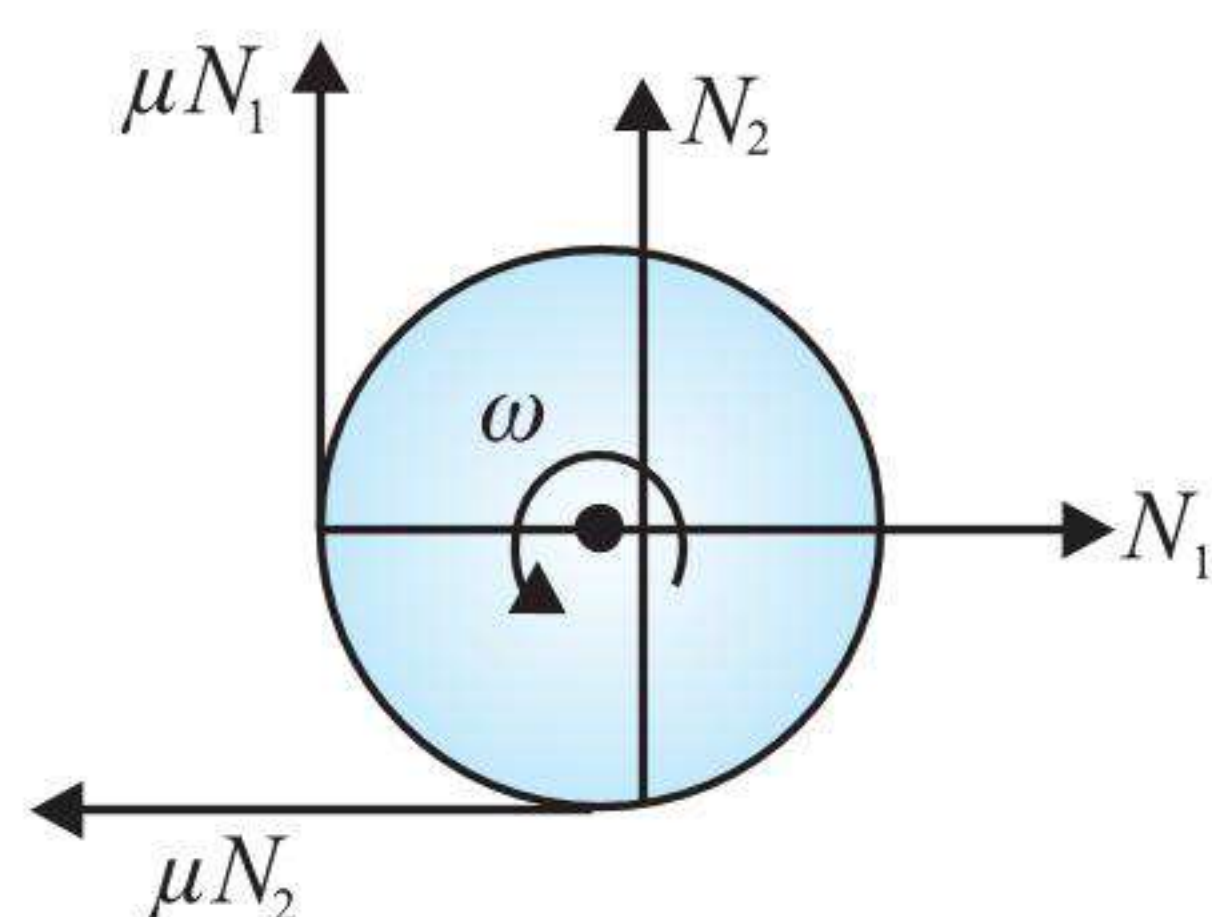
71. (4) The sphere is on the verge of toppling when a line of action of weight passes through the edge.

$$\cos \theta = \frac{(R-h)}{R} \Rightarrow h = R - R \cos \theta$$



72. (2) In the figure, for translational equilibrium in vertical direction,

$$\mu N_1 + N_2 = mg \quad \dots(i)$$



In horizontal direction,

$$N_1 = \mu N_2 \quad \dots(ii)$$

Solving, Eqs. (i) and (ii), $N_2 = \frac{mg}{1 + \mu^2}$

The required friction is $\mu N_2 = f_a = \frac{3}{10} mg$

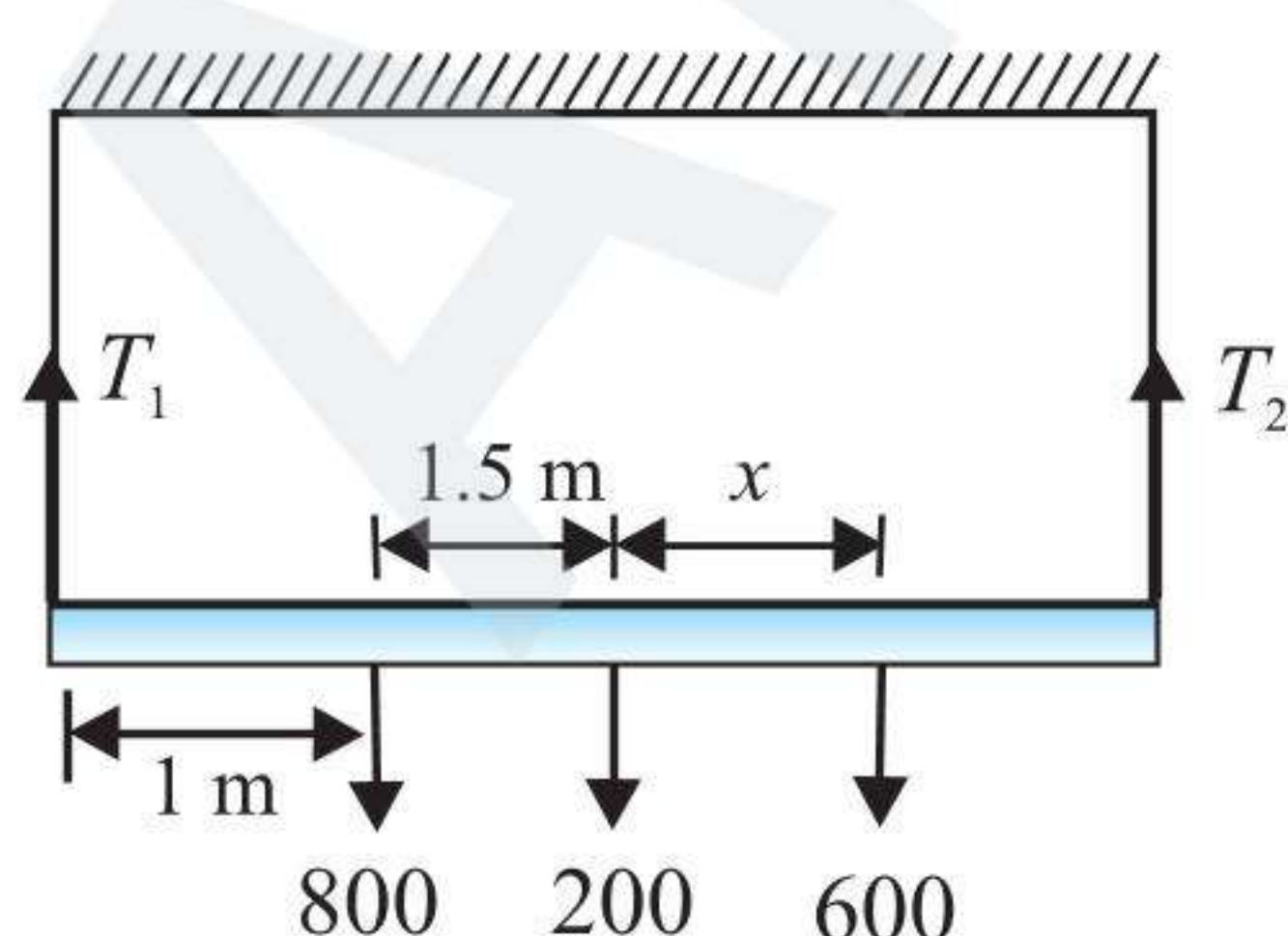
In the figure, the sphere has the tendency to move towards the right;

$$N_1 = 0; N_2 = mg$$

$$f_b = \mu N_2 = \frac{mg}{3}$$

which gives $\frac{f_a}{f_b} = \frac{9}{10}$

73. (3) For vertical equilibrium, $T_1 + T_2 = 1600 \text{ N} \quad \dots(i)$



Take moment about O,

$$T_1 \times 2.5 + 600x = T_2 \times 2.5 + 800 \times 1.5 \quad \dots(ii)$$

Solving Eqs. (i) and (ii)

$$T_1 = 1040 - 120x$$

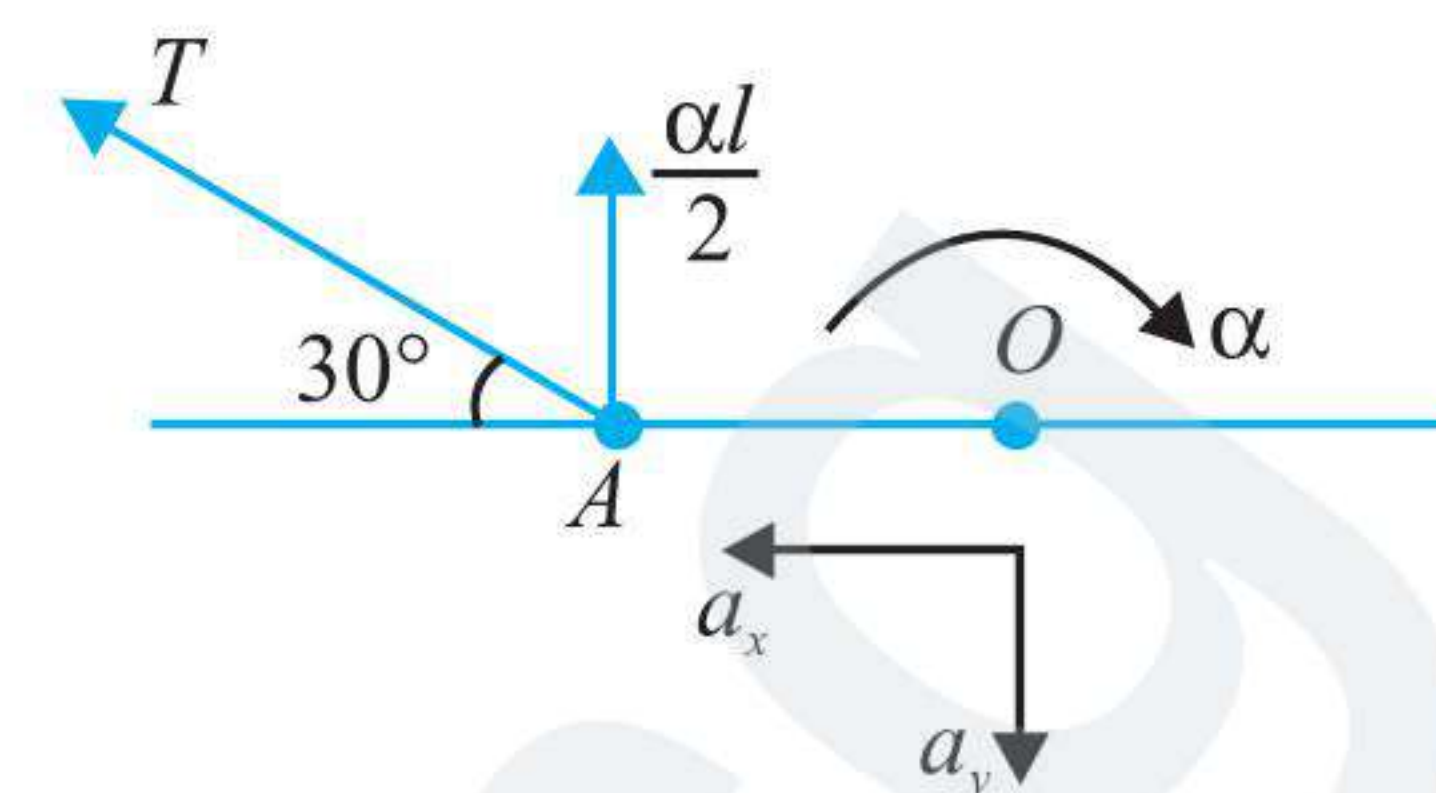
$$\text{and } T_2 = 560 + 120x$$

For safe working, $T_1 < 1040 \text{ N}$ and $T_2 < 1040 \text{ N}$

$x > 0$ and $x < 2$, so the range is $0 < x < 2$.

74. (4) Acceleration of point A along the string is zero. Therefore,

$$a_x \cos 30^\circ - a_y \cos 60^\circ + \frac{\alpha l}{2} \cos 60^\circ = 0$$



75. (1) $T = mg$

$$f = Mg \sin \theta + T \sin \theta$$

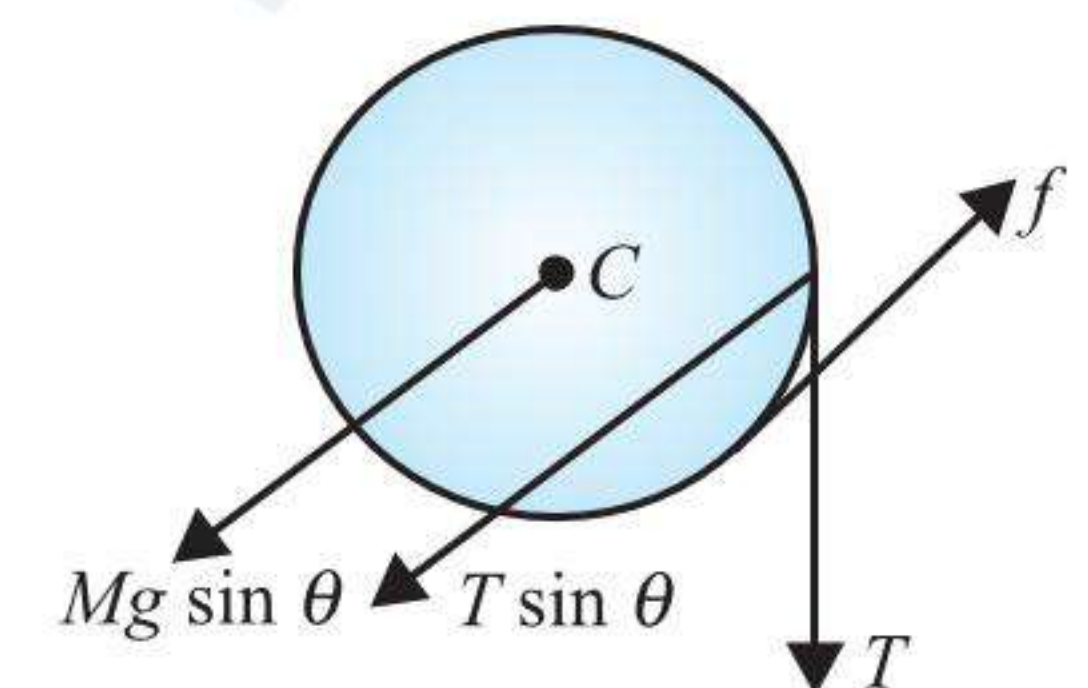
$$\Rightarrow f = Mg \sin \theta + mg \sin \theta$$

Balancing torque about C:

$$Tr = fr$$

$$\Rightarrow mg = Mg \sin \theta + mg \sin \theta$$

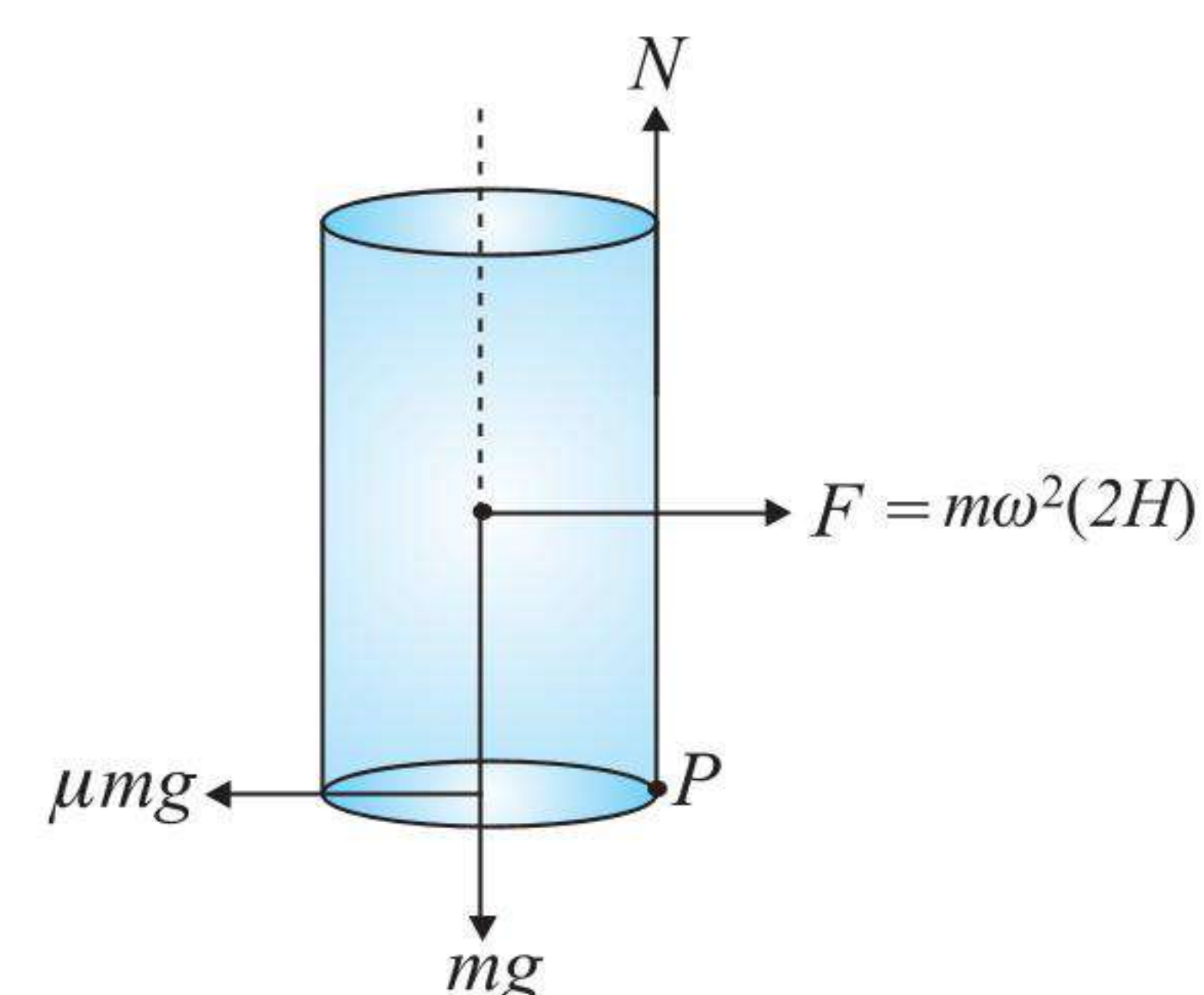
$$\Rightarrow m = \frac{M \sin \theta}{1 - \sin \theta}$$



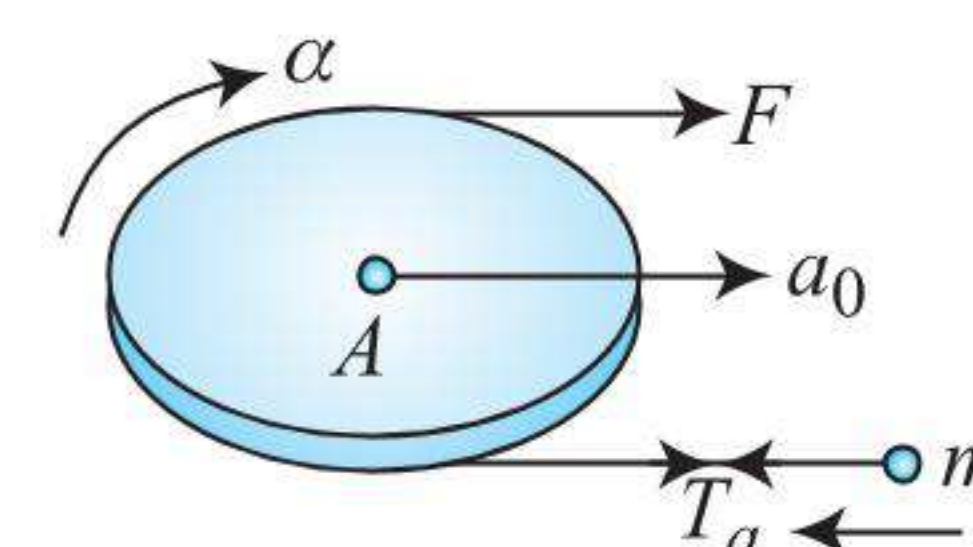
76. (4) $N = mg \quad \dots(i)$

Net moment of all the forces about point P is zero.

$$F \frac{H}{2} = mg \frac{H}{8}, \quad \omega = \sqrt{\frac{g}{8H}}$$



77. (1) For no slipping condition of thread acceleration of point A on the circumference of the disc = acceleration of the small body



$$a = R\alpha - a_0 \quad \dots(i)$$

Other equations are

$$T = ma \quad \dots(ii)$$

$$F + T = ma_0 \quad \dots(iii)$$

$$\text{And } (F - T)R = \frac{1}{2} mR^2 \cdot \alpha$$

$$\Rightarrow F - T = \frac{1}{2} mR\alpha \quad \dots(iv)$$

$$\text{Using (i) in (iv), } F - T = \frac{1}{2} m(a + a_0)$$

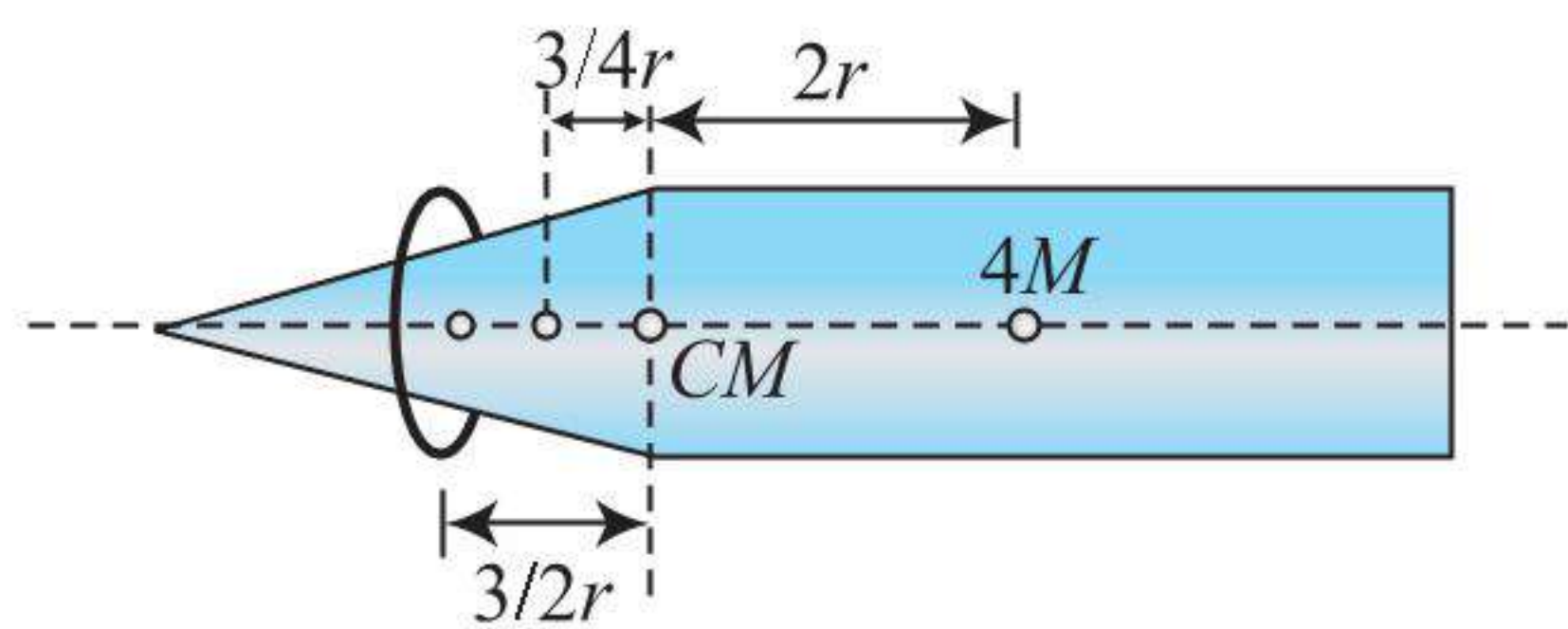
$$\text{Using (ii) in this, } F = \frac{2}{3} ma + \frac{1}{2} ma \quad \dots(v)$$

$$\text{From (ii) and (iii), } F = ma_0 - ma \quad \dots(vi)$$

$$\text{Solving (v) and (vi), } a = \frac{F}{4m}$$

78. (2) In critical case the COM of the system will lie at the centre of the circular base of the conical part (Just above A ; at O)

$$\text{Volume of cone} = \frac{1}{3} \pi r^2 (3r) = \pi r^3$$



$$\text{Volume of cylinder} = \pi r^2 (4r) = 4\pi r^3$$

$\therefore$ Mass of cylinder is $4M$

COM of the cone is at a distance of $\frac{h}{4} = \frac{3R}{4}$ from O.

Let mass of the ring = m

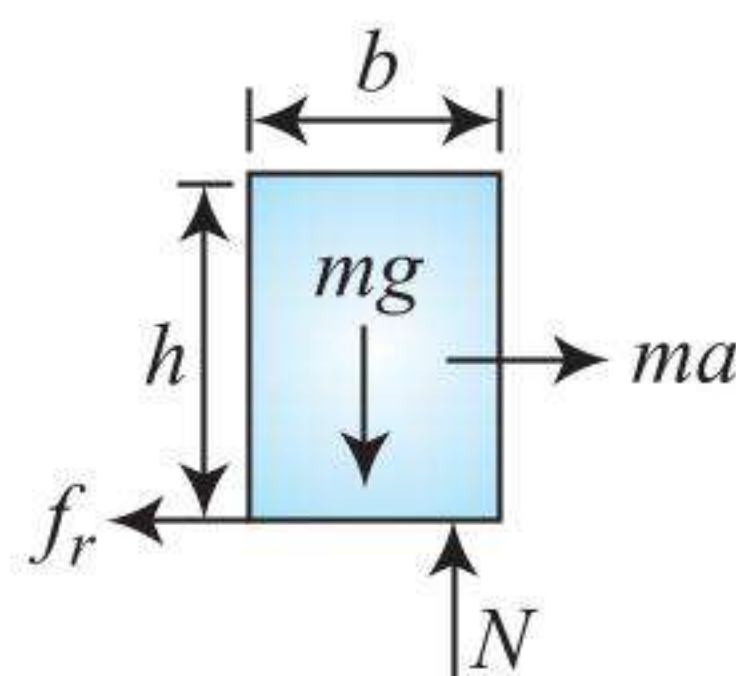
$$m \times 1.5r + M \times 0.75r = 4M \times 2r \quad \therefore m = \frac{29}{6} M$$

79. (4) From FBD of the block, taking torque about right edge from where normal reaction is passing.

$$ma \frac{h}{2} \leq mg \frac{b}{2} \Rightarrow a \leq \frac{b}{h} g$$

Final velocity of truck is zero. So that

$$0 = v^2 - 2 \left(\frac{b}{h} g \right) l \Rightarrow l = \frac{hv^2}{2bg}$$



Multiple Correct Answers Type

1. (2), (3)

We know that $K = \sqrt{\frac{I}{M}}$, $I \propto M$ hence radius of gyration does not depend upon mass of the body. Further I depends upon distribution of mass of body and also on the axis of rotation.

Hence K also depends upon these factors.

2. (3), (4)

All points in the body, in plane perpendicular to the axis of rotation, revolve in concentric circles. All points lying on the circle of same radius have same speed (and also same magnitude of acceleration) but different directions of velocity (also different directions of acceleration).

Hence there cannot be two points in the given plane with same velocity or with same acceleration. As mentioned above, points lying on circle of same radius have same speed.

Angular speed of body at any instant w.r.t. any point on the body is same by definition.

3. (1), (2), (3)

According to the theorem of perpendicular axis:

Moment of inertia of the plate about an axis passing through centre O and perpendicular to the plate is $I_0 = I_1 + I_2 = I_3 + I_4$.

(Because diagonals 1 and 2 as well as 3 and 4 are mutually perpendicular)

Now, according to symmetry,

$$I_1 = I_2 = I \text{ (say)}$$

$$\text{and } I_3 = I_4 = I' \text{ (say)}$$

$$\text{For a square plate, } I_0 = \frac{M(L^2 + L^2)}{12} = \frac{ML^2}{6}$$

$$\text{Hence, } I = I' = \frac{ML^2}{12}$$

$$\text{i.e., } I_1 = I_2 = I_3 = I_4 = \frac{ML^2}{12}$$

$$\text{Hence, } I_0 = I_1 + I_2 = I_3 + I_4 = I_1 + I_3$$

i.e., options (1), (2) and (3) are correct.

4. (1), (2), (3)

$$210 - T = 21a \quad \dots(i)$$

Again, $\tau = I\alpha$

$$TR = \frac{Ia}{R}$$

$$T = \frac{Ia}{R^2} = \frac{\frac{1}{2} \times 2I \times R^2 \times a}{R^2} = \frac{1}{2} \times 21a$$

$$21a = 2T$$

From Eq. (i), $10 - T = 2T$

$$\text{or } 3T = 210$$

$$\text{or } T = 70 \text{ N}$$

Again from Eq. (i), $210 - 70 = 21a$

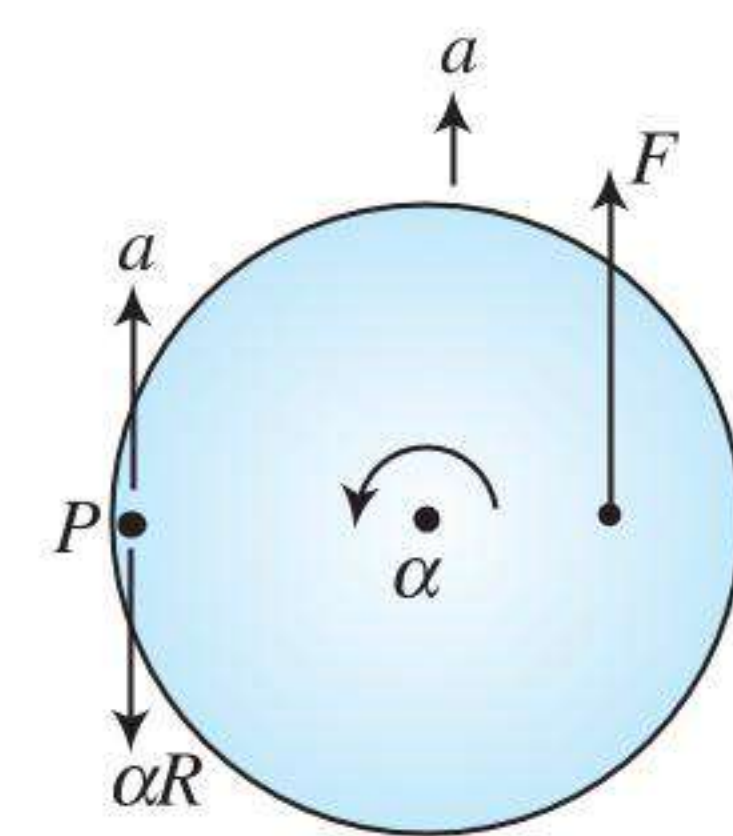
$$a = \frac{140}{21} \text{ ms}^{-2} = \frac{20}{3} \text{ ms}^{-2}$$

5. (1), (2), (3)

$$F = ma$$

$$F \frac{R}{2} = \frac{1}{2} mR^2 \alpha$$

$$a_p = a - \alpha R = 0$$



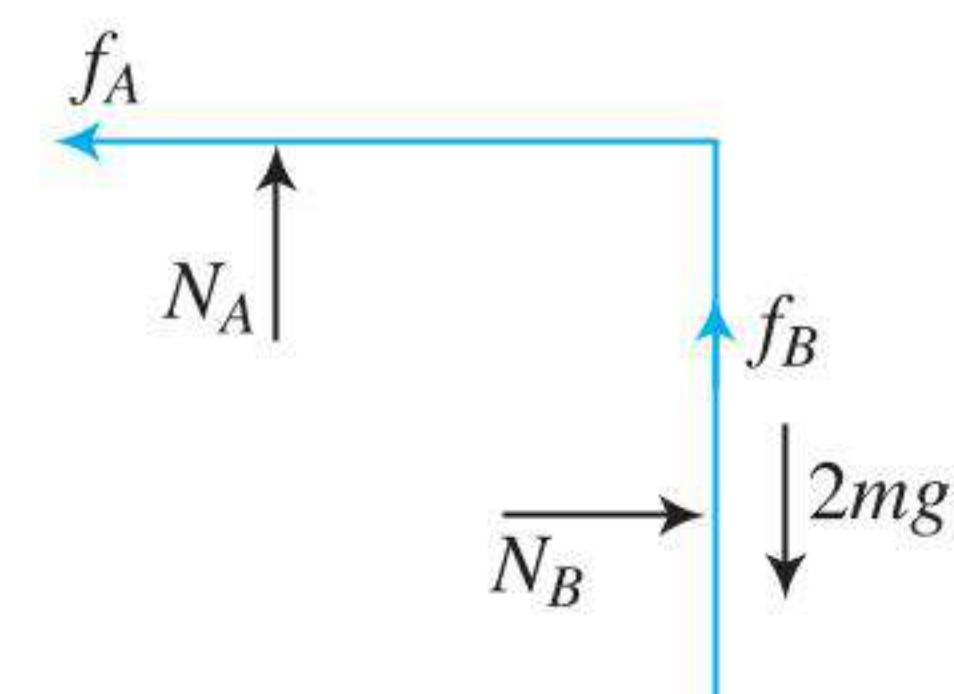
6. (1), (3)

$$N_A + f_B = 2mg$$

$$N_B = f_A$$

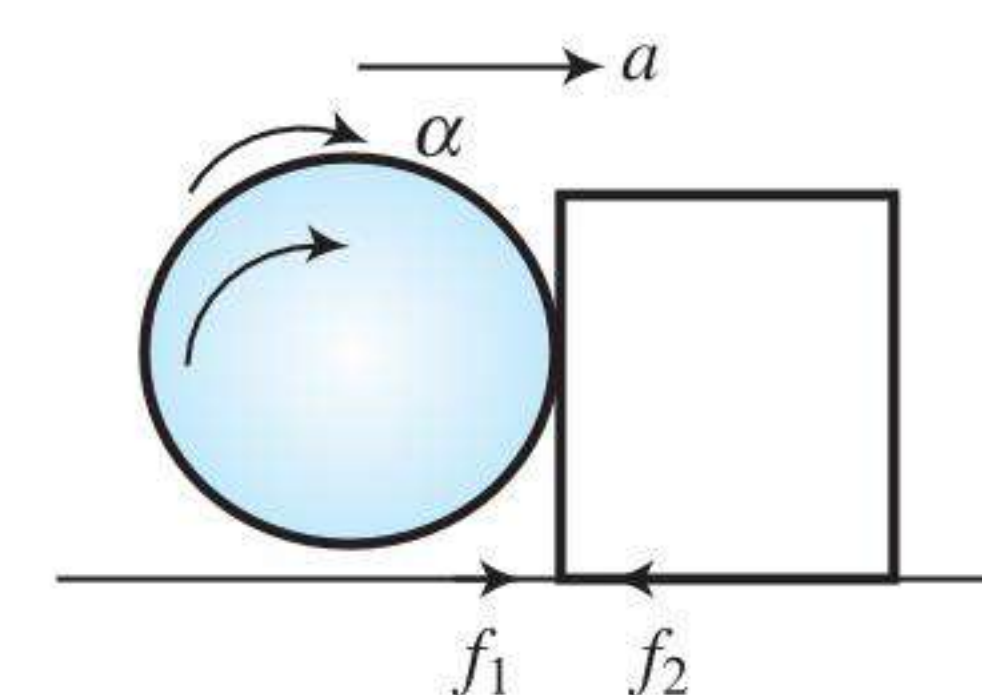
$$mgR - f_A R - f_B R = 0$$

$$mg = f_A + f_B \text{ and given } f_A = f_B$$



7. (3), (4)

$$a = \frac{f_1 - f_2}{m_1 + m_2} = \frac{0.4 \times 6g - 0.5 \times 3g}{6 + 3}$$



$$\Rightarrow a = 1 \text{ m s}^{-2}$$

$$\alpha = \frac{\tau_{\text{applied}} - \tau_{\text{friction}}}{I} = \frac{6 - f_1 R}{\frac{1}{2} 6(R)^2}$$

$$= \frac{6 - 0.4 \times 6g(0.2)}{\frac{1}{2} \times 6 \times (0.2)^2} = 10 \text{ rad s}^{-2}$$

8. (1), (2), (3)

Parallel axis theorem, check the distance carefully. $I_D = I_B$ (symmetric)

$$I_B = I_A + M \left(\frac{a}{2} \right)^2 = \frac{5}{12} Ma^2$$

$$I_C = I_A + M \left(\frac{a}{\sqrt{2}} \right)^2 = \frac{2}{3} Ma^2$$

9. (1), (4)

$$mgr = fR \quad \dots(i)$$

$$N_1 \sin \theta + f = mg \quad \dots(ii)$$

$$N_1 \cos \theta = N_2 \quad \dots(iii)$$

$$\text{From (i), } f = \frac{mgr}{R}$$

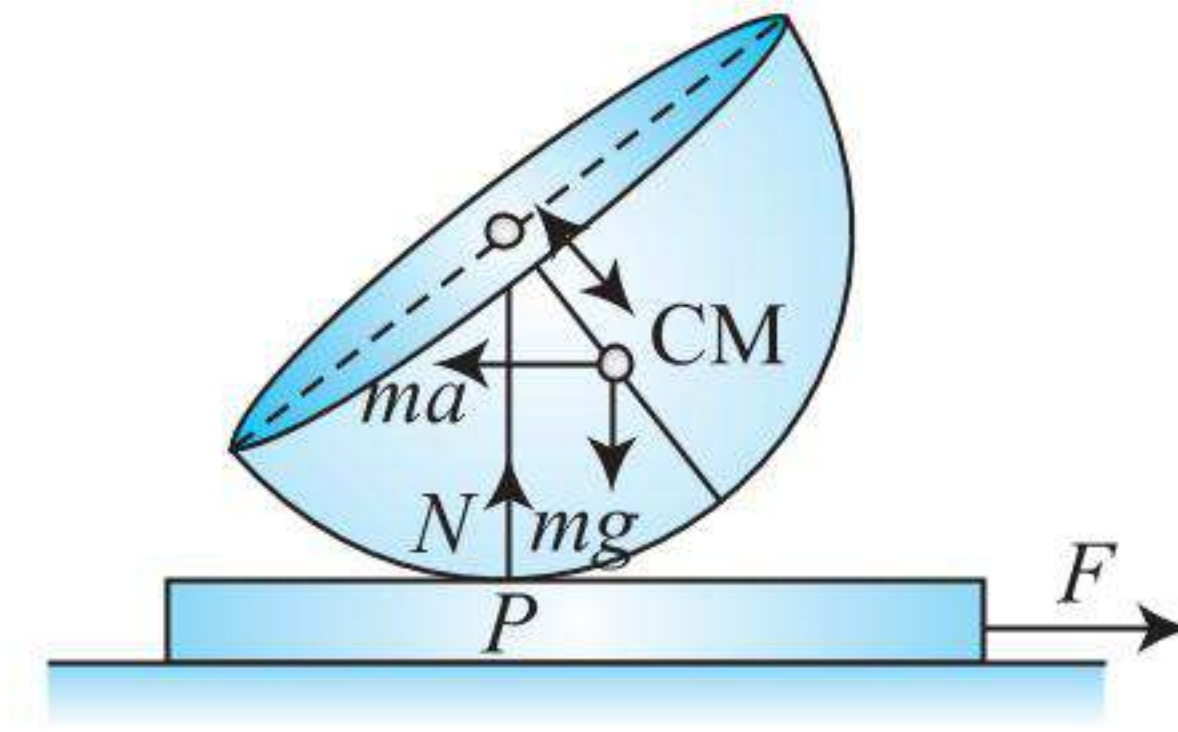
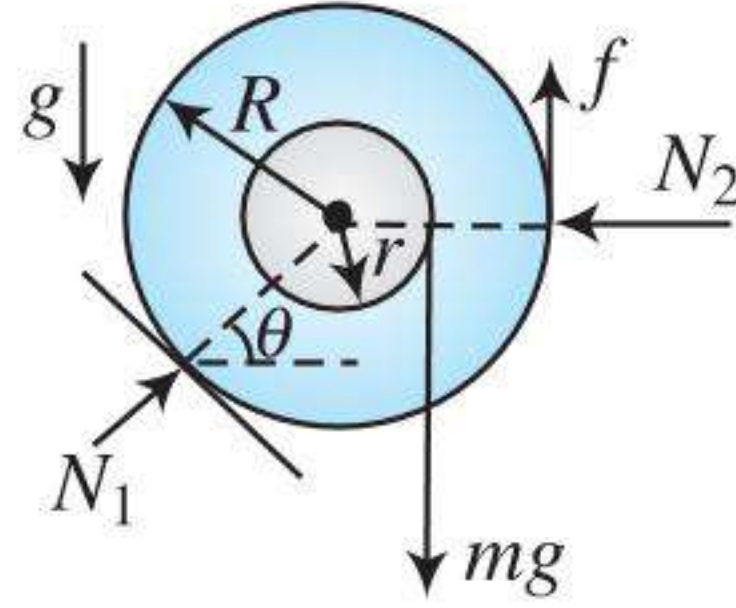
From (ii) and (iii),

$$N_2 = \frac{mg - f}{\tan \theta} = \frac{mg}{\tan \theta} \left[1 - \frac{r}{R} \right]$$

$$\text{Net force at B: } F_B = \sqrt{f^2 + N_2^2}$$

For minimum value of μ : $f \leq \mu N_2$

$$\Rightarrow \frac{mgr}{R} \leq \frac{\mu mg}{\tan \theta} \left[1 - \frac{r}{R} \right] \Rightarrow \mu \geq \frac{\tan \theta}{R/r - 1}$$



$$a = \frac{g \sin \theta}{2 - \cos \theta} \text{ when } \theta = 37^\circ$$

$$a = 5 \text{ m/s}^2$$

$$\text{and } f = ma$$

$$\mu N = ma$$

$$\Rightarrow \mu \cdot mg = ma$$

$$\Rightarrow \mu = \frac{a}{g} = \frac{1}{2} \text{ and } F - \mu(2mg) = 2ma \quad \dots(ii)$$

$$\text{So, } F = 100 \text{ N}$$

15. (1), (3)

Torque about 'P':

$$F\sqrt{3}a = mg \frac{a}{2}$$

$$\Rightarrow F = \frac{mg}{2\sqrt{3}}$$

Torque about 'O':

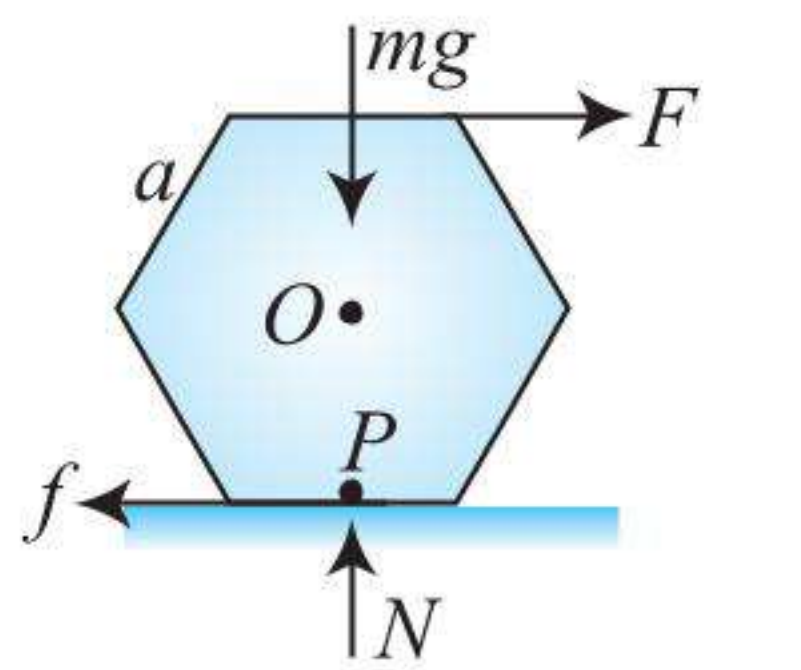
$$\frac{mg}{2\sqrt{3}} \left(\frac{\sqrt{3}}{2} a \right) + \mu mg \left(\frac{\sqrt{3}}{2} a \right) = mg \frac{a}{2}$$

$$\Rightarrow \mu = \frac{1}{2\sqrt{3}}$$

When $\mu = 2\mu_{\min} = \frac{1}{\sqrt{3}}$ and $F = \frac{mg}{\sqrt{3}}$, taking torque about P

$$\frac{mg}{\sqrt{3}} \cdot \sqrt{3}a - mg \frac{a}{2} = \frac{17}{12} ma^2 \alpha$$

$$\Rightarrow \alpha = \frac{6}{17} \frac{g}{a}$$



10. (1), (3)

In case of pure rotation, the axis of rotation should be at rest. The axis of rotation may exist anywhere about which all particles of body will be in a circular motion with all centers lying on the axis of rotation.

11. (2), (4)

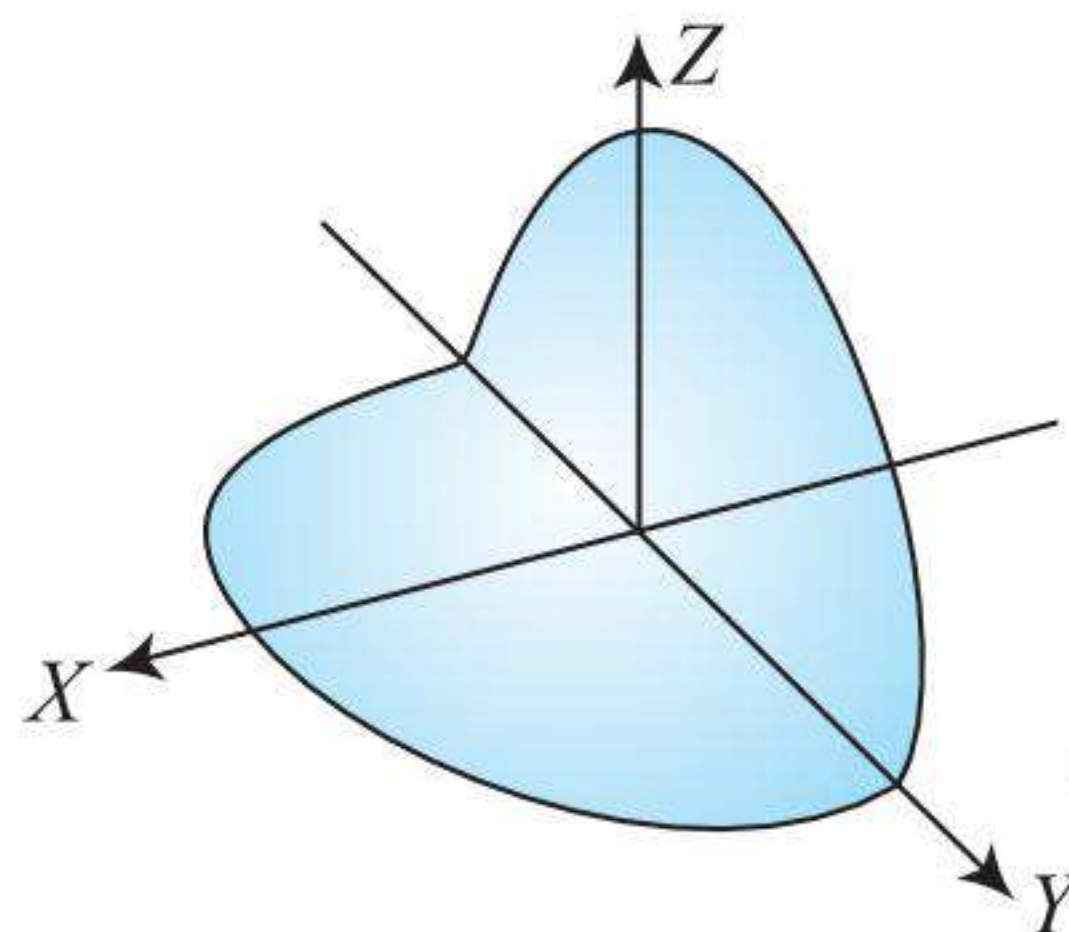
As the sphere is rotating uniformly the tangential acceleration of any particle on sphere will be zero and all particles will only have normal acceleration.

The particles on the diameter do not have any linear acceleration as they have zero radius of rotation.

The particles on the surface have the different radius of rotation hence different linear speeds.

12. (2), (3)

From figure we have



$$I_Y = \frac{1}{2} mR^2 \text{ and } I_X = I_Z = \frac{3}{4} mR^2$$

13. (1), (2), (4)

$$I_{xx} = m \left(\frac{a}{2} \right)^2 + m \left(\frac{a}{2} \right)^2 + m \left(\frac{a}{2} \right)^2 + m \left(\frac{a}{2} \right)^2 = ma^2$$

$$I_{yy} = m \left(\frac{a}{2} \right)^2 + m \left(\frac{a}{2} \right)^2 + m \left(\frac{a}{2} \right)^2 + m \left(\frac{a}{2} \right)^2 = ma^2$$

$$I_{AA'} = m \left(\frac{a}{\sqrt{2}} \right)^2 + m \left(\frac{a}{\sqrt{2}} \right)^2 + 0 + 0 = ma^2$$

$$I_{zz} = \left(m \left(\frac{a}{\sqrt{2}} \right)^2 \right) \times 4 = 2ma^2$$

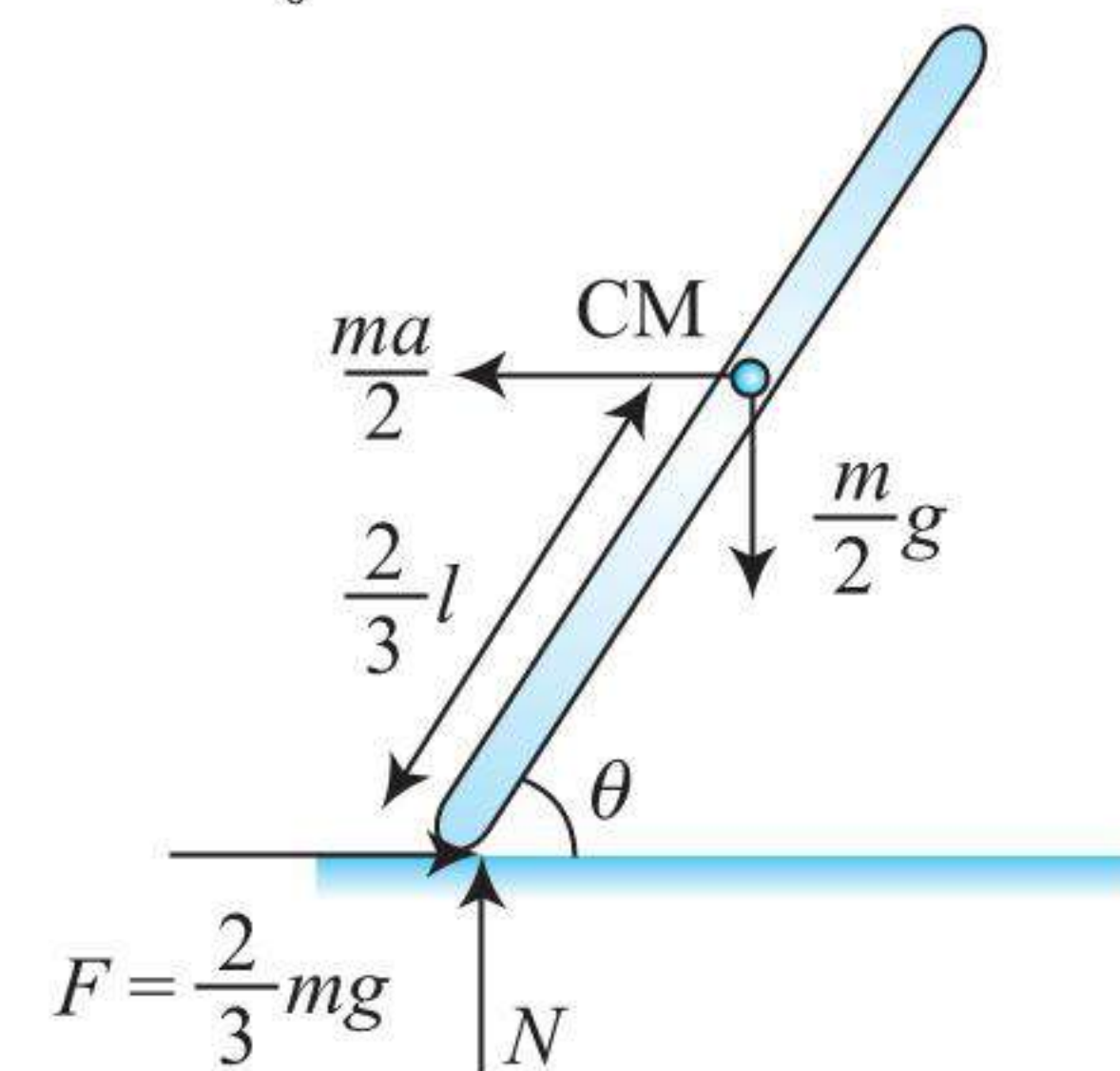
14. (1), (3)

From the plank frame net torque about 'P'

$$ma \left(R - \frac{R}{2} \cos \theta \right) = mg \cdot \frac{R}{2} \sin \theta \quad \dots(i)$$

16. (1), (2), (3)

$$\text{Mass of the rod} = \int_0^l \frac{m}{l^2} r dr = \frac{m}{2}$$



$$\text{Position of centre of mass} = \frac{\int_0^l \left(\frac{m}{l^2} r dr \right) r}{\int_0^l \frac{m}{l^2} r dr} = \frac{2}{3} l$$

$$N = \frac{mg}{2} \quad \dots(i)$$

$$F = \frac{2mg}{3} = \frac{ma}{2} \Rightarrow a = \frac{4}{3} g \quad \dots(ii)$$

Torque about centre of mass, $F \sin \theta = N \cos \theta$

$$\tan \theta = \frac{g}{a} = \frac{3}{4}$$

$$\theta = 37^\circ$$

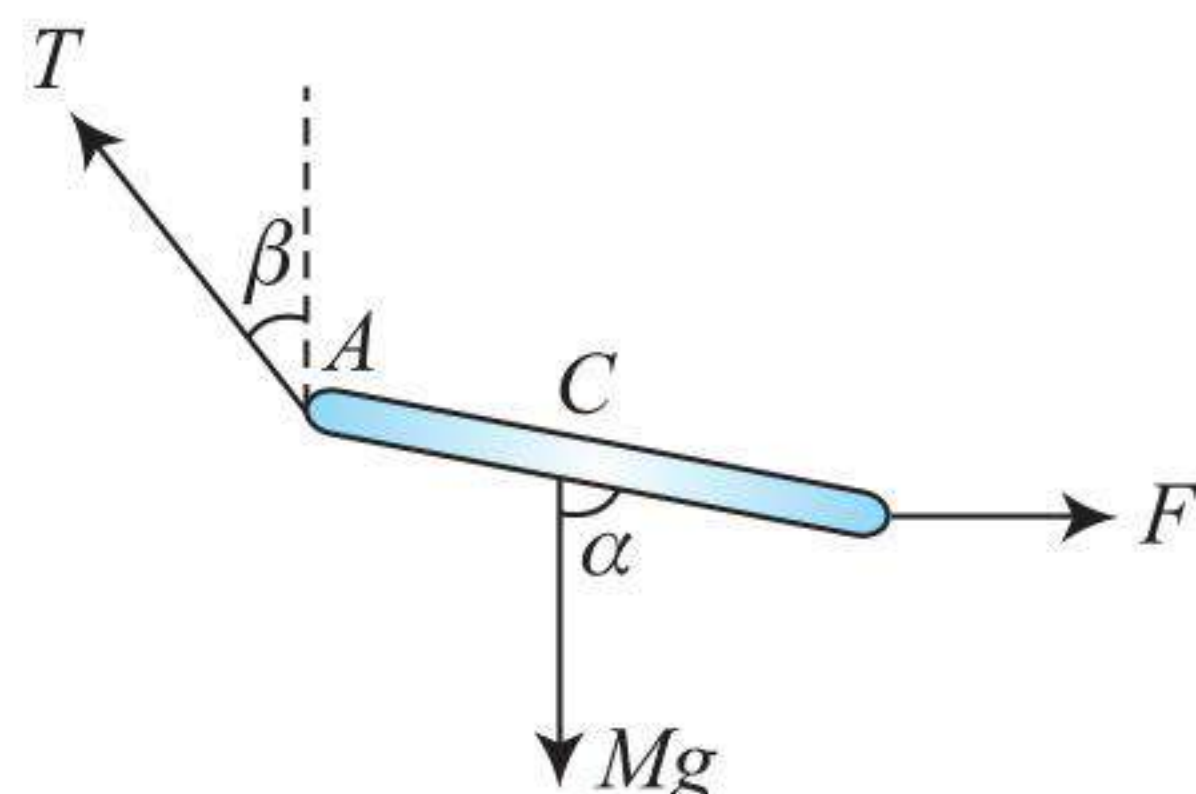
17. (1), (2), (3)

Horizontal and vertical equilibrium of the stick requires -

$$F = T \sin \beta \Rightarrow Mg = T \sin \beta \quad \dots(i)$$

$$\text{and } Mg = T \cos \beta \quad \dots(ii)$$

Solving (i) and (ii), $\tan \beta = 1 \Rightarrow \beta = 45^\circ$



$$T = \sqrt{2} Mg$$

Rotational equilibrium requires torque about C to be zero ($\tau_c = 0$)

$$T \cos \beta \frac{l}{2} \sin \alpha - T \sin \beta \frac{l}{2} \cos \alpha = F \frac{l}{2} \cos \alpha$$

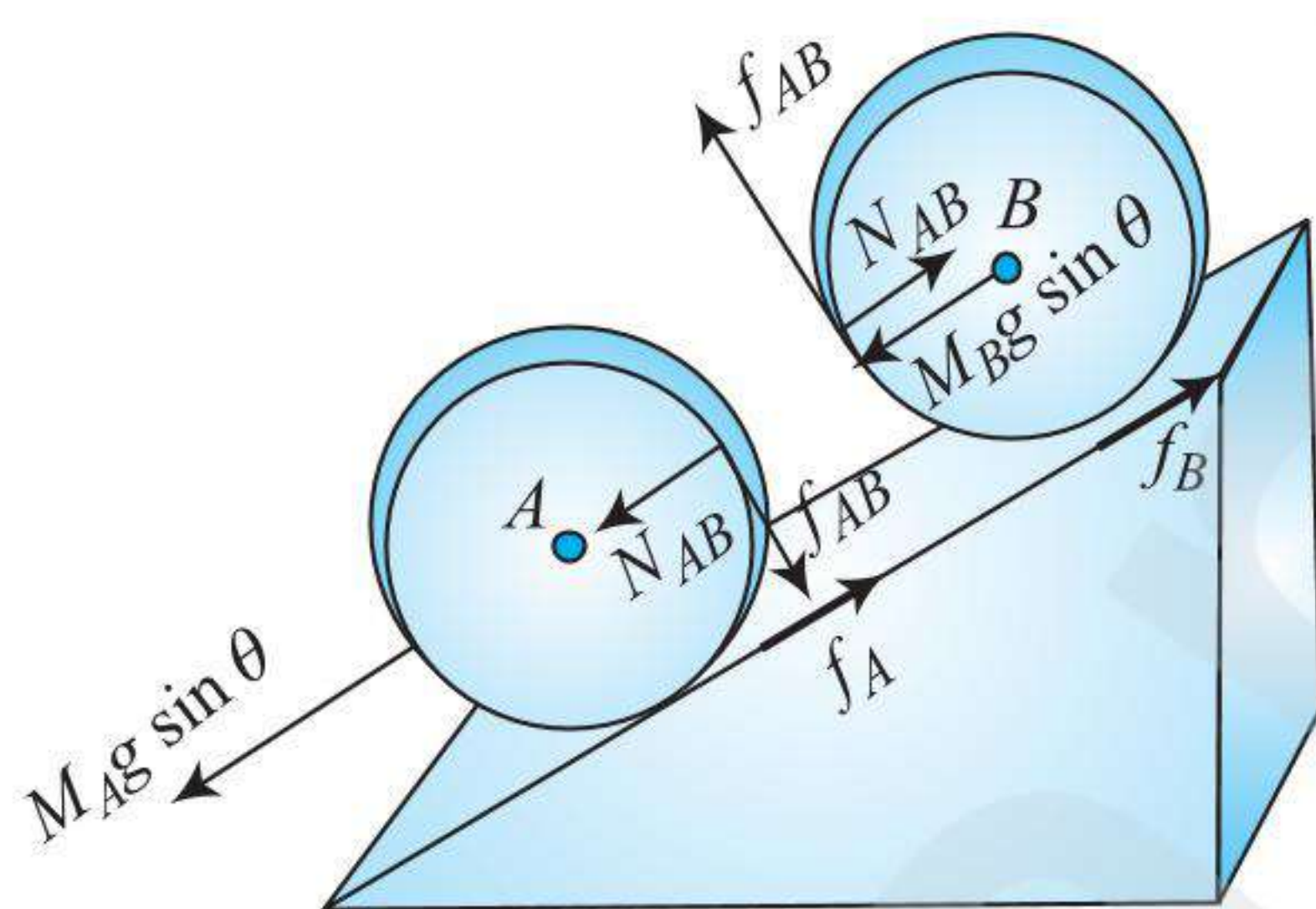
$$T \sin \alpha - T \cos \alpha = \sqrt{2} F \cos \alpha \quad \left[\because \cos \beta = \sin \beta = \frac{1}{\sqrt{2}} \right]$$

$$\sqrt{2} Mg (\sin \alpha - \cos \alpha) = \sqrt{2} Mg \cdot \cos \alpha$$

$$\therefore \tan \alpha = 2 \Rightarrow \alpha = \tan^{-1}(2)$$

18. (1), (2), (3)

The diagram shows the relevant forces on the two cylinders.



f_{AB} = friction between the two cylinders;

f_A = friction between A and the incline;

f_B = friction between B and the incline

For rotational equilibrium of the cylinders

$$f_A = f_{AB} \text{ [for A]}; f_B = f_{AB} \text{ [for B]} \therefore f_A = f_B = f_{AB}$$

For translational equilibrium of (A + B)

$$f_A + f_B = (M_A + M_B)g \sin \theta$$

$$\therefore f_A = f_B = f_{AB} = \left(\frac{M_A + M_B}{2} \right) g \sin \theta$$

For A to be in equilibrium

$$N_{AB} + M_A g \sin \theta = \left(\frac{M_A + M_B}{2} \right) g \sin \theta$$

$$N_{AB} = \left(\frac{M_B - M_A}{2} \right) g \sin \theta$$

For the two cylinders to be in contact

$$N_{AB} > 0 \Rightarrow M_B > M_A$$

19. (1), (2), (4)

From perpendicular axis theorem,

$$I_z = I_1 + I_2 \quad \dots(i)$$

Moment of inertia of disc about a diameter axis, $I_d = \frac{mR^2}{4} = I_3$

Moment of inertia of the disc about an axis passing through the tangent, $I_t = \frac{mR^2}{4} + mR^2 = \frac{5mR^2}{4}$

$$\text{As } I_1 = I_2 = I_t = \frac{5mR^2}{4} \quad \dots(ii)$$

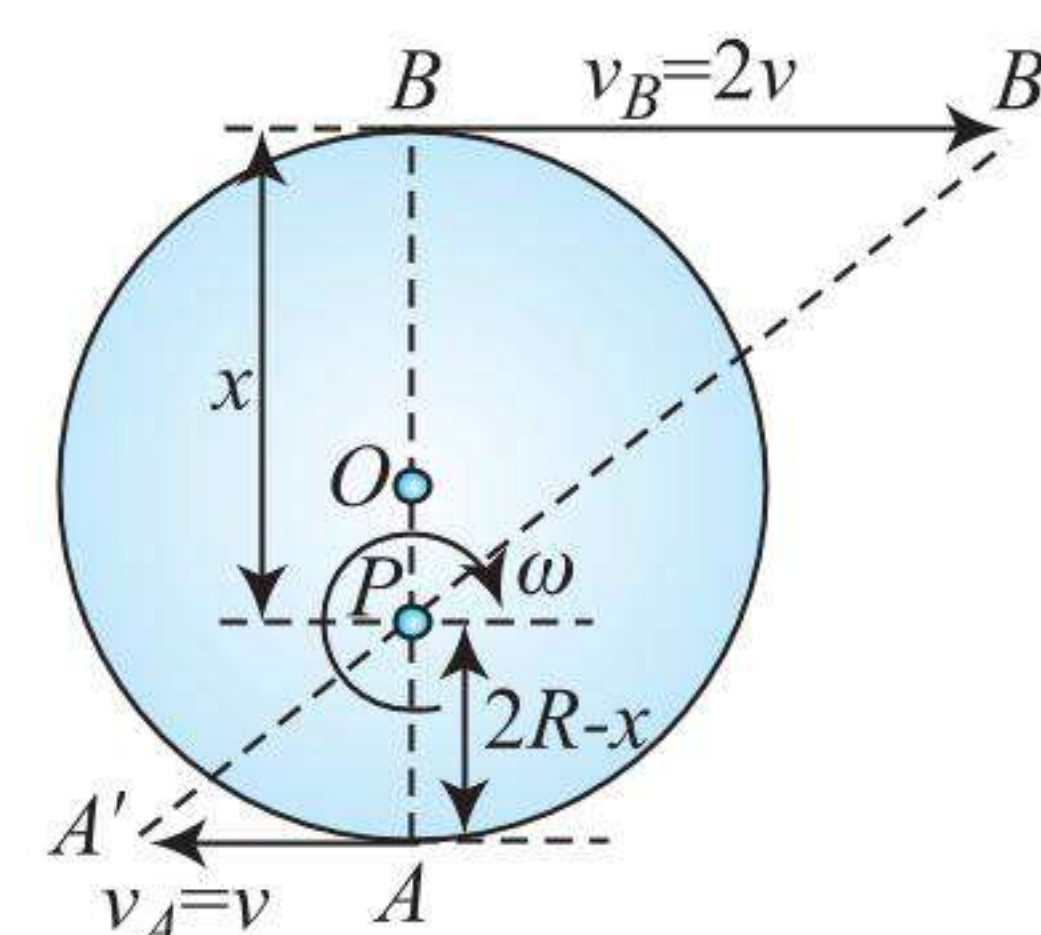
$$\text{From (i) and (ii), } I_z = 2 \times \frac{5mR^2}{4} = \frac{5mR^2}{2}$$

20. (1), (3)

Since, there is no relative sliding between the cylinder and the planks 1 and 2, the points A and B of the disc will move with velocities equal to the velocities of the respective surfaces.

$$\vec{v}_A = -v \hat{i} \text{ and } \vec{v}_B = -2v \hat{i}$$

Joining A and B and A' and B' we find the point P as IAR. Then we have the similar triangles PAA' and PBB'. Using the properties of similar triangles we have



$$\omega = \frac{BB'}{AA'} = \frac{BP}{AP}$$

where $AA' = v_A = v$, $BB' = v_B = 2v$, $AP = 2R - x$ and $BP = x$

$$\text{Then, we have } \frac{2v}{v} = \frac{x}{2R - x}$$

$$\text{This gives, } x = \frac{4R}{3}$$

Hence, IAR is located at a distance of $4R/3$ below the top of the disc.

$$\text{As we know } \omega = \frac{v_{AB}}{AB}$$

$$v_{AB} = |\vec{v}_A - \vec{v}_B| = |-v \hat{i} - 2v \hat{i}| = 3v \text{ and } AB = 2R$$

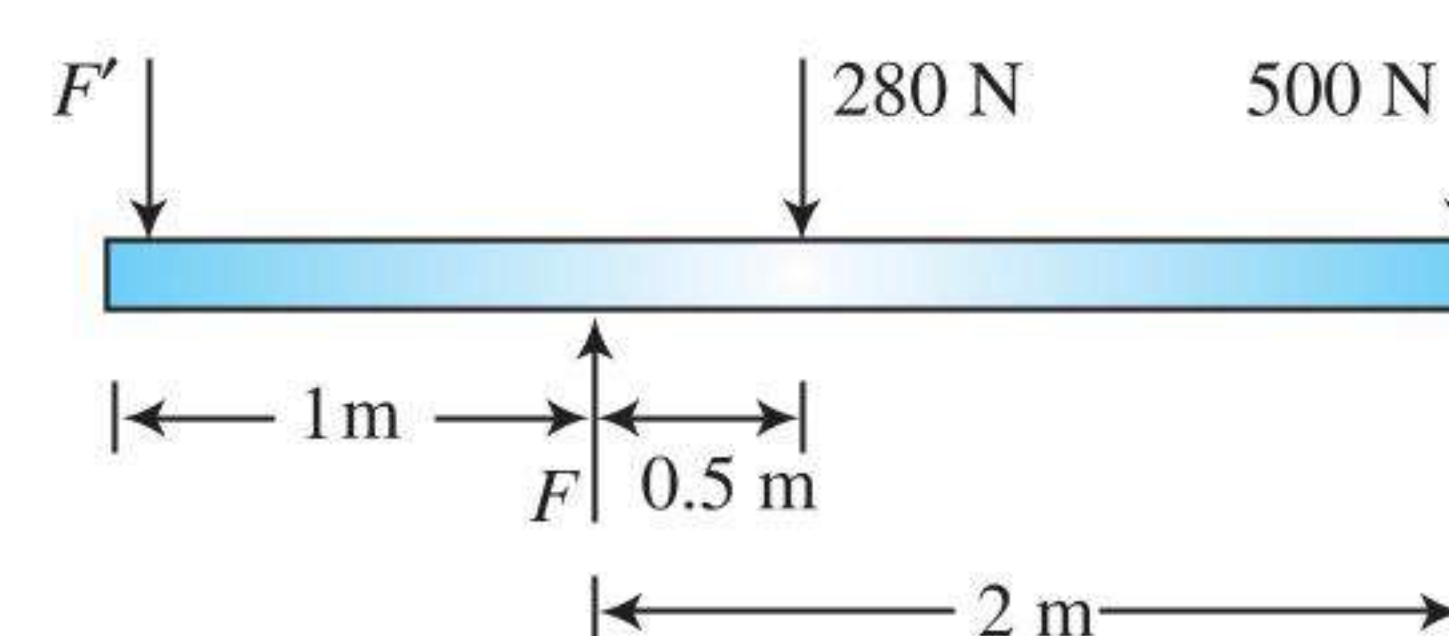
$$\text{This gives } \omega = \frac{3v}{2R} \text{ clockwise}$$

Linked Comprehension Type

For Problem 1-2

1. (3), 2. (4)

Take torques about the left end of the board inward



The force F at the support point is found from

$$F(1.00 \text{ m}) = +(280 \text{ N})(1.50 \text{ m}) + (500 \text{ N})(3.00 \text{ m})$$

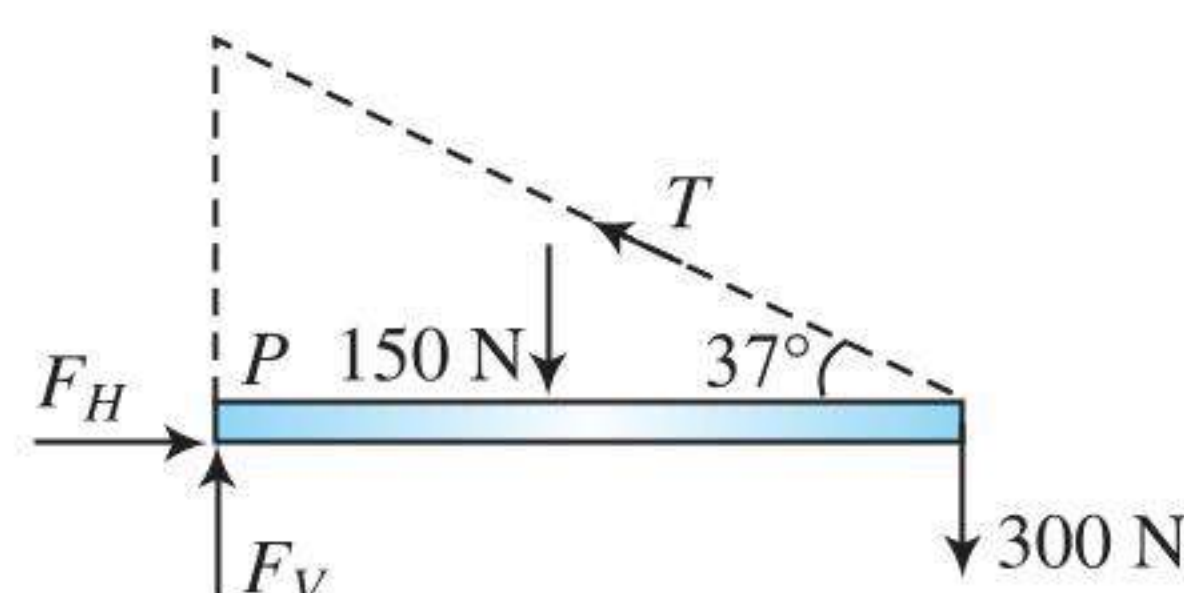
$$\Rightarrow F = 1920 \text{ N}$$

The net force must be zero, so the force at the left end is $(1920 \text{ N}) - (500 \text{ N}) - (280 \text{ N}) = 1140 \text{ N}$, downward.

For Problems 3–4

3. (4), 4. (2)

Taking torques about the pivot (P)



$$(4 \text{ m})(3/5)T = (4 \text{ m})(300 \text{ N}) + (2 \text{ m})(150 \text{ N})$$

$$\text{so } T = 625 \text{ N}$$

The horizontal force must balance the horizontal component of the force exerted by the rope, or $T(4/5) = 500 \text{ N}$.

The vertical force is $300 \text{ N} + 150 \text{ N} - T(3/5) = 75 \text{ N}$, upwards.

For Problem 5–6

$$5. (3) Mg - T = ma \quad \dots(i)$$

$$TR = \frac{MR^2}{2} \cdot \frac{a}{R} \quad \dots(ii)$$

From energy conservation

$$mgh = \frac{1}{2} \frac{MR^2\omega^2}{2} + \frac{1}{2} m\omega^2 R^2$$

$$\sqrt{\frac{4mgh}{(M+2m)R^2}} = \omega$$

$$6. (4) mgh = \frac{1}{2} \frac{MR^2}{2} \cdot \frac{v^2}{R^2} + \frac{1}{2} mv^2$$

$$4mgh = Mv^2 + 2mv^2 \Rightarrow v = \sqrt{\frac{4mgh}{M+2m}}$$

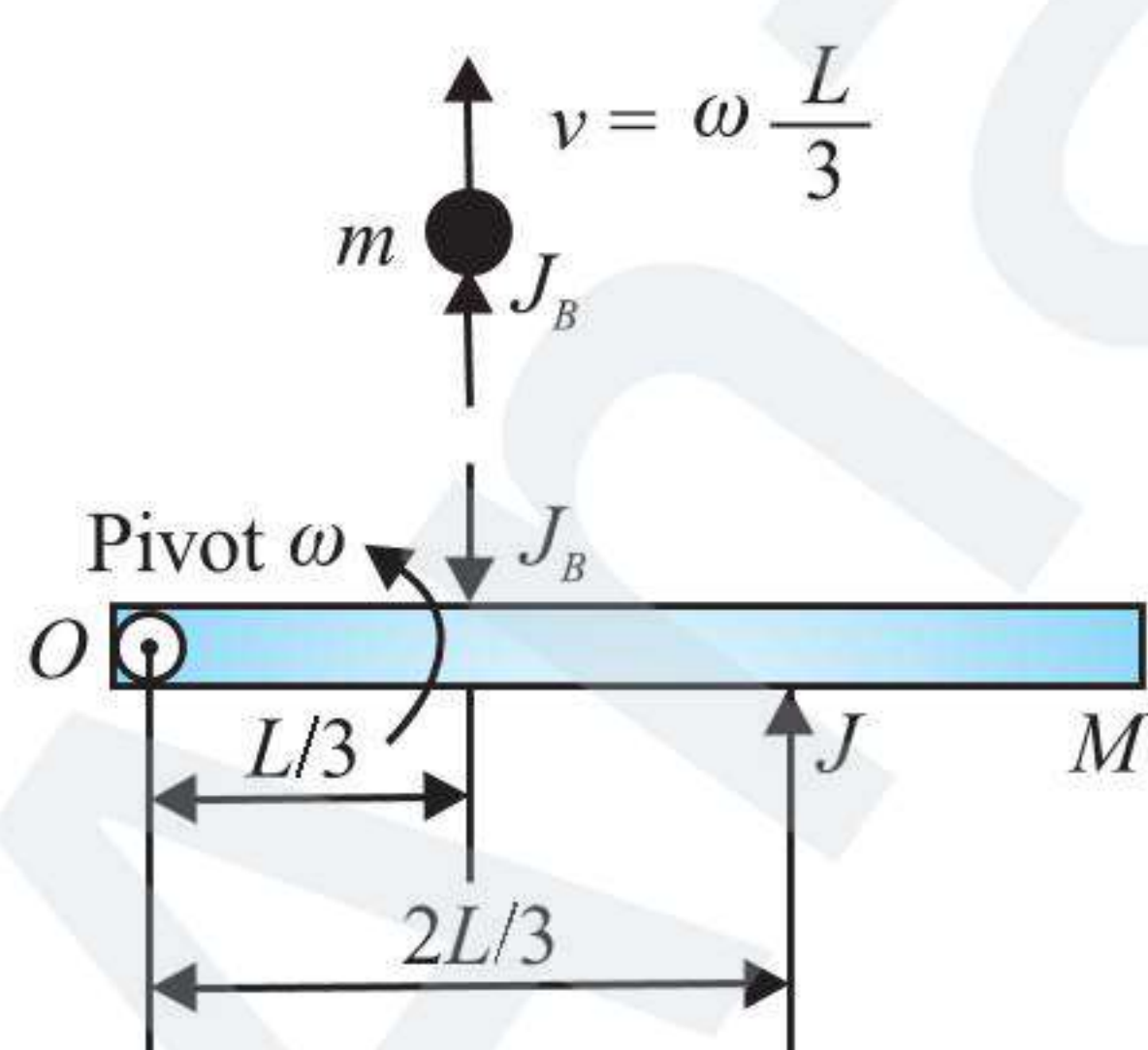
which is independent of R

For Problems 7–9

7. (2), 8. (4), 9. (1)

Let the system starts with angular velocity ω . Angular velocity of the ball will also be ω as it remains struck to the rod.

Velocity of the ball $v = \omega \frac{L}{3}$



For the rod, angular impulse = change in angular momentum:

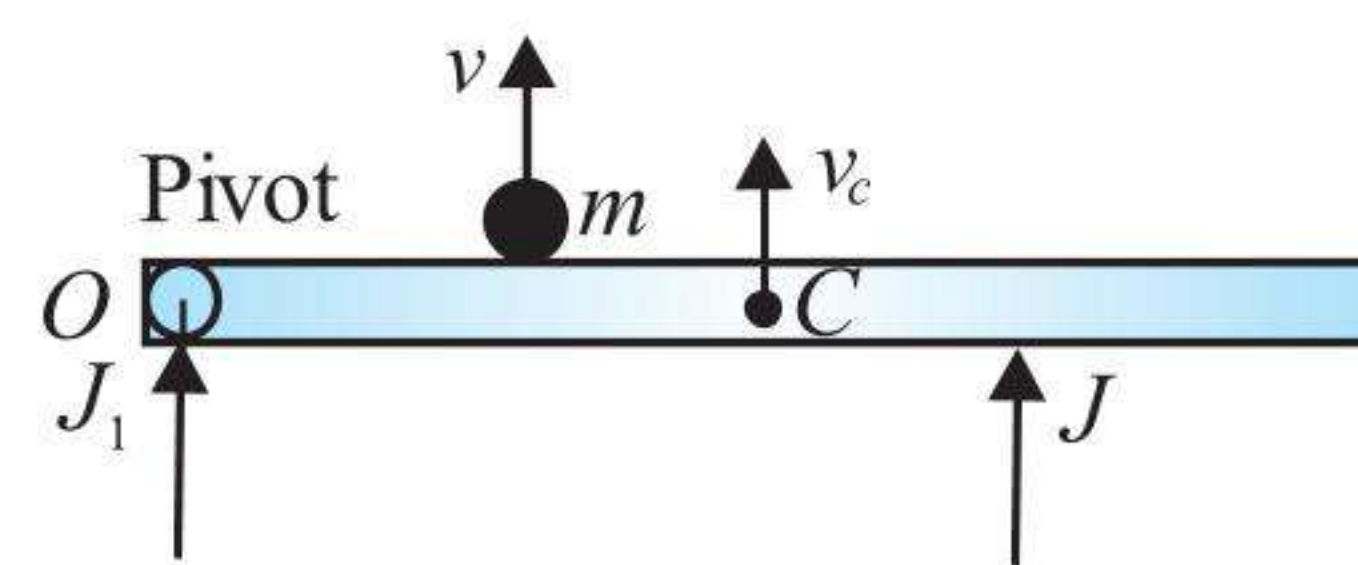
$$J \frac{2L}{3} - J_b \frac{L}{3} = \frac{ML^2}{3} \omega \quad \dots(i)$$

For the ball, impulse = change in linear momentum

$$J_b = mv = m\omega \left(\frac{L}{3} \right) \quad \dots(ii)$$

From Eqs. (i) and (ii), $\omega = \frac{6J}{(m+3M)L}$

$$\text{and } J_b = \frac{2mJ}{(m+3M)}$$



Let impulse on the pivot be J_1 .

For the rod and ball system,

total impulse = change in linear momentum

$$J + J_1 = Mv_c + mv$$

$$\text{Solve to get: } J_1 = \frac{mJ}{m+3M}$$

For Problems 10–12

10. (2) Let $OA = x$, $OB = y$

$$x^2 + y^2 = l^2 \quad \text{and} \quad x = l \cos \theta$$

Differentiate $x^2 + y^2 = l^2$ with respect to time.

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$v_B = \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} = -v_A \cot \theta = \frac{4}{5} v_0 \downarrow$$

11. (1) Differentiate $x = l \cos \theta$ with respect to time

$$\frac{dx}{dt} = -l \sin \theta \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = -\frac{v_A}{l \sin \theta}$$

(negative sign indicates that θ is decreasing)

$$\omega = \left| \frac{d\theta}{dt} \right| = \frac{v_A}{l \sin 37^\circ} = \frac{5v_0}{3l}$$

$$12. (4) x_{CM} = \frac{x}{2}$$

$$\Rightarrow v_{CM_x} = \frac{1}{2} \frac{dx}{dt} = \frac{v_0}{2} (\rightarrow)$$

$$\Rightarrow y_{CM} = \frac{y}{2}$$

$$\Rightarrow v_{CM_y} = \frac{1}{2} \frac{dy}{dt} = \frac{v_B}{2} = \frac{2}{3} v_0 \downarrow$$

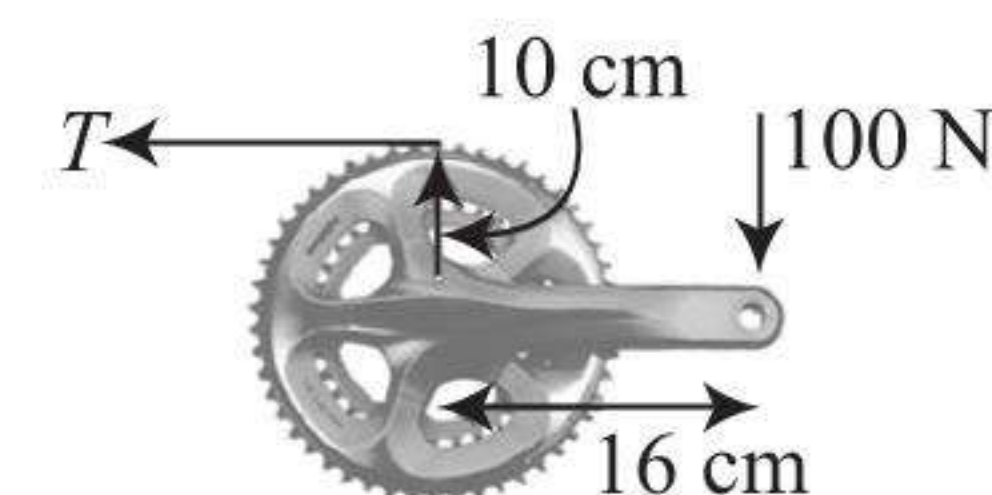
$$v_{CM} = \sqrt{\frac{v_0^2}{4} + \frac{4v_0^2}{9}} = \frac{5}{6} v_0 \text{ at } \tan^{-1} \left(\frac{4}{3} \right) \text{ below horizontal}$$

For Problems 13–17

13. (3) As angular velocity of the disc is constant i.e., $\sum \tau = 0$

$$100 \text{ N} \times 16 \text{ cm} = T \times 10 \text{ cm}$$

$$T = 160 \text{ N}$$



14. (1) As angular acceleration of the rear wheel is zero therefore net torque on the wheel is zero.

15. (3) Power delivered = $\vec{F} \cdot \vec{v}$

where $\vec{v}$ is velocity of the point of application of the force.

$$v = 16 \text{ cm} \times 2\pi.2 (= R\omega)$$

$$= 0.64 \pi \text{ m s}^{-1}$$

$$P = 100 \times 0.64\pi = 64\pi \text{ W}$$

$$16. (2) RN = rn \Rightarrow n = \frac{10 \text{ cm} \times 2}{4 \text{ cm}} = 5 \text{ cy/s}$$

So rear wheel rotates 5 cycles/s.

$$\text{Hence } V = \frac{35}{100} \times 2\pi \times 5 = 3.5\pi \text{ m s}^{-1}$$

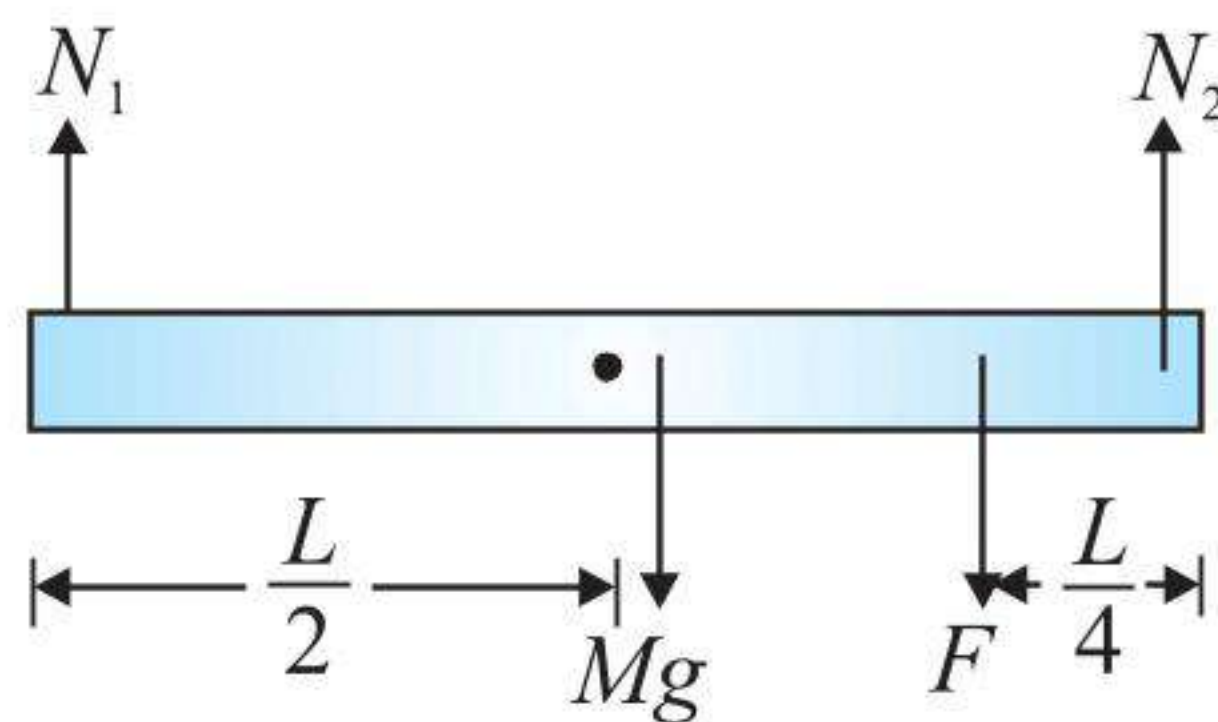
17. (4) As $\sum \tau = 0$

$$160 \text{ N} \times 4 \text{ cm} = f \times 35 \text{ cm}$$

$$f = \frac{160 \times 4 \text{ cm}}{35 \text{ cm}} = 18.3 \text{ N}$$

For Problems 18–20

18. (3)



As the rod is in equilibrium,

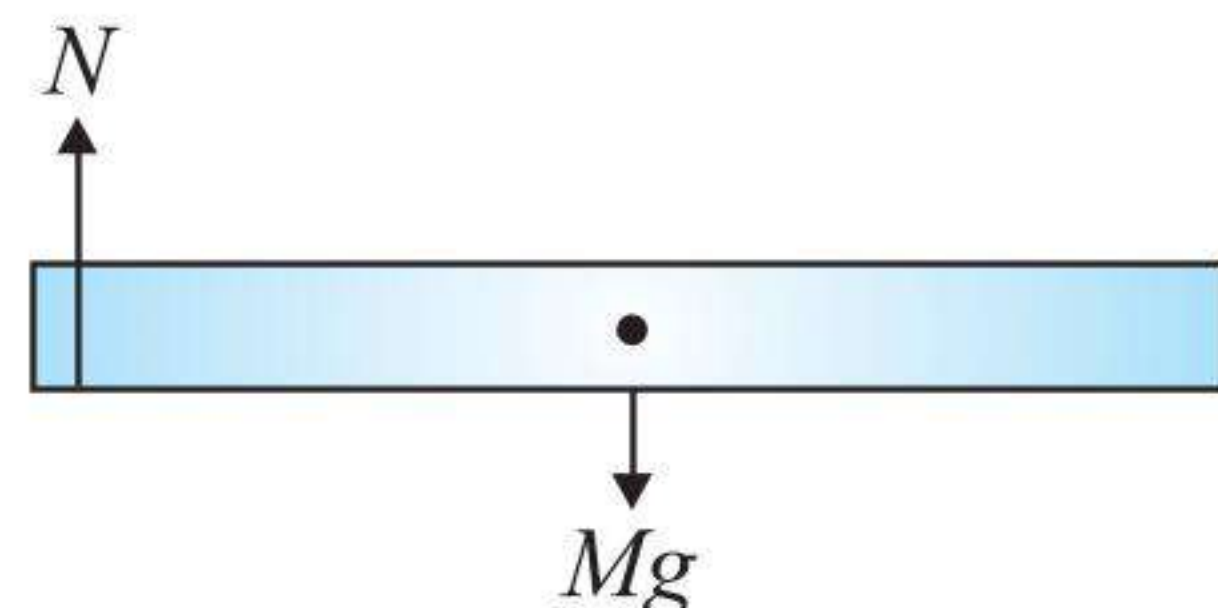
$$\therefore \sum F_x = 0; \sum F_y = 0; \sum \tau = 0$$

Taking torque about hinge 2, we get

$$N_1 \times L = Mg \times \frac{L}{2} + F \times \frac{L}{4}$$

$$\therefore N_1 = \frac{Mg}{2} + \frac{F}{4} = 11 \text{ N}$$

19. (3)



Now, rod is not in equilibrium,

$$\therefore F = Ma_{\text{CM}}$$

$$\Rightarrow Mg - N = Ma$$

$$\text{and } \tau = I\alpha$$

$$\Rightarrow Mg\left(\frac{L}{2}\right) = \frac{ML^2}{3} \frac{a}{(L/2)}$$

$$\Rightarrow a = \frac{3}{4}g$$

From Eqs. (i) and (ii),

$$N = \frac{Mg}{4} = 5 \text{ N}$$

20. (1) By conservation of energy,

$$Mg\left(\frac{L}{2}\right) = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{ML^2}{3}\right)\omega^2$$

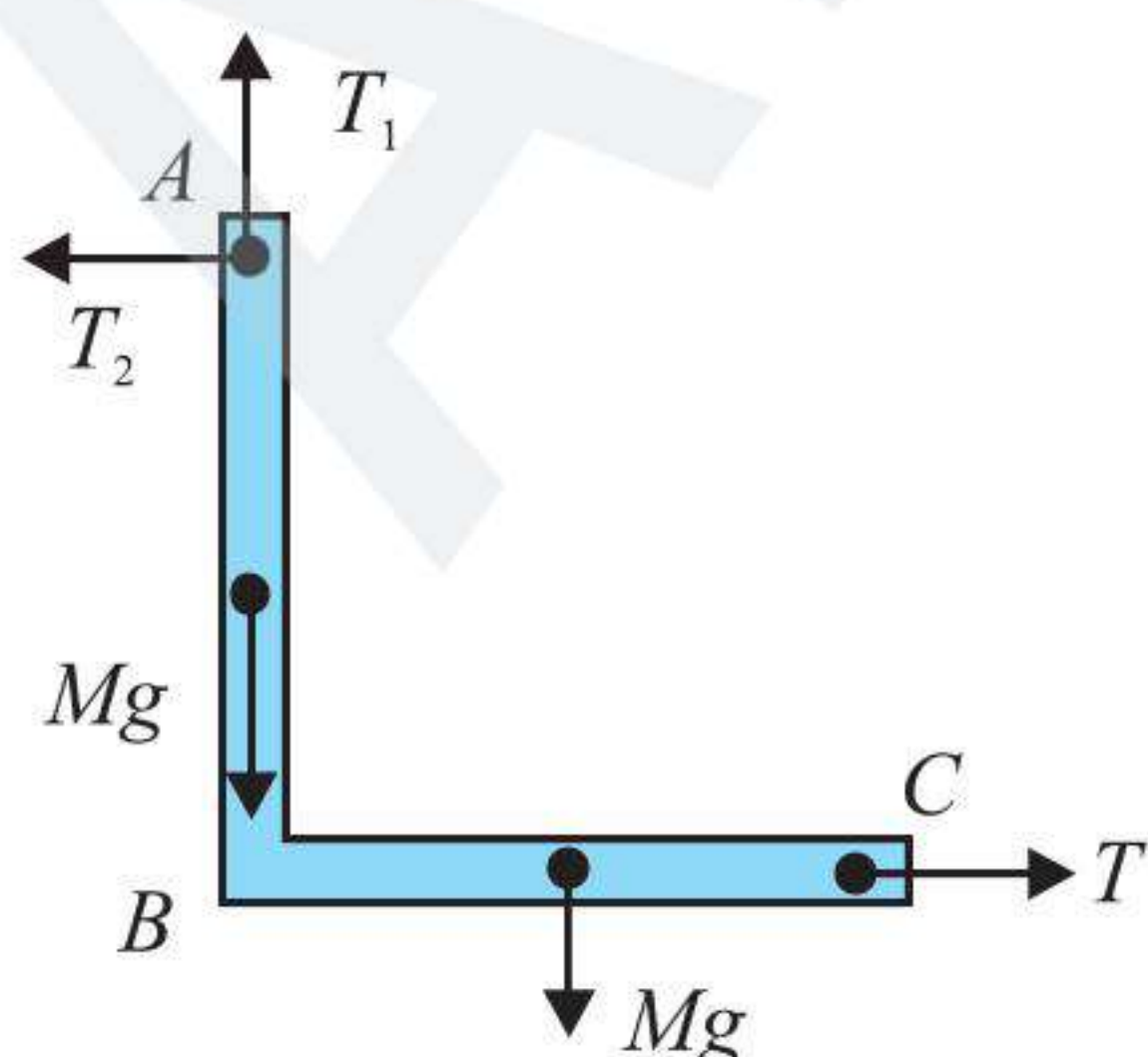
$$\therefore \omega = \sqrt{\frac{3g}{L}}$$

$$\text{Acceleration} = \omega^2 L = 3g = 30 \text{ m/s}^2$$

For Problems 21–23

 21. (4) Taking torque about A: $Mg \frac{L}{2} = TL$

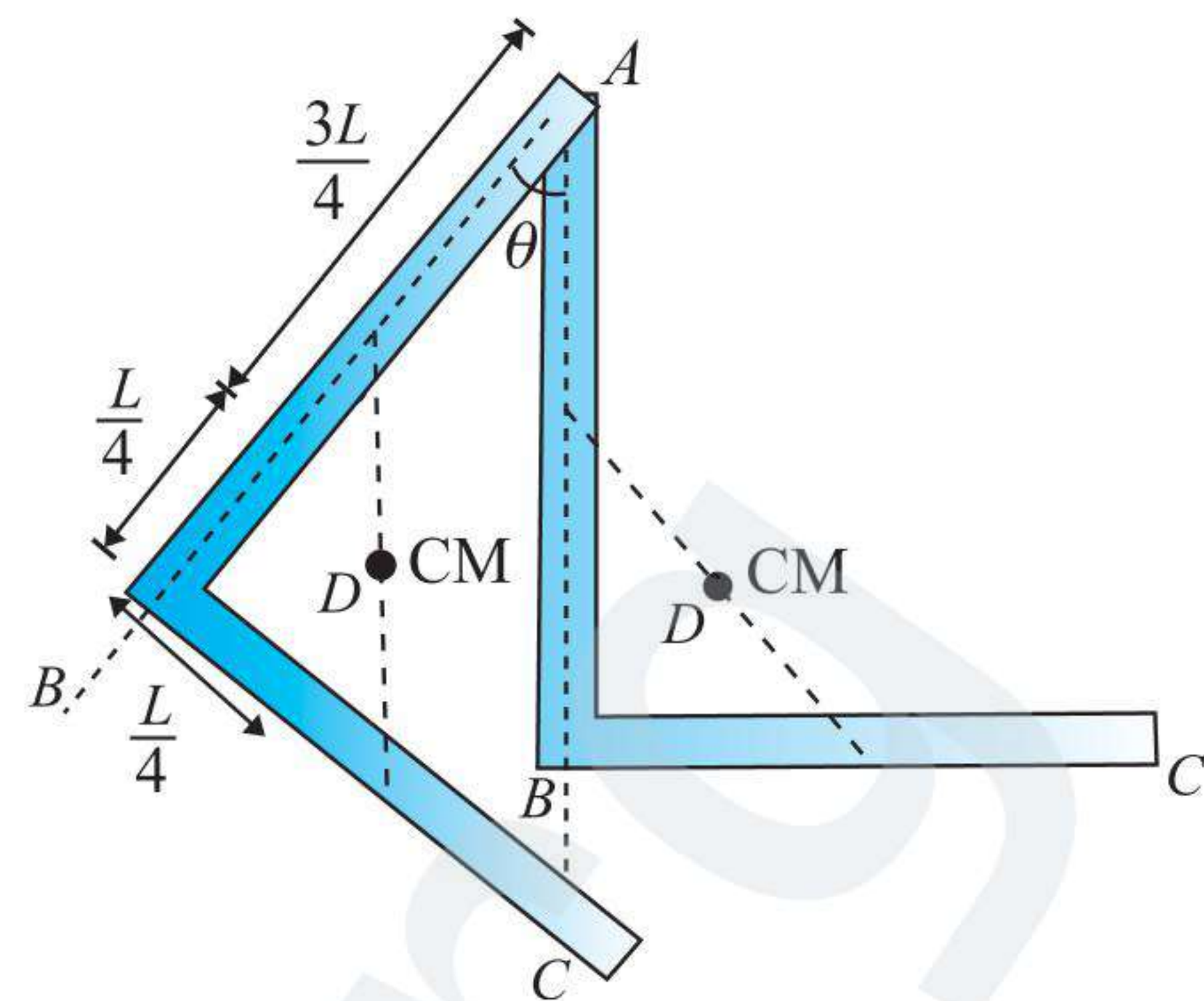
$$\Rightarrow T = \frac{Mg}{2}$$



$$22. (4) T_1 = 2Mg, T_2 = T = \frac{Mg}{2}$$

$$\text{Net reaction} = \sqrt{T_1^2 + T_2^2} = \sqrt{(2Mg)^2 + (Mg/2)^2} = \frac{\sqrt{17}Mg}{2}$$

23. (1) D is the CM of the rods



In triangle ABE:

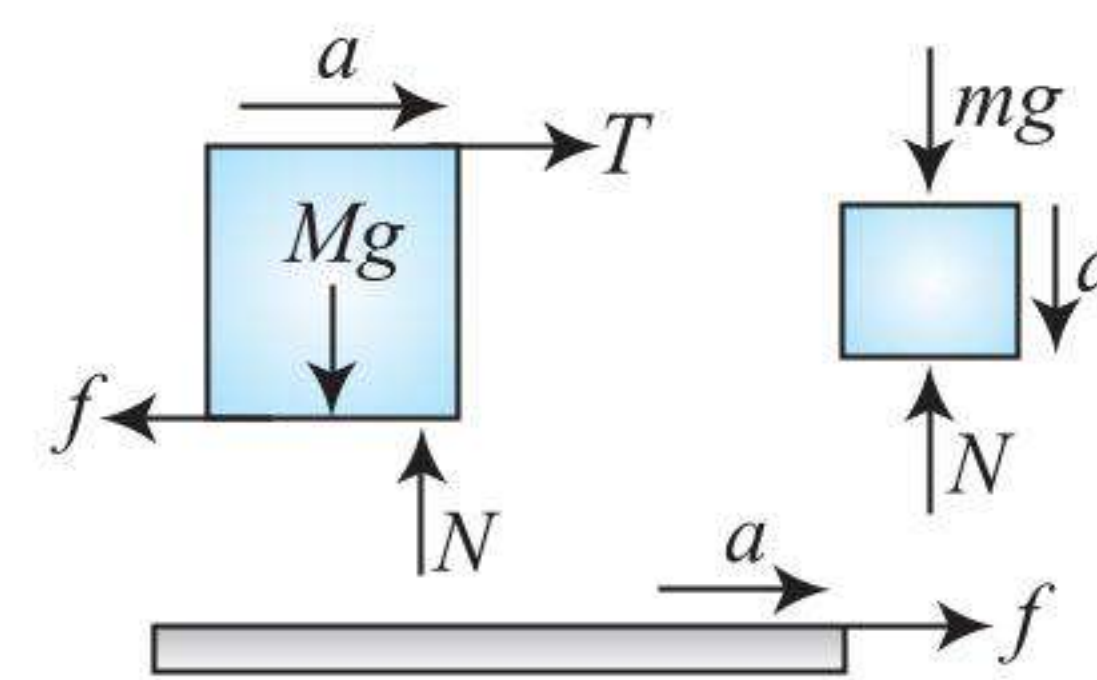
$$\tan \theta = \frac{L/4}{3L/4}$$

$$\Rightarrow \tan \theta = \frac{1}{3} \Rightarrow \theta = \tan^{-1}(1/3)$$

 Here D is the centre of mass. In equilibrium, D will be vertically below A. So in equilibrium, AB will make angle θ , as calculated above, with the vertical.

For Problems 24–25

24. (4) Drawing free body diagrams of all the three (block, plank and mass)



Since we are taking the limiting case.

 Therefore $f =$ maximum friction between the block and the plank.

$$\text{or } f = \mu N = \mu Mg = \mu(4)g$$

$$f = 4\mu g \quad \dots(i)$$

Now writing equations of motion for all three

$$mg - T = ma \quad (m = 1 \text{ kg})$$

$$\text{or } g - T = a \quad \dots(ii)$$

$$T - f = Ma \quad (M = 4 \text{ kg})$$

$$\text{or } T - 4\mu g = 4a \quad \dots(iii)$$

$$f = Ma \quad \text{or } f = 4a \quad \text{or } 4\mu g = 4a$$

$$\text{or } a = \mu g \quad \dots(iv)$$

 Solving Eqs. (ii), (iii) and (iv) for μ , we get $\mu = \frac{1}{9}$

 Therefore, minimum value of μ should be $\frac{1}{9}$ or $\mu \geq \frac{1}{9}$

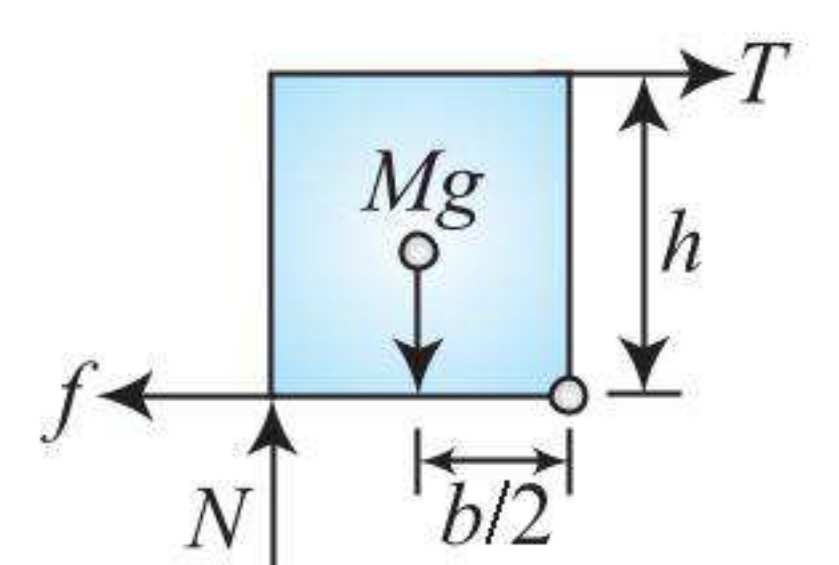
25. (2) At the time of toppling, normal reaction N will pass through the front edge as shown below.

 So, torque of T about P $\leq$ torque of Mg about P

$$\text{or } Th \leq (Mg)\left(\frac{b}{2}\right) \quad \text{or } T \leq \left(\frac{Mg}{2}\right)\left(\frac{b}{h}\right)$$

 From Eqs. (ii), (iii) and (iv) we can find that, $T = \frac{8}{9}g$

$$\text{Therefore, } \frac{8}{9}g \leq \left(\frac{Mg}{2}\right)\left(\frac{b}{h}\right)$$



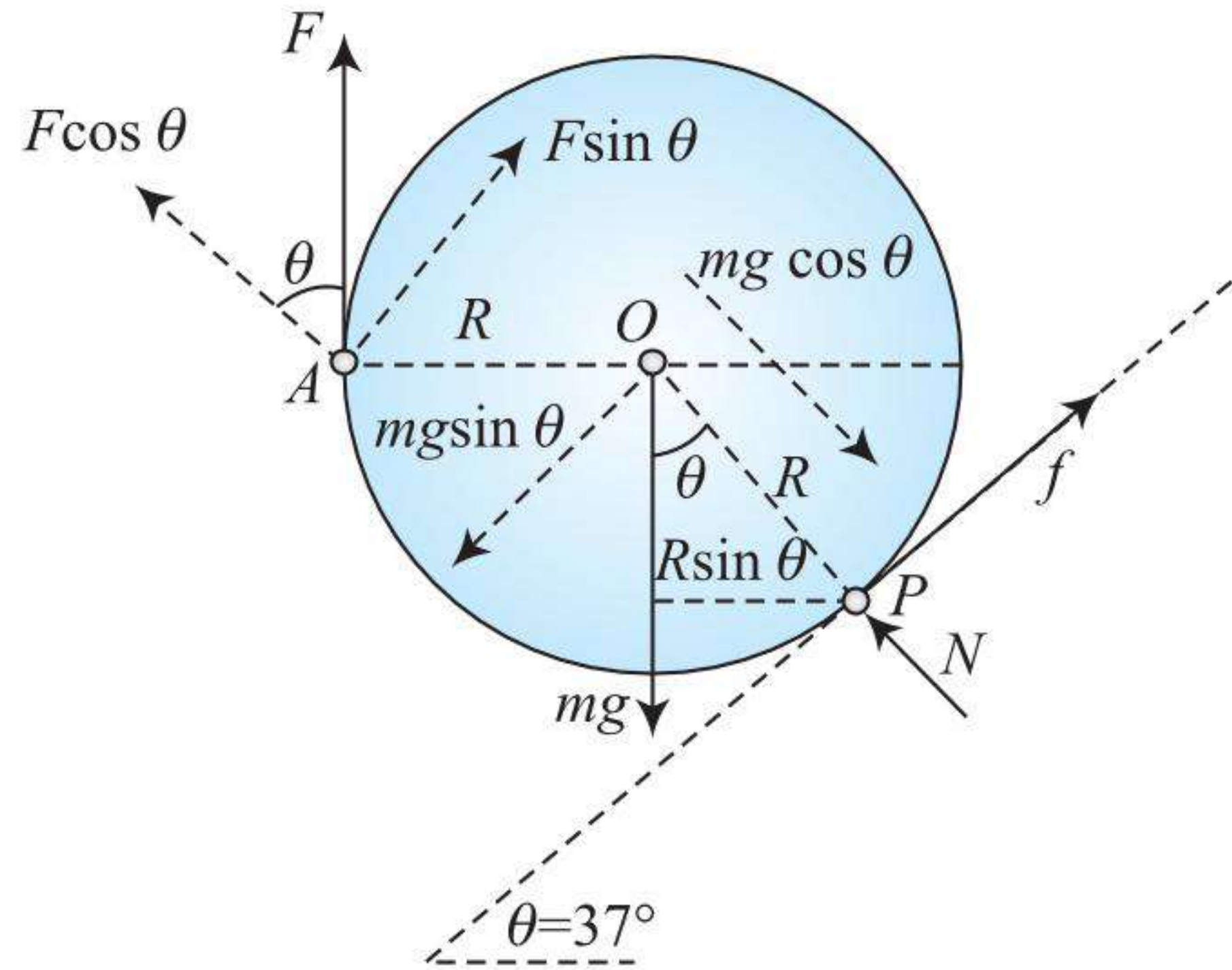
$$\text{or } \frac{b}{h} \geq \frac{16}{9M} \quad (M = 4 \text{ kg})$$

$$\text{or } \left(\frac{b}{h}\right) \geq \frac{4}{9}$$

Therefore, minimum value of b/h so that the block does not topple is $4/9$.

For Problems 26–28

26. (3) For rotational equilibrium of the cylinder.



Torque about P: $F(R + R \sin \theta) = mgR \sin \theta$

$$F\left(1 + \frac{3}{5}\right) = mg \times \frac{3}{5} \Rightarrow F = \frac{3mg}{8}$$

27. (3) For translational equilibrium

$$f + F \sin \theta = mg \sin \theta$$

$$\Rightarrow f = (mg - F) \sin \theta \quad \dots(i)$$

$$N + F \cos \theta = mg \cos \theta$$

$$\Rightarrow N = (mg - F) \cos \theta \quad \dots(ii)$$

For no sliding $f \leq \mu_s N$

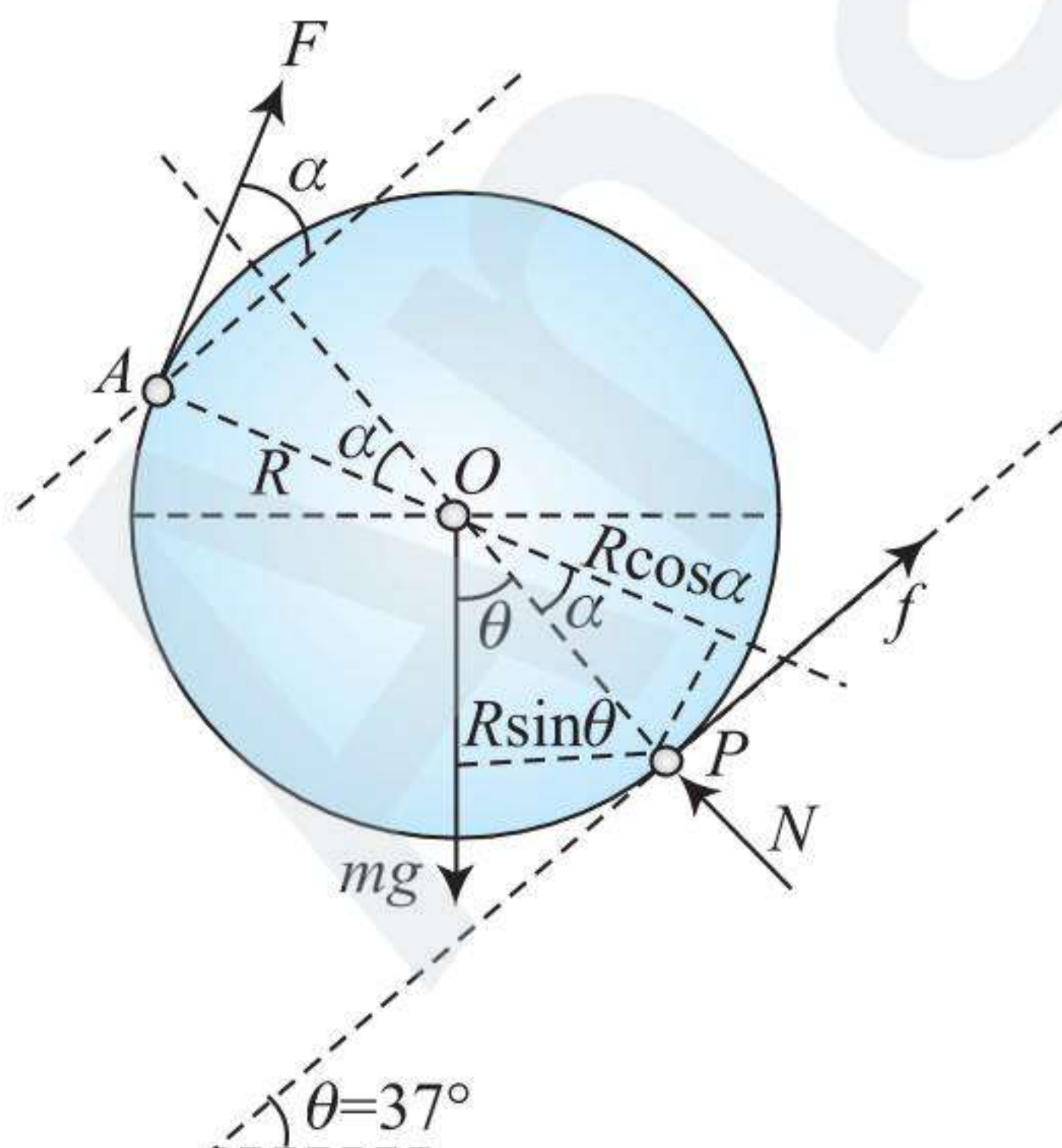
$$(mg - F) \sin \theta \leq \mu_s (mg - F) \cos \theta$$

$$\text{or } \mu_s \geq \tan \theta$$

$$\text{As } \theta = 37^\circ. \text{ It means } \mu_s \geq \frac{3}{4}$$

The minimum value of coefficient of static friction is $\frac{3}{4}$.

28. (1) For rotational equilibrium, balancing torque about P



$$F(R + R \cos \alpha) = mgR \sin \theta$$

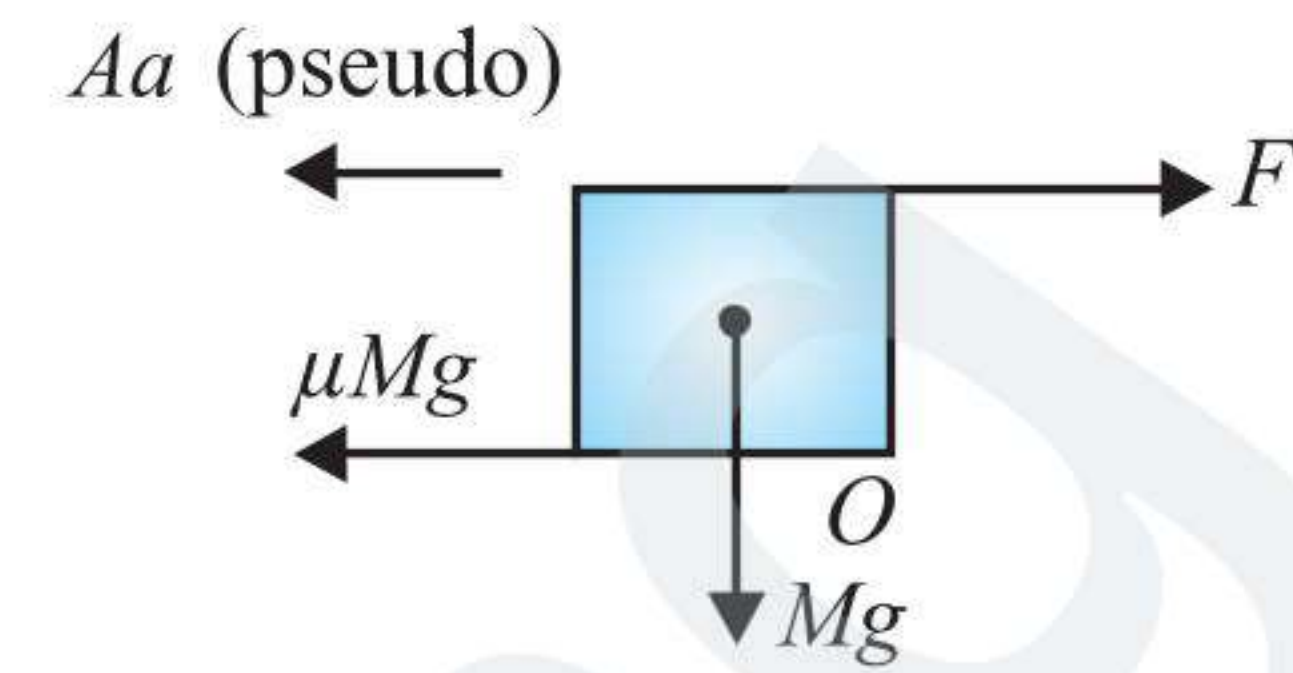
$$\Rightarrow F = \frac{mg \sin \theta}{(1 + \cos \alpha)} = \frac{3mg}{5(1 + \cos \alpha)} \quad \dots(i)$$

For F to be minimum $(1 + \cos \alpha)$ should be maximum, for this $\alpha = 0$

$$\text{and } F_{\min} = \frac{3}{5} \frac{mg}{(1 + \cos \theta)} = \frac{3}{10} mg$$

Matrix Match Type

1. i. $\rightarrow$ a.; ii. $\rightarrow$ d.; iii. $\rightarrow$ a.; iv. $\rightarrow$ c.



The force required to topple the block about O,

$$Fa = \frac{Ma}{g} g + \frac{Ma}{2} \times 1$$

$$F = M + \frac{M}{2} = 55 \text{ N}$$

$$\text{and } f_{\max} = \mu mg = 0.1 \times 10 \times 10 = 10 \text{ N}$$

Force between 0–10 N will do neither sliding or slipping.

Force between 10–55 N will cause slipping.

Force above 55 N will topple the block.

2. i. $\rightarrow$ c.; ii. $\rightarrow$ a.; iii. $\rightarrow$ a.; iv. $\rightarrow$ b.

Total mass: M

$$\text{Mass of cutout portion: } m_1 = \frac{M}{4}$$

$$I_1 = \frac{1}{2} MR^2 - \frac{1}{2} m_1 \left(\frac{R}{2}\right)^2 = \frac{15}{32} MR^2$$

$$I_2 = I_1 + \frac{3M}{4} \left(\frac{R}{2}\right)^2 = \frac{21}{32} MR^2$$

$$\text{Also } I_3 + I_4 = I_2$$

$$I_2 = I_1 + \frac{3M}{4} \left(\frac{R}{2}\right)^2$$

$$I_3 = \frac{I_1}{2} + \frac{3M}{4} \left(\frac{R}{2}\right)^2$$

$$\Rightarrow I_2 - I_3 = \frac{I_1}{2}$$

3. i. $\rightarrow$ b.; ii. $\rightarrow$ a.; iii. $\rightarrow$ c.; iv. $\rightarrow$ a., b., d.

i. Pure translational motion

ii. Pure rotational motion about centre of mass

iii. No motion as $\sum \vec{F} = 0$ and $\sum \vec{\tau} = 0$

iv. Combined rotation and translational motion

4. i. $\rightarrow$ b.; ii. $\rightarrow$ d.; iii. $\rightarrow$ d.; iv. $\rightarrow$ a.

For sphere, all the three axis are diameter, so $I_x = I_y = I_z$

Perpendicular axis theorem is valid for two dimensional bodies.

It is not valid for spheres.

5. (2)

$$(i) K = \sqrt{\frac{I}{m}} = \sqrt{\frac{m(2a)^2}{3m}} = \frac{2}{\sqrt{3}} a$$

$$(ii) K = \sqrt{\frac{m(2a)^2}{12m}} = \frac{a}{\sqrt{3}}$$

$$(iii) K = \sqrt{\frac{ma^2}{3m}} = \frac{a}{\sqrt{3}}$$

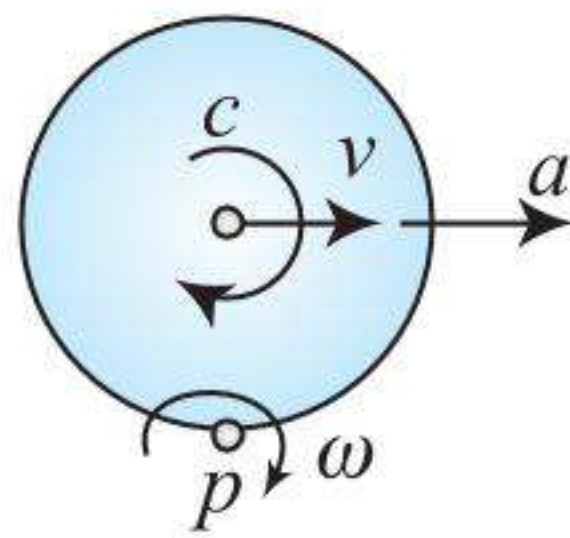
$$(iv) K = \sqrt{\frac{ma^2}{12m}} = \frac{a}{\sqrt{12}}$$

6. (1) (i) The centre of rolling body is rotating about O. Hence

$$\omega_{CO} = \frac{v}{(R-r)}.$$

- (ii) As the body is rolling point P is I.C.R

$$\text{Hence } \omega_{CP} = \frac{v}{r}$$



- (iii) As point P is ICR, have

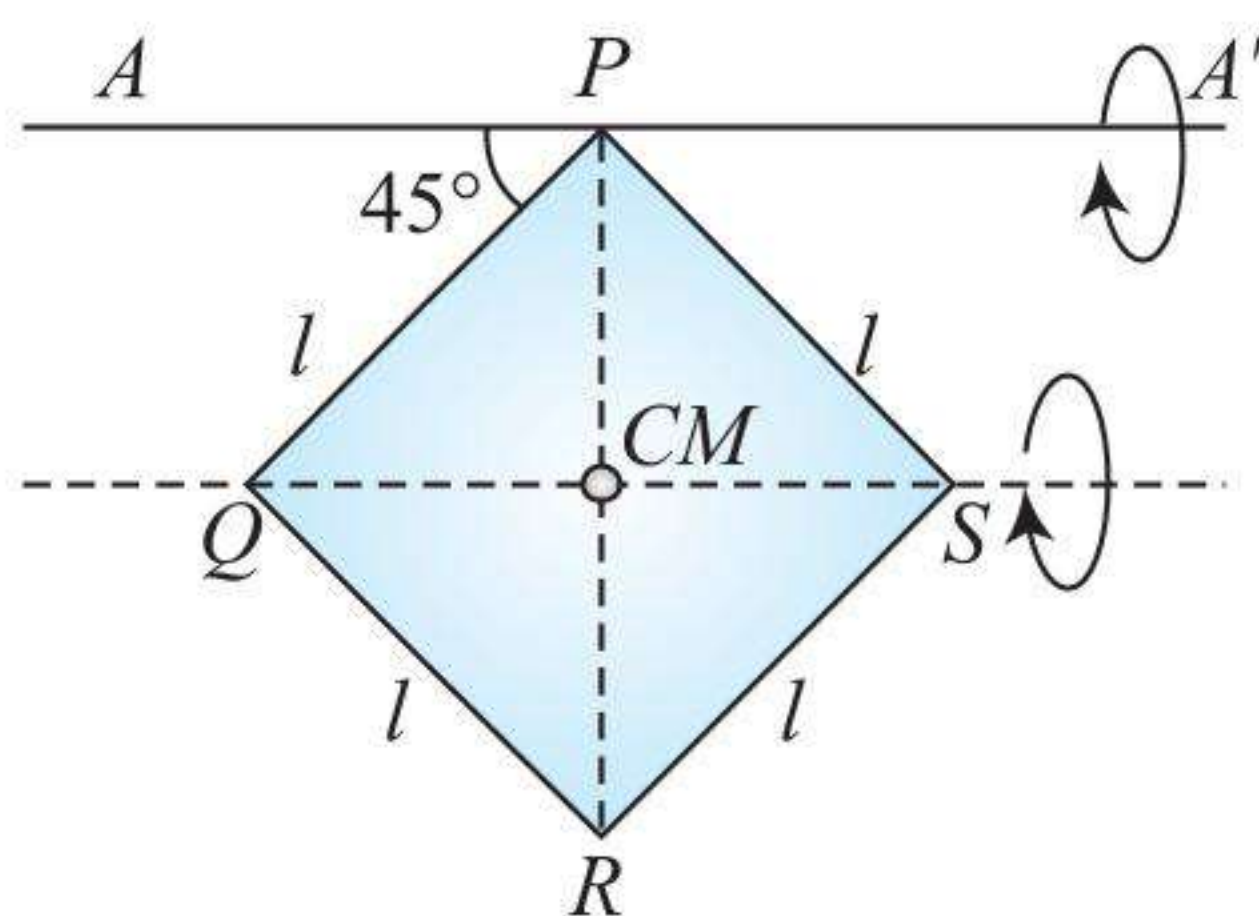
$$a_{C,P} = \alpha_{CP} \times r \Rightarrow \alpha_{CP} = \frac{\alpha_{cp}}{r} = \frac{a}{r}$$

- (iv) The centre of the rolling body is rotating about O

$$\text{Hence } \alpha_{CO} = \frac{a}{(R-r)}$$

7. (3)

- (i)



Moment of inertia of the frame about an axis passing through QS

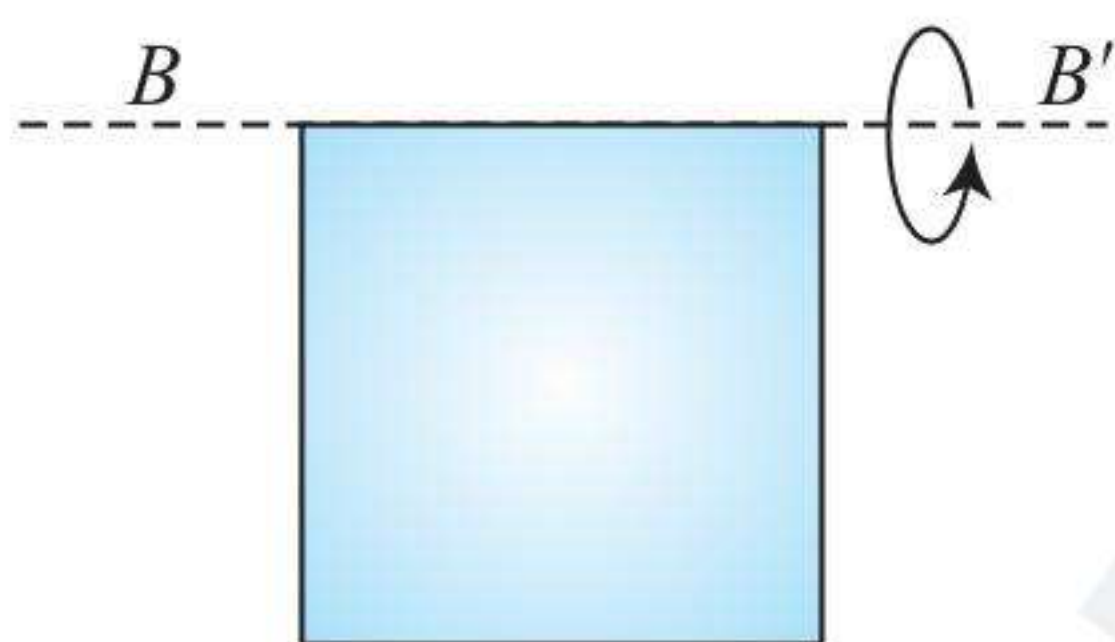
$$I_{QS} = 4 \times \frac{1}{3} ml^2 \sin^2 45^\circ = \frac{2}{3} ml^2$$

Using parallel axis theorem

$$I_{AA'} = I_{QS} + 4m \left(\frac{l}{\sqrt{2}} \right)^2 = \frac{2}{3} ml^2 + 2ml^2$$

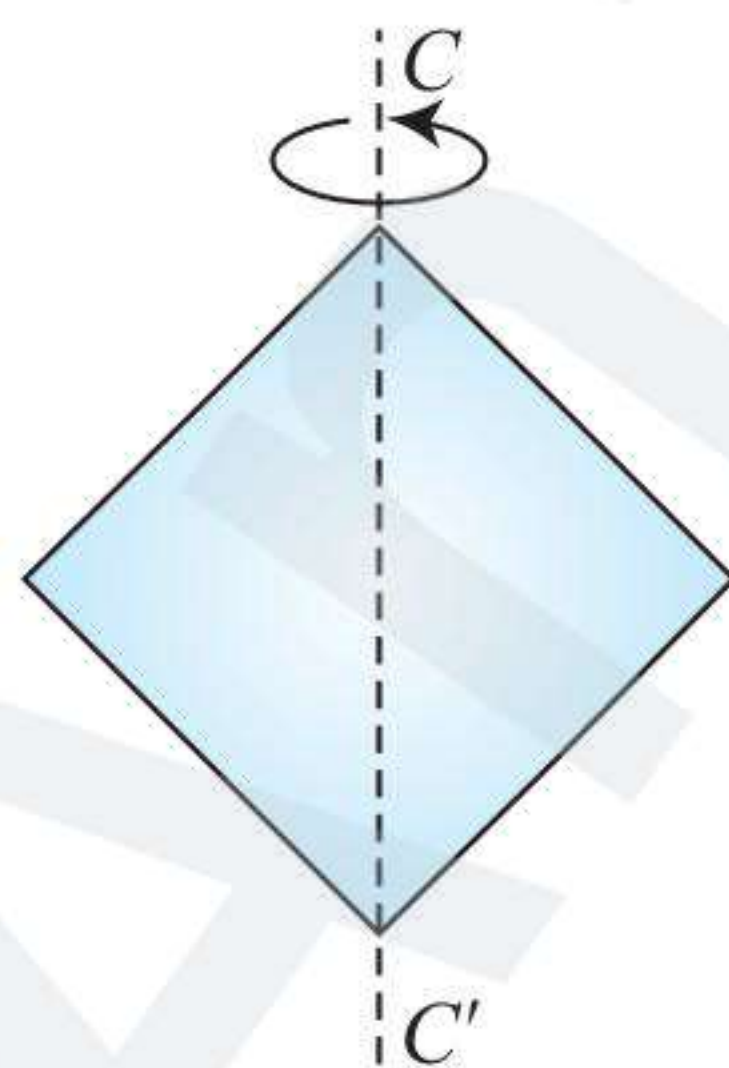
$$\Rightarrow I_{AA'} = \frac{8}{3} ml^2$$

- (ii)



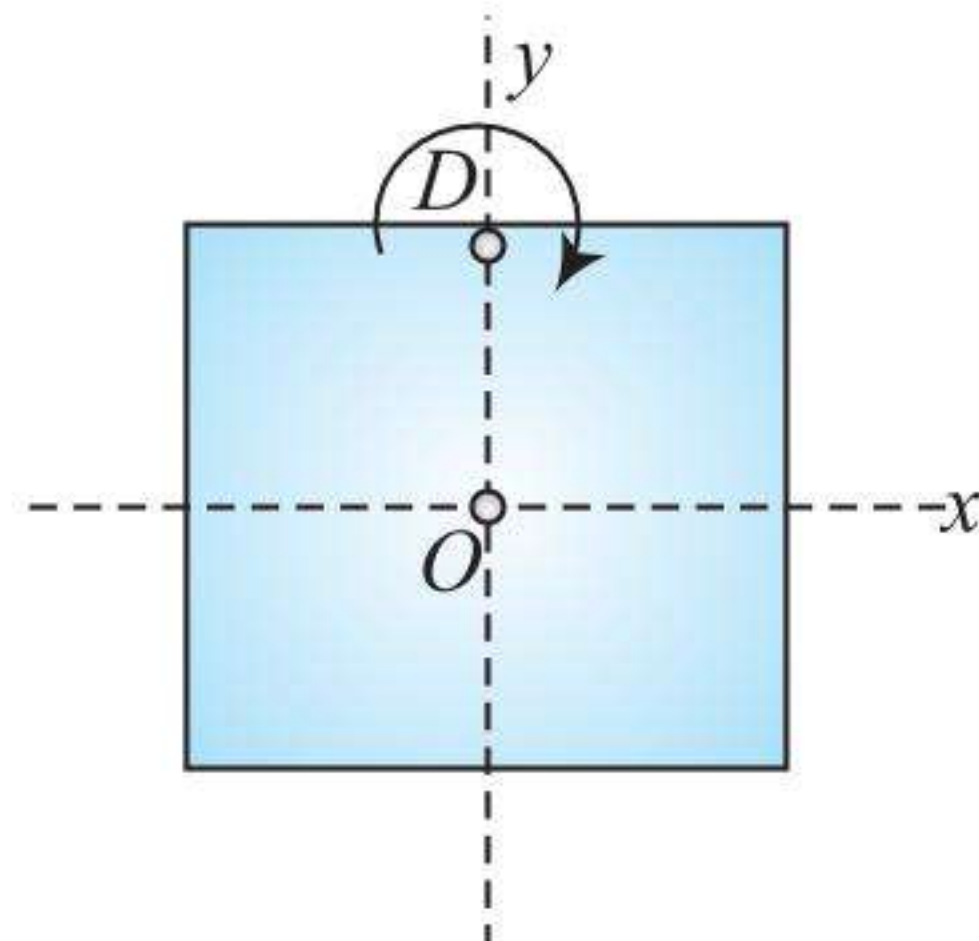
$$I_{BB'} = \frac{2ml^2}{3} + ml^2 = \frac{5ml^2}{3}$$

- (iii)



$$I_{CC'} = 4 \times \frac{1}{3} ml^2 \sin^2 45^\circ = \frac{2}{3} ml^2$$

- (iv)



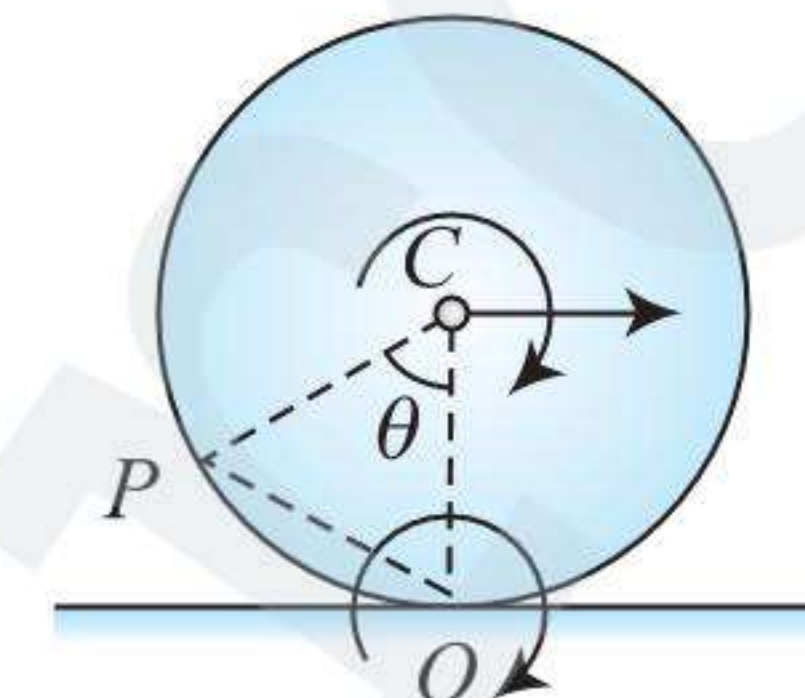
$$I_x = I_y = 2 \left[\frac{1}{12} ml^2 + m \left(\frac{l}{2} \right)^2 \right] = \frac{2ml^2}{3}$$

$$I_z = I_x + I_y \Rightarrow I_y = \frac{4ml^2}{3}$$

Moment of inertia about an axis passing through D

$$I_D = I_z + 4m \left(\frac{l}{2} \right)^2 = \frac{4}{3} ml^2 + ml^2 = \frac{7ml^2}{3}$$

8. (2) As the disc is rolling point O should be ICR.



$$v_P = \omega(PO) = \omega 2R \sin \frac{\theta}{2} = 2v \sin \frac{\theta}{2}$$

As $v = \omega R$

$$(i) \text{ If } \theta = 60^\circ \Rightarrow v_P = 2v \sin \frac{60^\circ}{2} = v$$

$$(ii) \text{ If } \theta = 90^\circ \Rightarrow v_P = 2v \sin \frac{90^\circ}{2} = \sqrt{2} v$$

$$(iii) \text{ If } \theta = 120^\circ \Rightarrow v_P = 2v \sin \frac{120^\circ}{2} = \sqrt{3} v$$

$$(iv) \text{ If } \theta = 180^\circ \Rightarrow v_P = 2v \sin \frac{180^\circ}{2} = 2v$$

9. (2) For (i): Acceleration of l w.r.t. centre of mass = $r\alpha\hat{i} - \omega^2 r\hat{j}$

$$\Rightarrow \vec{a}_1 = r\alpha\hat{i} - \omega^2 r\hat{j} + R\alpha\hat{i} = (R+r)\alpha\hat{i} - \omega^2 r\hat{j}$$

$$\text{For (ii): } \vec{a}_2 = -r\alpha\hat{j} - \omega^2 r\hat{i} + R\alpha\hat{i} = (R\alpha - \omega^2 r)\hat{i} - r\alpha\hat{j}$$

$$\text{For (iii): } \vec{a}_3 = -r\alpha\hat{i} + \omega^2 r\hat{j} + R\alpha\hat{i} = (R\alpha - r\alpha)\hat{i} + \omega^2 r\hat{j}$$

$$\text{For (iv): } \vec{a}_4 = r\alpha\hat{j} + \omega^2 r\hat{i} + R\alpha\hat{i} = (R\alpha + \omega^2 r)\hat{i} + r\alpha\hat{j}$$

10. (4)

$$(i) I_O = I_1 + I_2$$

$$= \frac{m_1 l_1^2}{3} + \frac{m_2 l_2^2}{3} = \frac{1}{3} \left(\frac{2m}{3} \right) (2l)^2 + \frac{1}{3} \left(\frac{m}{3} \right) l^2 = ml^2$$

$$(ii) I_O = ml^2 + m \left(\frac{l^2}{12} \right) + m \left(l^2 + \frac{l^2}{4} \right) = \frac{7}{3} ml^2$$

$$(iii) I_O = \frac{ml^2}{12} + m \left[\left(\frac{l}{4} \right)^2 + l^2 \right] = \frac{55}{48} ml^2$$

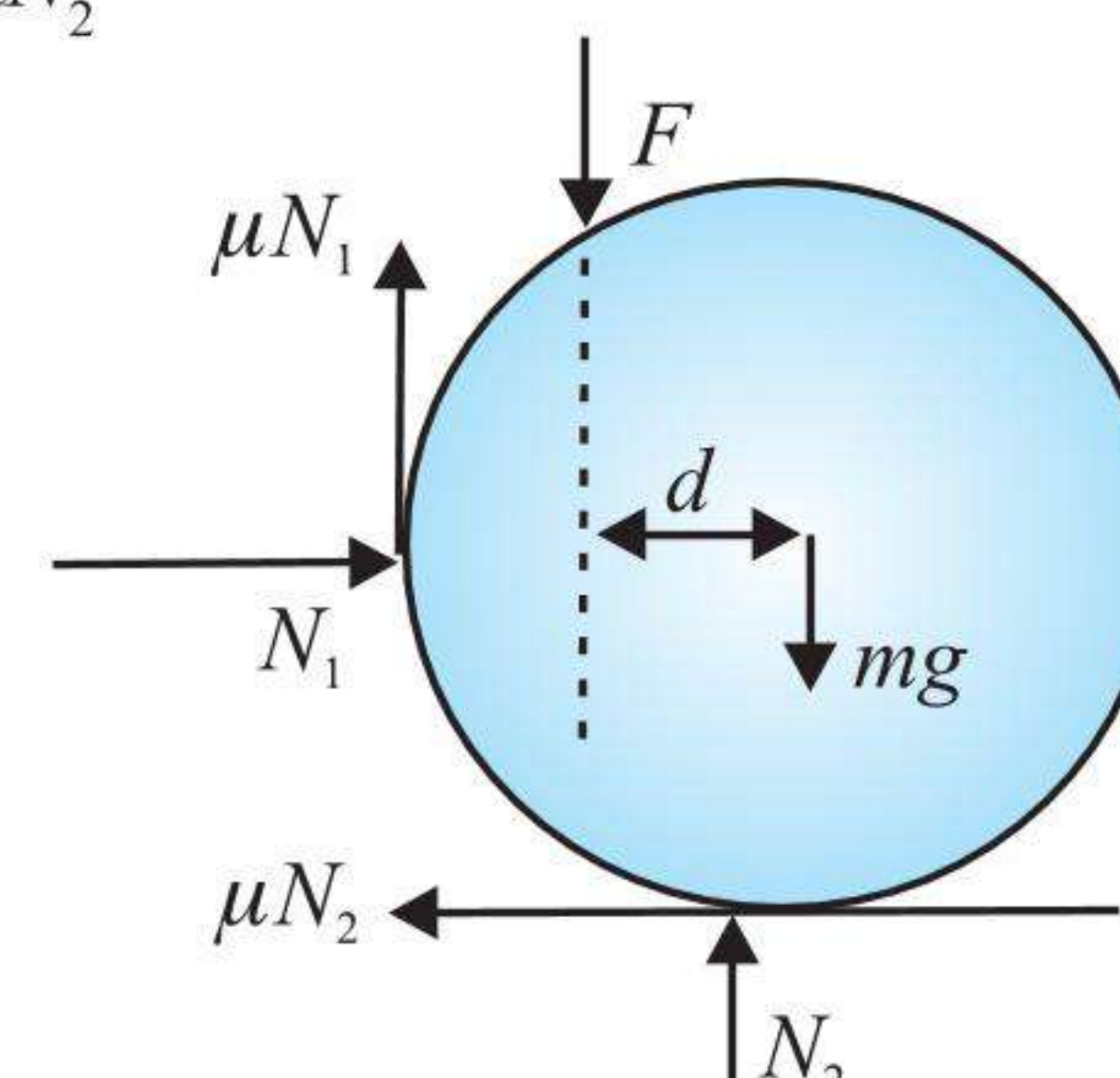
$$(iv) I_O = I_1 + I_2 = \frac{ml^2}{3} + \frac{ml^2}{12} + ml^2 = \frac{17ml^2}{12}$$

Numerical Value Type

1. (6) For equilibrium of the cylinder in the horizontal direction,

$$N_1 = \mu N_2$$

...(i)



In the vertical direction,

$$N_2 + \mu N_1 = F + Mg \quad \dots(ii)$$

Solving these two equations with $F = 40 \text{ N}$, $M = 2 \text{ kg}$ and $\mu = 1/3$, we get

$$N_1 = 18 \text{ N}$$

and $N_2 = 54 \text{ N}$

Now, $Fd = \mu(N_1 + N_2)r$

$$\therefore d = \frac{M(N_1 + N_2)r}{F} = \frac{(1/3)(18 + 54)(0.1)}{40} = 0.06 \text{ m} = 6 \text{ cm}$$

2. (5) When the string is cut, the weight of the rod constitutes torque about the hinge, so

$$\tau_A = mg \frac{l}{2} \quad \dots(i)$$

According to Newton's second law,

$$\tau_A = I\alpha \quad \dots(ii)$$

where α is the angular acceleration of the rod about the end A.

From Eqs. (i) and (ii), we get

$$I\alpha = mg \frac{l}{2} \text{ or } \alpha = \frac{mg \frac{l}{2}}{I}$$

Here $I = ml^2/3$, therefore

$$\therefore \alpha = \frac{mgl/2}{ml^2/3} = \frac{3g}{2l}$$

Acceleration of the CM of the rod is

$$a_{\text{CM}} = \alpha r = \frac{3g}{2l} \times \frac{l}{2} = \frac{3g}{4}$$

Again by Newton's second law,

$$mg - R_A = ma_{\text{CM}}$$

$$\text{or } mg - R_A = m \times \frac{3g}{4}$$

$$\therefore R_A = \frac{Mg}{4} = \frac{2 \times 10}{4} = 5 \text{ N}$$

$$3. (6) T \frac{l}{2} = \frac{ml^2}{6} \times \alpha; mg - T = ma$$

$$\alpha = \frac{3T}{ml} = \frac{3}{ml} (mg - ma)$$

Constraint relation: Acceleration of A in the vertical direction should be zero

$$\frac{l}{2} \alpha = a$$

$$\Rightarrow \alpha = \frac{2a}{l} = \frac{3(g-a)}{l}$$

$$5a = 3g \Rightarrow a = 3 \times \frac{g}{5} = 6 \text{ m/s}^2$$

4. (4) Maximum acceleration of cart so that cylinder does not slip:

$$a_m = \mu g = 0.5 \times 10 = 5 \text{ ms}^{-2}$$

For tipping over: Let acceleration of cart is a .

Considering torque about A:

$$Ma \times 5 \geq mg \times 2$$

$$\Rightarrow a \geq 2g/5 = 4 \text{ ms}^{-2}$$

5. (2.00) We have $F_2 - F_1 = 2 \Rightarrow F_2 = 10 \text{ N}$ Balancing torque,

$$F_1 \frac{l}{2} = F_2 \left(\frac{l}{2} - 0.2 \right) \Rightarrow l = 2 \text{ m}$$

6. (1.0) Let x be the distance of centre point C of rod from D. Then, $F_2 - F_1 = ma$

$$\text{or } F_1 = 3 \text{ N}$$

Further, $\tau_C = 0$

$$\therefore F_2 x = F_1 (0.2 + x)$$

$$5x = F_1 (0.2 + x)$$

$$\therefore 5x = 3(0.2 + x) \text{ or } x = 0.3 \text{ m}$$

$$\therefore \text{Length of rod} = 2(x + 0.2) = 1.0 \text{ m}$$

7. (3.00) $a_1 = R\alpha - a_2 \Rightarrow 2 = 2\alpha - 4 \Rightarrow \alpha = 3 \text{ rad/s}^2$

8. (30.00) Drawing FBD

Taking torque about A

$$N_x l \cos \phi - mg \frac{l}{2} \sin \phi - T \frac{l}{3} \cos \phi = 0$$

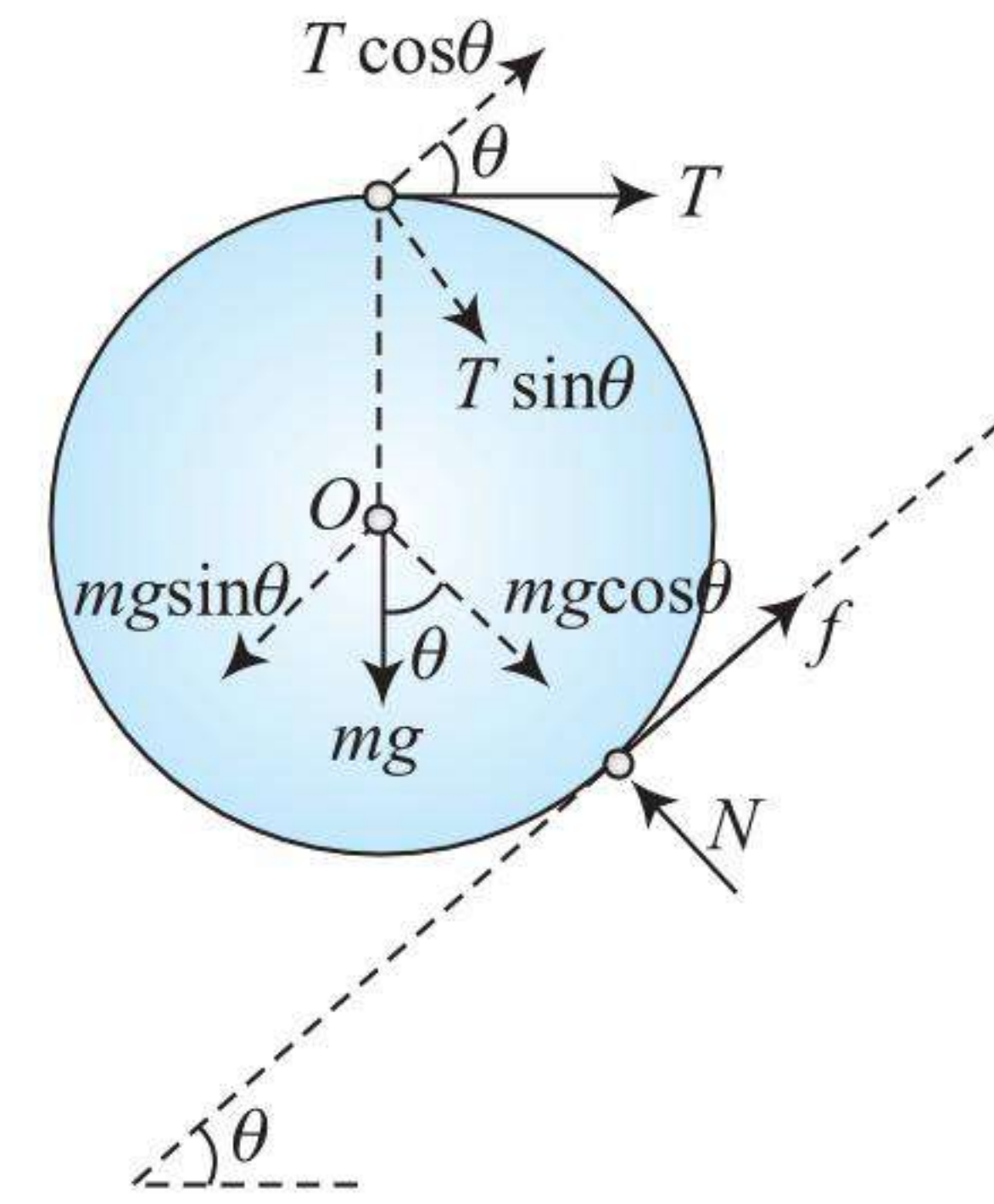
$$N_x - \frac{T}{3} = \frac{mg}{2} \tan \phi \quad \dots(i)$$

$$N_x = T \quad \dots(ii)$$

$$\text{Then } T = N_x = \frac{3mg}{4}$$

On substituting $m = 4 \text{ kg}$ we get $T = 30 \text{ N}$

9. (0.50) Drawing FBD of the cylinder



$$T = f \quad \dots(i)$$

$$T \cos \theta + f = mg \sin \theta \quad \dots(ii)$$

$$T \sin \theta + mg \cos \theta = N \quad \dots(iii)$$

$$f \leq \mu N \quad \dots(iv)$$

$$\frac{mg \sin \theta}{1 + \cos \theta} \leq \mu \left(\frac{mg \cos \theta}{1 - \mu \sin \theta} \right)$$

$$(1 - \mu \sin \theta) \sin \theta \leq \mu (1 + \cos \theta) \cos \theta$$

$$\mu [\cos \theta + \cos^2 \theta + \sin^2 \theta] \geq \sin \theta$$

$$\mu_{\min} \geq \frac{\sin \theta}{1 + \cos \theta} = \frac{\sin 53^\circ}{1 + \cos 53^\circ} = 0.50$$

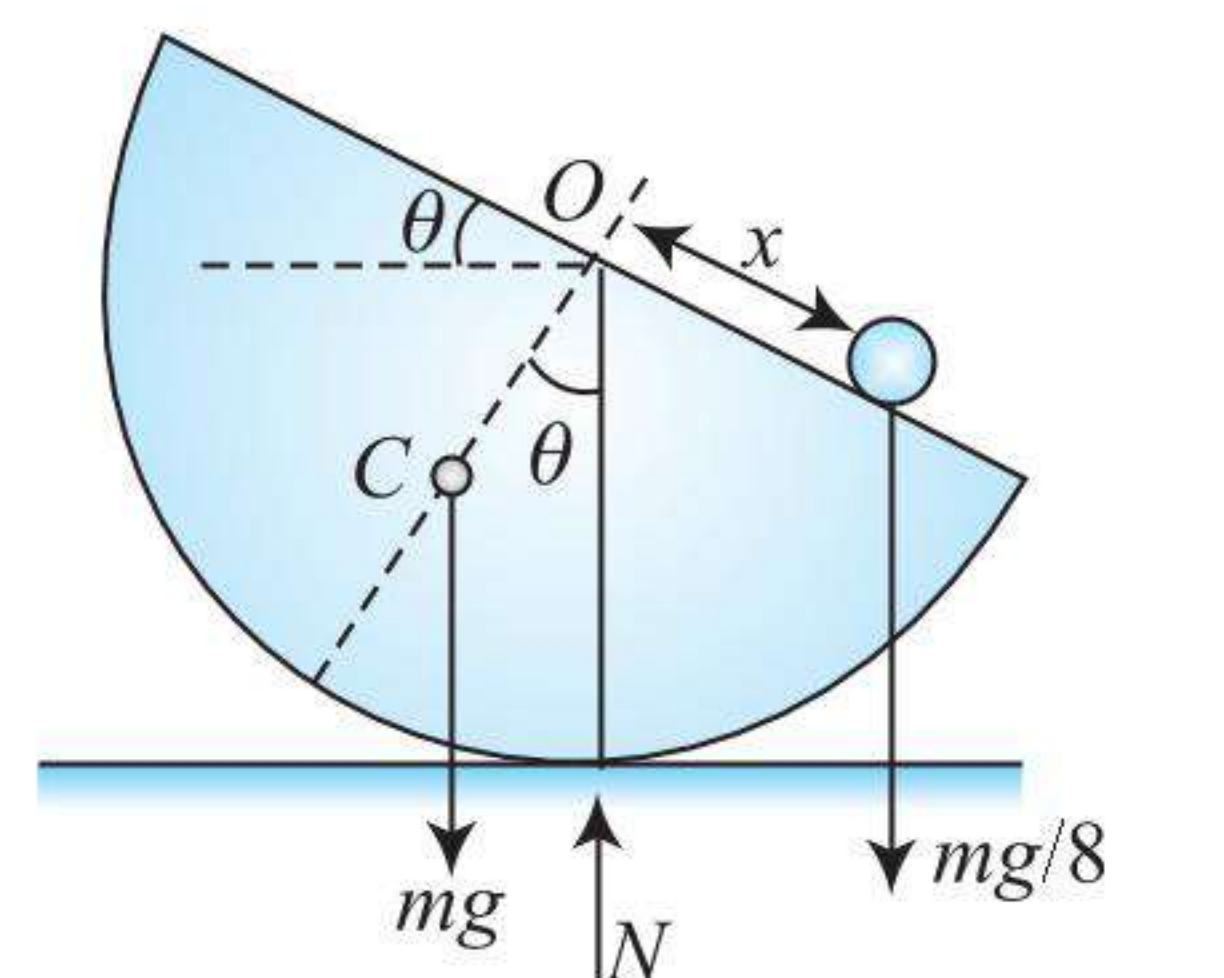
$$10. (6.00) \Sigma \tau_0 = 0 \Rightarrow mg \frac{3R}{8} \sin \theta = \frac{mgx}{8} \cos \theta$$

$$\therefore x = 3R \tan \theta$$

$$f = \frac{mg}{8} \sin \theta$$

$$\text{and } N = \frac{mg}{8} \cos \theta \text{ (for the object)}$$

$$\therefore \tan \theta = \frac{f}{N}$$



$$\therefore \tan \theta_m = \frac{f_m}{N} = \mu$$

$$\therefore x_m = 3R \tan \theta_m = 3R\mu = (3)(10)(0.2) = 6 \text{ m}$$

11. (5.00) Torque about A

$$F \frac{\sqrt{3}}{2} d = mg \frac{d}{2} + ma \frac{\sqrt{3}}{2} d \frac{1}{3}$$

$$\therefore F = \frac{mg}{\sqrt{3}} + \frac{ma}{3}$$

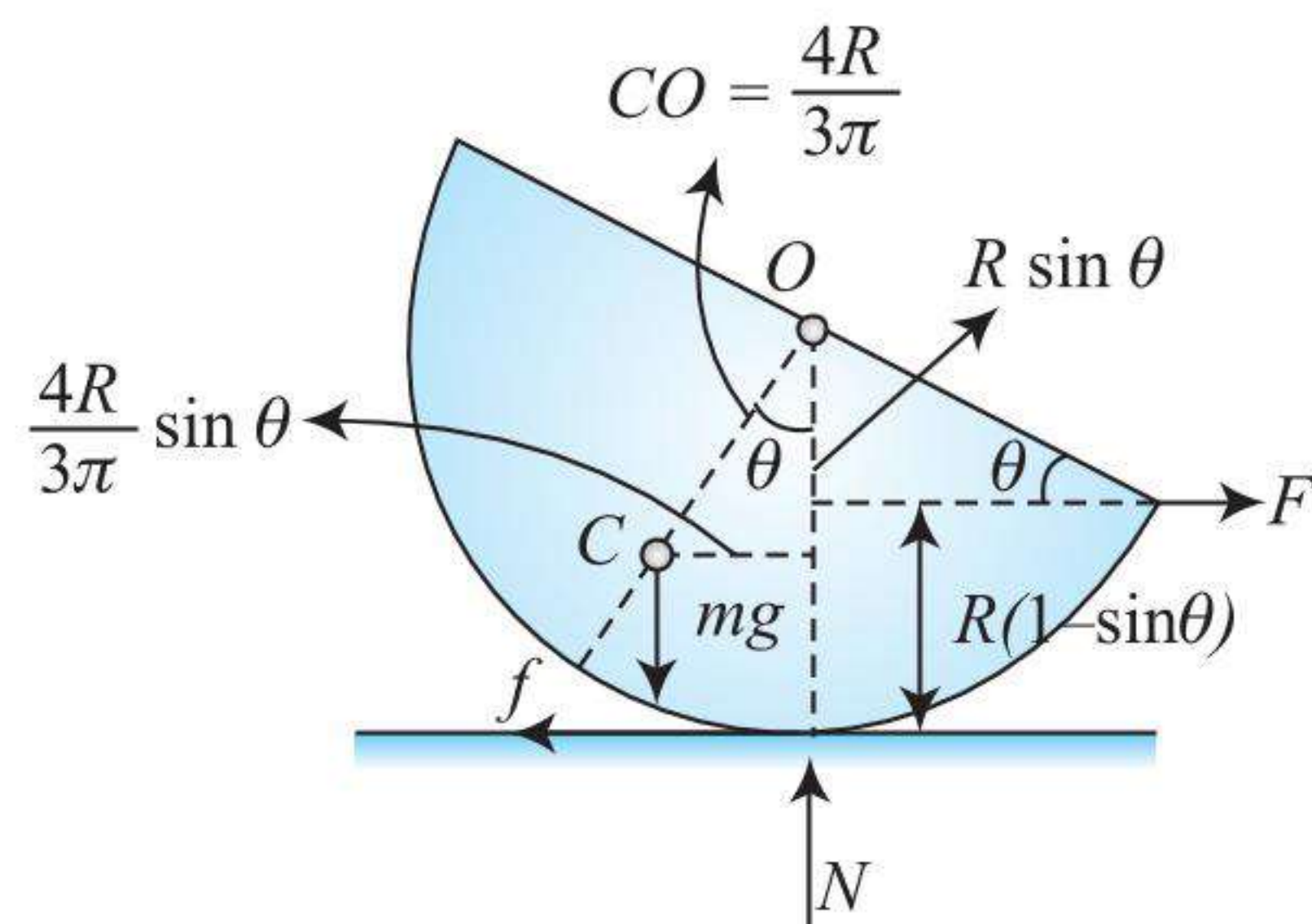
$$\therefore F_H = \frac{mg}{\sqrt{3}} - \frac{2ma}{3}$$

$$F_v = mg$$

$$\text{So, } F_A = 5 \text{ N}$$

12. (30) As the disc moves with uniform velocity $F = \mu mg$ and $N = mg$

The force F and friction force will form a couple.



Balancing torque about centre of mass

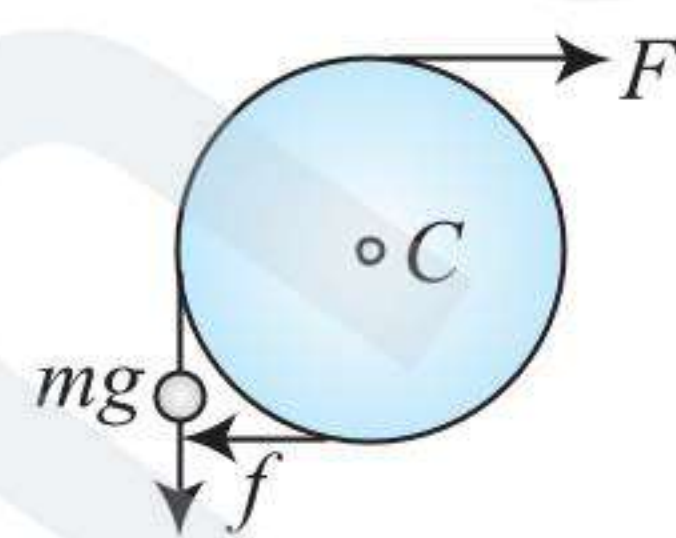
$$\mu mg R(1 - \sin \theta) = mg \frac{4R}{3\pi} \sin \theta$$

$$\Rightarrow \sin \theta = \frac{1}{2} \Rightarrow \theta = 30^\circ$$

13. (25.00) For cylinder $F = f$ (friction)

$$\text{and } FR + fR = mgR$$

$$\Rightarrow 2F = mg \Rightarrow F = \frac{mg}{2} = \frac{5 \times 10}{2} = 25 \text{ N}$$



14. (0.50) Net torque about A = 0

$$f_s \frac{3R}{2} = Mg \cdot \frac{R}{2}$$

$$\therefore f_s = \frac{Mg}{3}$$

...(i)

$$\left[\therefore PA = \frac{3R}{2} \tan 45^\circ = \frac{3R}{2} \right]$$

$$\text{Horizontal equilibrium: } N = \frac{T}{\sqrt{2}}$$

...(ii)

$$\text{Vertical equilibrium: } \frac{T}{\sqrt{2}} + f_s = Mg$$

$$T = \frac{2\sqrt{2}}{3} Mg$$

...(iii)

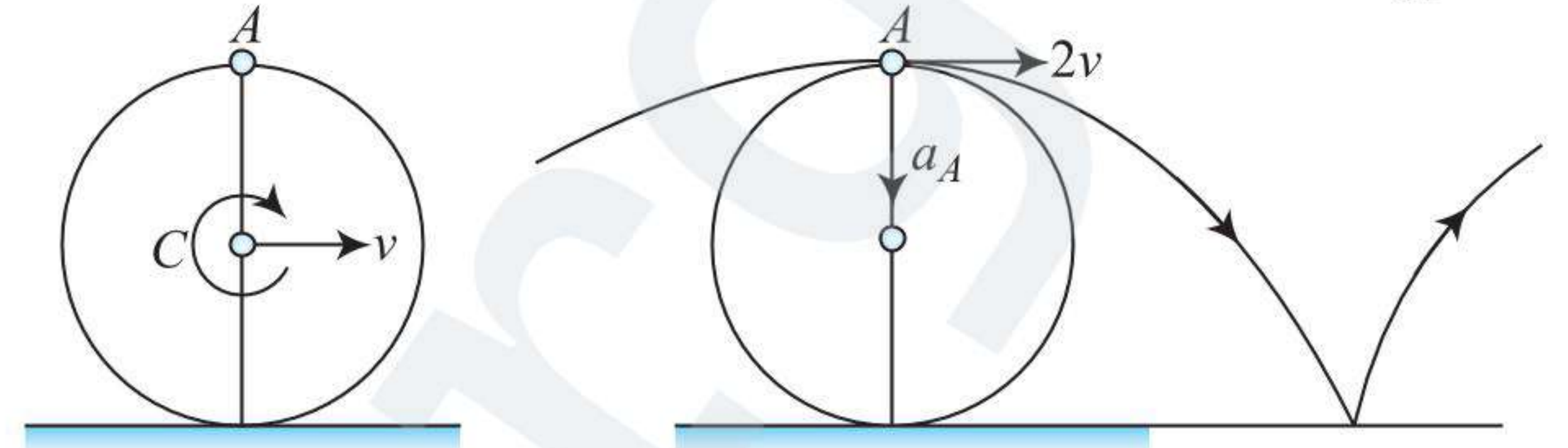
$$\text{Put in (ii) } N = \frac{2}{3} Mg$$

$$f_s \leq \mu_s N \Rightarrow \frac{Mg}{3} \leq \mu_s \cdot \frac{2}{3} Mg \Rightarrow \frac{1}{2} \leq \mu_s$$

$$\text{Hence, } \mu_{\min} = 0.50$$

15. (4.00) Radius of curvature of the path is the radius of the circle that matches up with the path locally at a point.

Let's first find the acceleration of point A with respect to the COM of the wheel. An observer at COM sees that point A is spinning about it. Acceleration of A is $\frac{v^2}{2R}(\downarrow)$ with respect to COM. Because the COM does not have any acceleration hence acceleration of point A with respect to the ground is $a_A = \frac{v^2}{R}(\downarrow)$

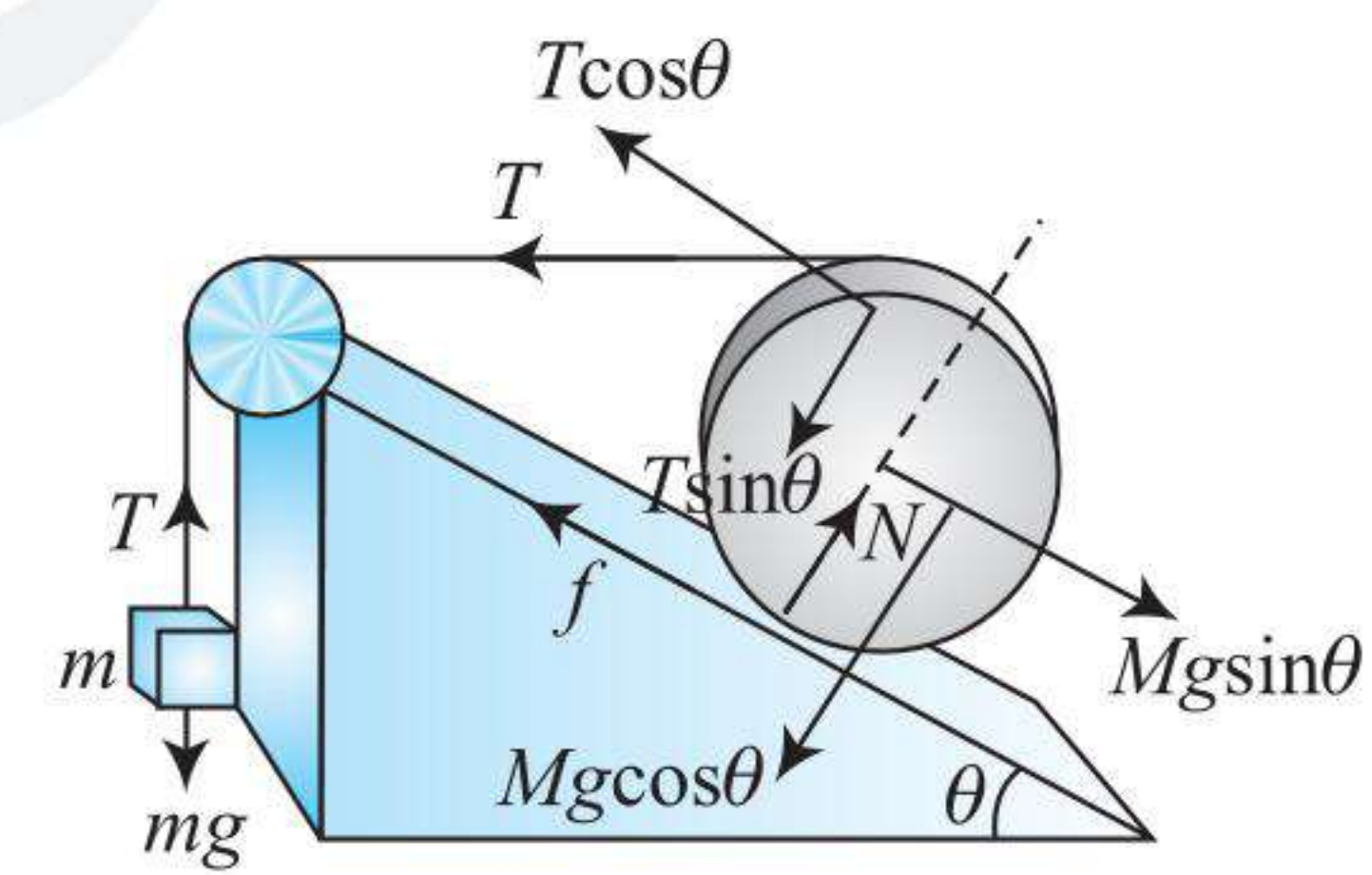


For a person on ground, speed of point A is $v_A = v + \omega R = 2v$.

If the radius of curvature of the path of A is r then $a_A = \frac{v_A^2}{r}$ i.e.,

$$\frac{v^2}{R} = \frac{(2v)^2}{r} \Rightarrow r = 4R = 4 \text{ m}$$

16. (30.00) Let the mass of the block be m



For equilibrium of the block, $T = mg$... (i)

For rotational equilibrium of the cylinder, net torque about its centre must be zero.

$$\therefore TR = fR. \quad [R = \text{radius of cylinder}]$$

$$\Rightarrow mg = f \quad \dots (ii)$$

For translational equilibrium of the cylinder

$$N = Mg \cos \theta + T \sin \theta$$

$$\Rightarrow N = Mg \cos \theta + mg \sin \theta \quad \dots (iii)$$

$$\text{and } f + T \cos \theta = Mg \sin \theta \quad \dots (iv)$$

$$\Rightarrow f(1 + \cos \theta) = Mg \sin \theta$$

$$\Rightarrow f = \frac{Mg \sin \theta}{(1 + \cos \theta)} \Rightarrow mg = \frac{Mg \sin \theta}{1 + \cos \theta}$$

$$m = \frac{M \sin \theta}{1 + \cos \theta} \quad \dots (v)$$

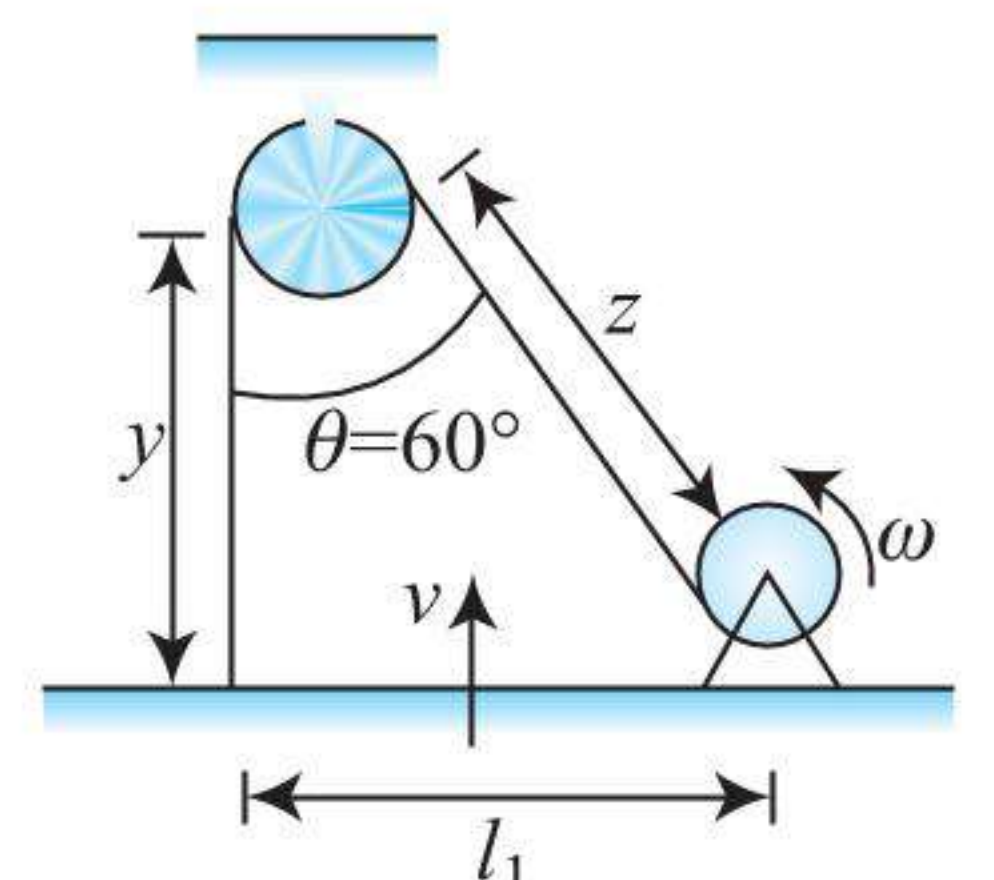
On substituting the values of M and θ we get $m = 30 \text{ kg}$

17. (1.00) Total length of the string at any time t

$$L = y + z = y + \sqrt{y^2 + l_1^2}$$

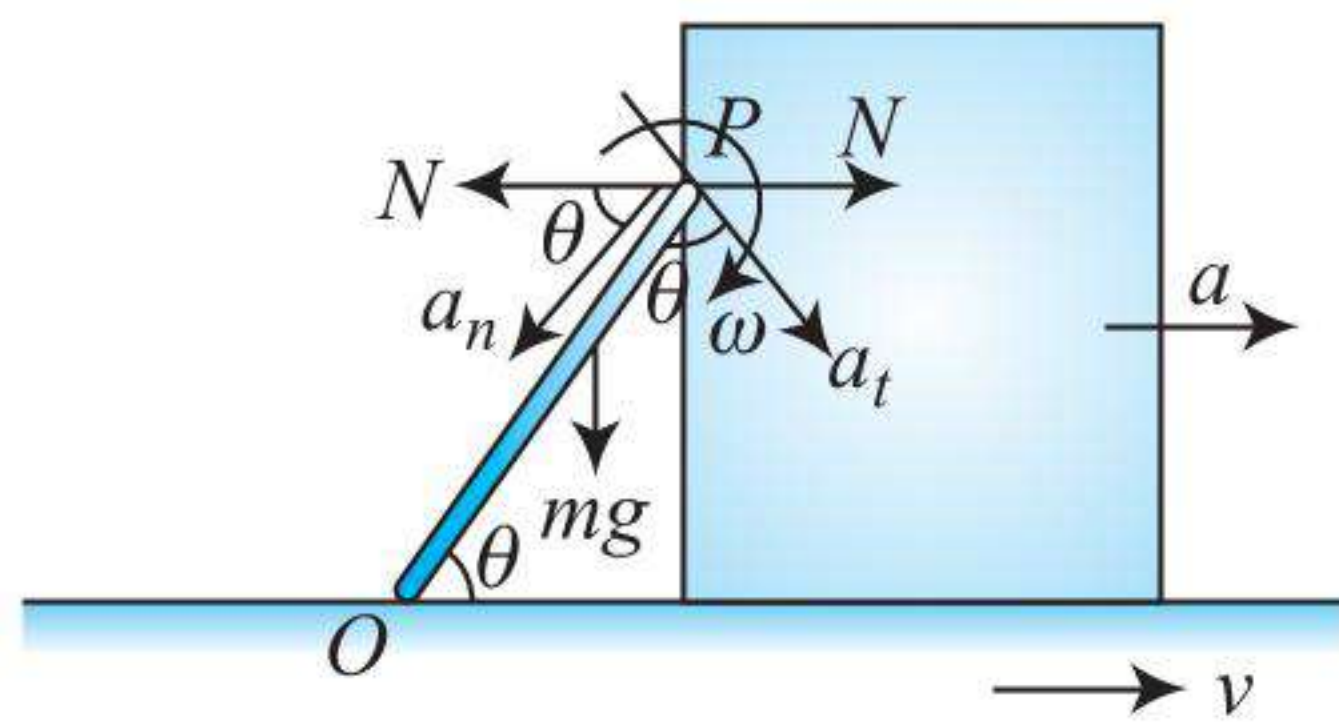
$$\frac{dL}{dt} = \frac{dy}{dt} + \frac{y}{\sqrt{y^2 + l_1^2}} \frac{dy}{dt}$$

$$\omega R = v(1 + \cos \theta), v = \frac{\omega R}{1 + \cos \theta}$$



On substituting the values of $\omega = 15 \text{ rad/s}$, $\theta = 60^\circ$ and $R = 0.10 \text{ m}$, we get $v = 1 \text{ m/s}$

18. (4)



Decrease in potential energy of rod = increase in rotational kinetic energy of rod + translational kinetic energy of block.

But $v = v_p \sin \theta = (l\omega) \sin \theta$

$$\therefore mg(l - l \sin \theta) = \frac{1}{2}(ml^2)\omega^2 + \frac{1}{2}M(\omega l \sin \theta)^2$$

From above equation we get,

$$a_n = l\omega^2 = \frac{2mg(1 - \sin \theta)}{m + M \sin^2 \theta} \quad \dots(i)$$

At the time of losing contact

$$a_t = l\alpha = l\left(\frac{\tau}{I}\right) = l\left[\frac{mgl \cos \theta}{ml^2}\right] = g \cos \theta \quad \dots(ii)$$

At the time of separation,

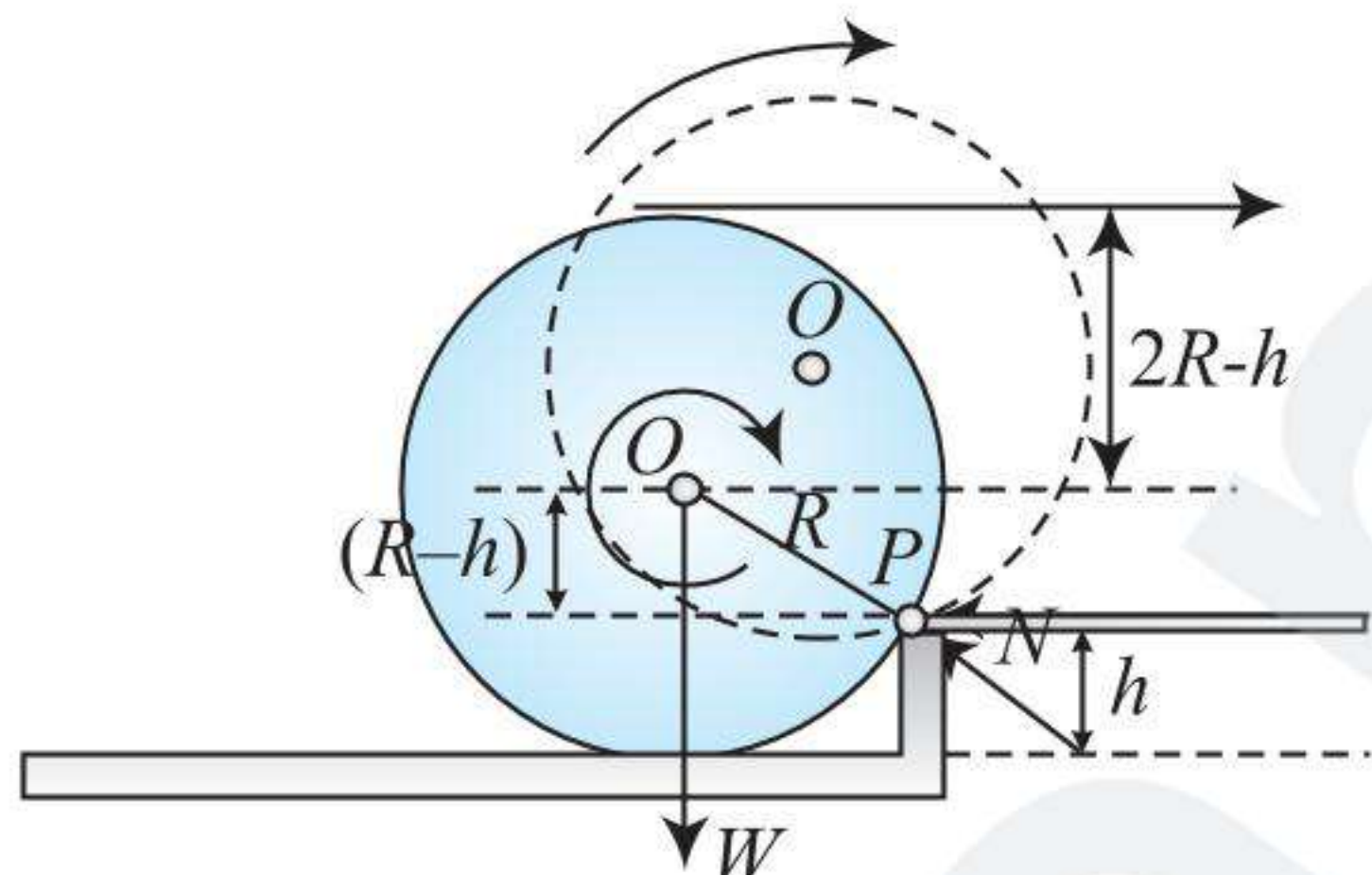
$$\Sigma F_x = 0 \text{ and } a = 0$$

$$\text{For block: } a_t \sin \theta - a_n \cos \theta = 0 \quad \dots(iii)$$

Now putting values of a_t and a_n from Eqs. (i) and (ii) in Eq. (iii) and then putting $\theta = 30^\circ$ in the equation we get,

$$\frac{M}{m} = 4$$

19. (50.00) From figure



$$d = \sqrt{R^2 - (R-h)^2} = \sqrt{2Rh - h^2}$$

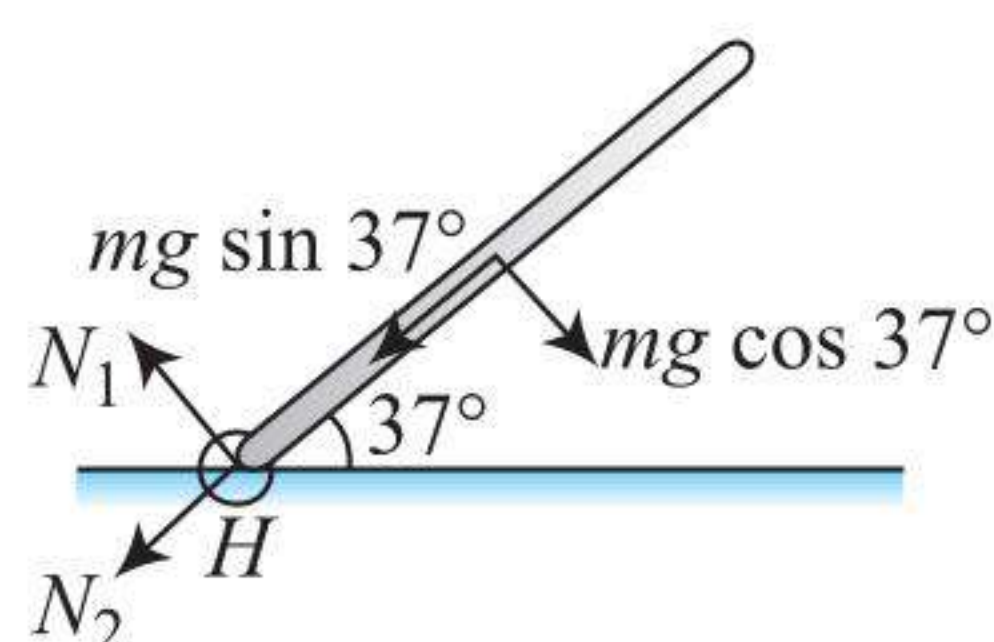
Taking torque about P, $\tau_F > \tau_W$

For minimum F, $mgd = F(2R - h)$

$$\begin{aligned} \text{or } F &= \frac{mg\sqrt{2Rh - h^2}}{(2R - h)} = \frac{mgh\sqrt{\frac{2R}{h} - 1}}{h\left(\frac{2R}{h} - h\right)} = \frac{mg}{\sqrt{\frac{2R}{h} - 1}} \\ &= \frac{15 \times 10}{\sqrt{10 - 1}} = 50 \text{ N} \end{aligned}$$

20. (40.00) Torque about hinge $\tau_H = I\alpha$

$$\begin{aligned} Mg \cos 37^\circ \frac{l}{2} &= \frac{ml^2}{3} \alpha \\ \Rightarrow \alpha &= 6g/5l \\ a_t &= \alpha \frac{l}{2} = \frac{3g}{5} \\ \Rightarrow Mg \cos 37^\circ - N_1 &= ma_t \end{aligned}$$



$$\therefore N_1 = \frac{mg}{5}$$

Angular velocity of rod is zero. So

$$N_2 = mg \sin 37^\circ = 3mg/5$$

$$N = \sqrt{N_1^2 + N_2^2} = \sqrt{\left(\frac{mg}{5}\right)^2 + \left(\frac{3mg}{5}\right)^2} = \frac{mg\sqrt{10}}{5}$$

$$\Rightarrow N = \frac{2\sqrt{10} \times 10 \times \sqrt{10}}{5} = 40 \text{ N}$$

Archives

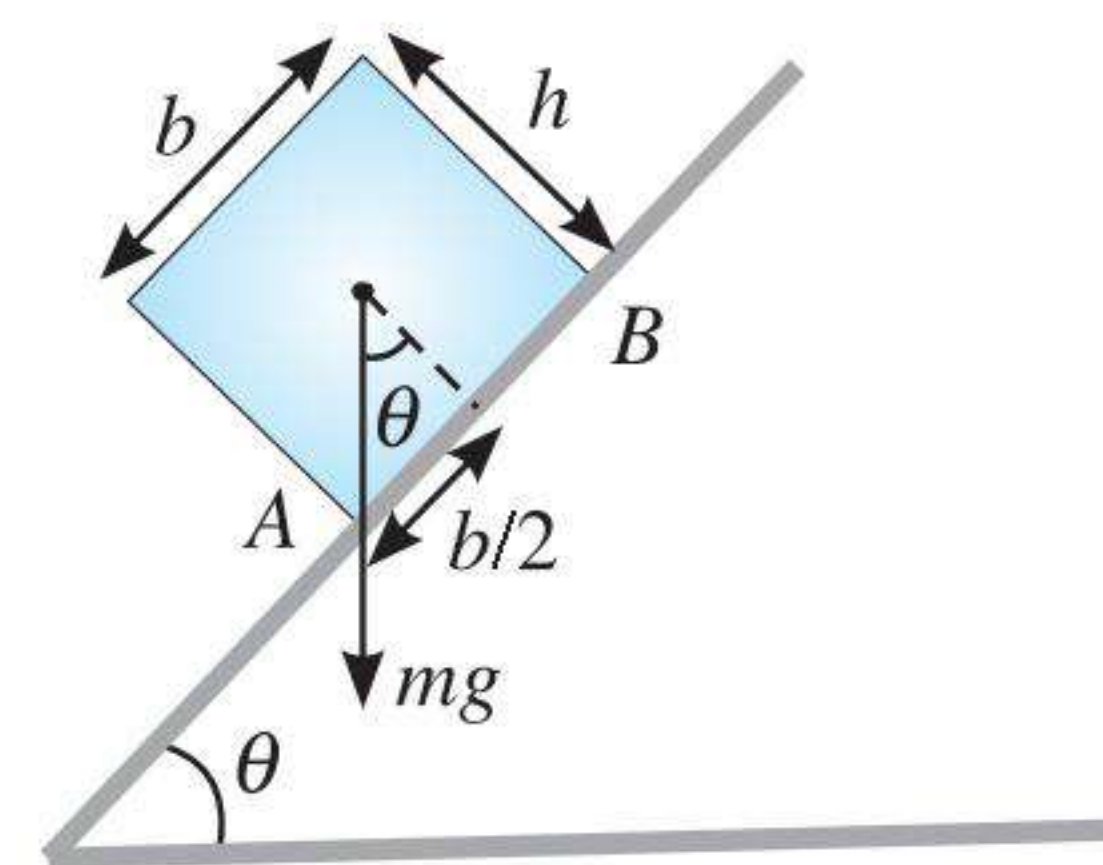
JEE Advanced

Single Correct Answer Type

1. (2) The maximum angle when the block will not slide should be equal to the angle of repose.

$$\phi = \tan^{-1} \mu = \tan^{-1} \sqrt{3} = 60^\circ$$

If the block does not topple, the line of action of weight should pass within the base of the block.



And for toppling, maximum angle of inclination.

$$\theta = \tan^{-1} \frac{a/2}{h/2}$$

$$\theta = \tan^{-1} \frac{(10/2)}{(15/2)} = \tan^{-1} \frac{2}{3}$$

$$\theta \approx 34^\circ < 60^\circ$$

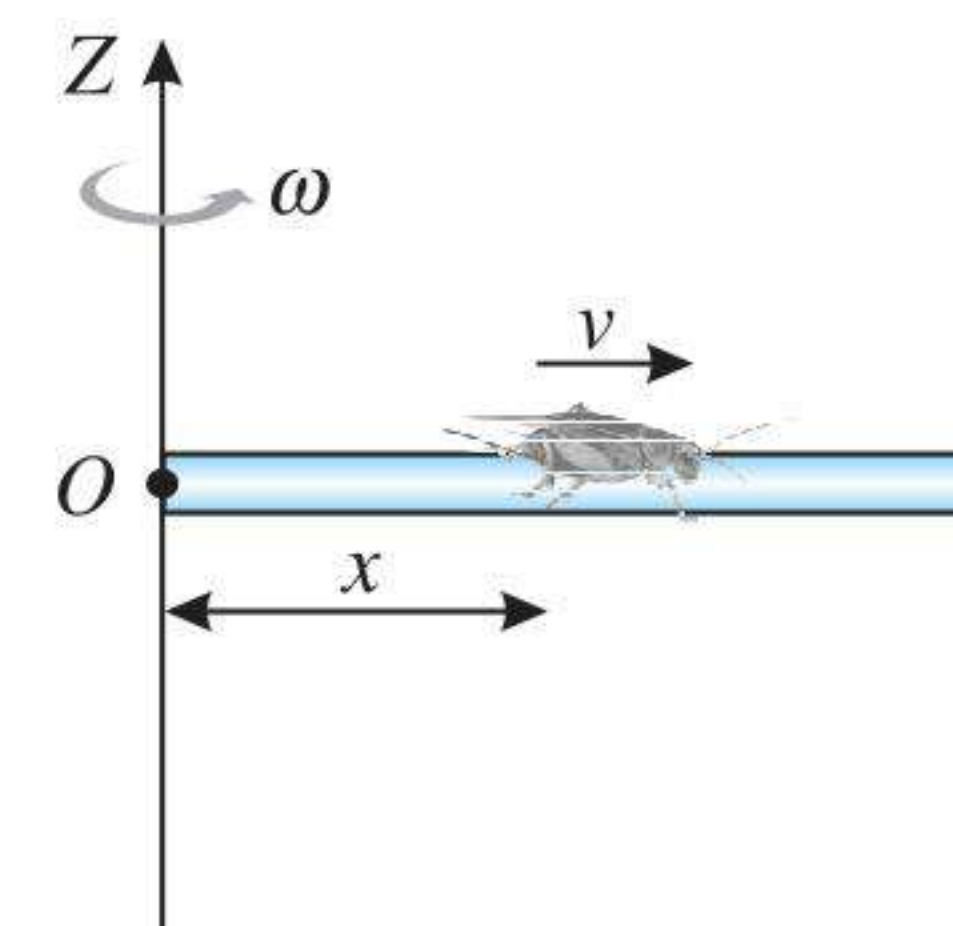
For 60° , mg is already outside the base.

$\therefore$ As the angle θ is increased first it will topple before it starts sliding.

2. (2) Let M and l be the mass and length of the rod respectively and m be the mass of the insect. Let insect be at a distance x from O at any instant of time t .

$$\therefore x = vt \quad \dots(i)$$

Angular momentum of the system about O ,



$$\begin{aligned} L &= \left[\frac{Ml^2}{3} + mx^2 \right] \omega = \left[\frac{Ml^2}{3} + m(vt)^2 \right] \omega \quad (\text{using (i)}) \\ &= \left[\frac{Ml^2}{3} + mv^2 t^2 \right] \omega \end{aligned}$$

$$\text{As } |\vec{\tau}| = \frac{dL}{dt} = \frac{d}{dt} \left[\frac{Ml^2}{3} + mv^2 t^2 \right] \omega$$

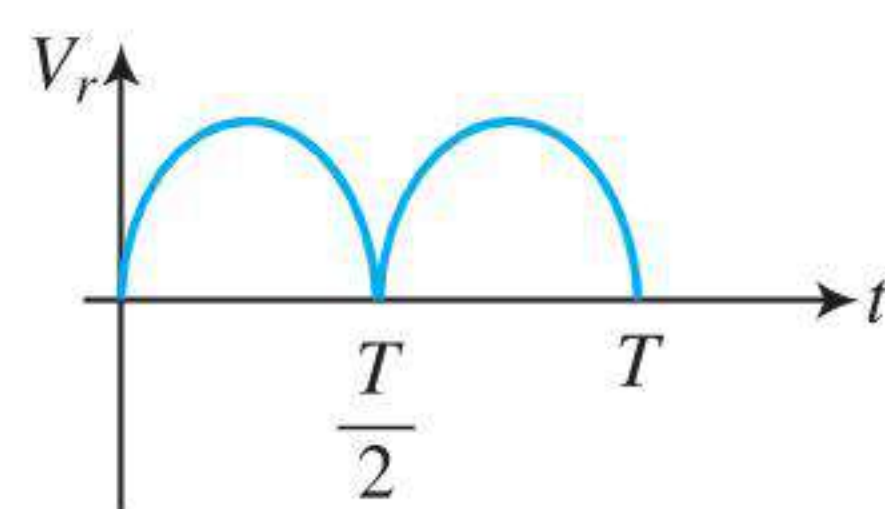
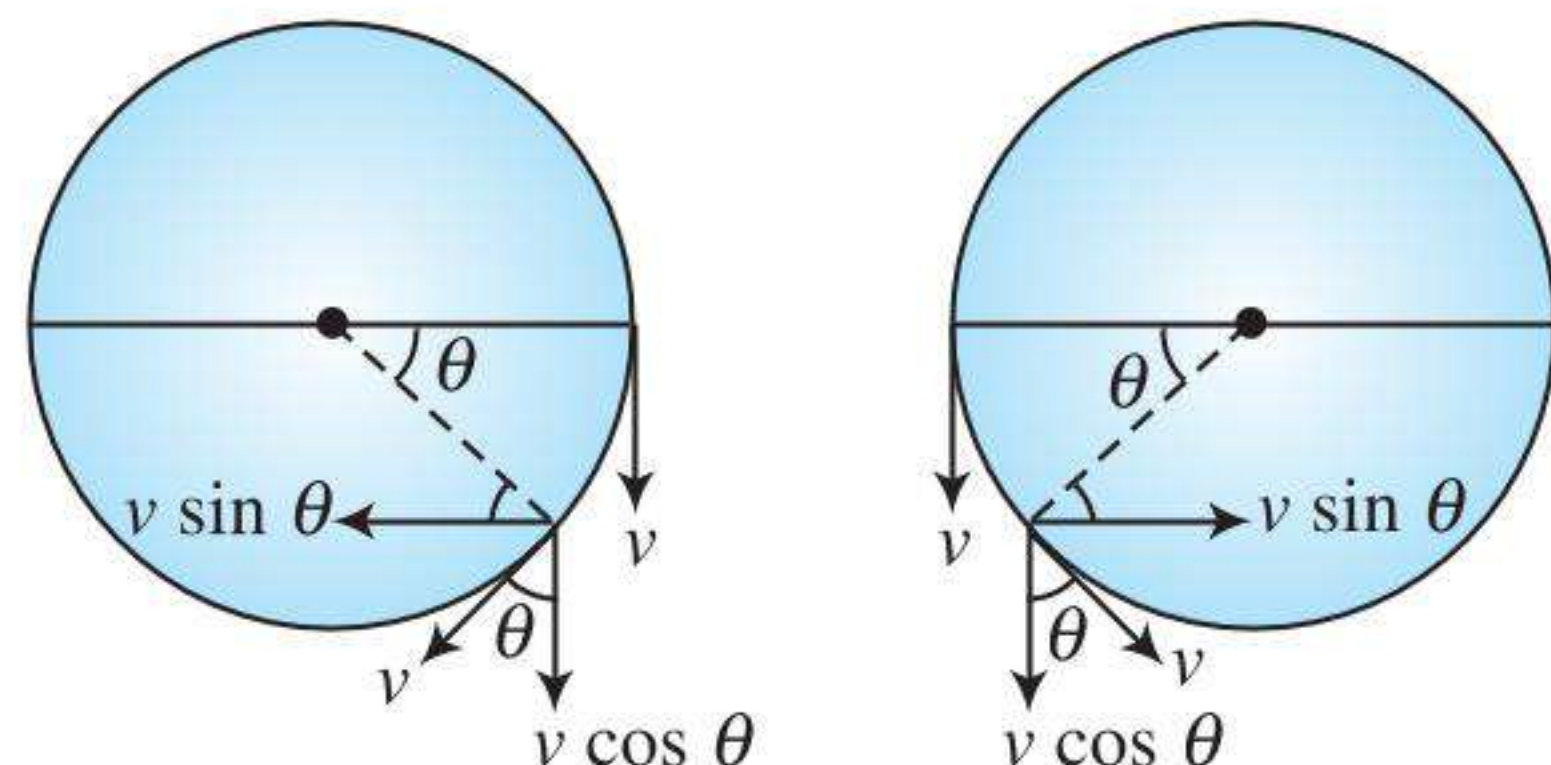
As ω and v remain constant.

$$\therefore |\vec{\tau}| = 2m\omega v^2 t$$

$$|\vec{\tau}| \propto t$$

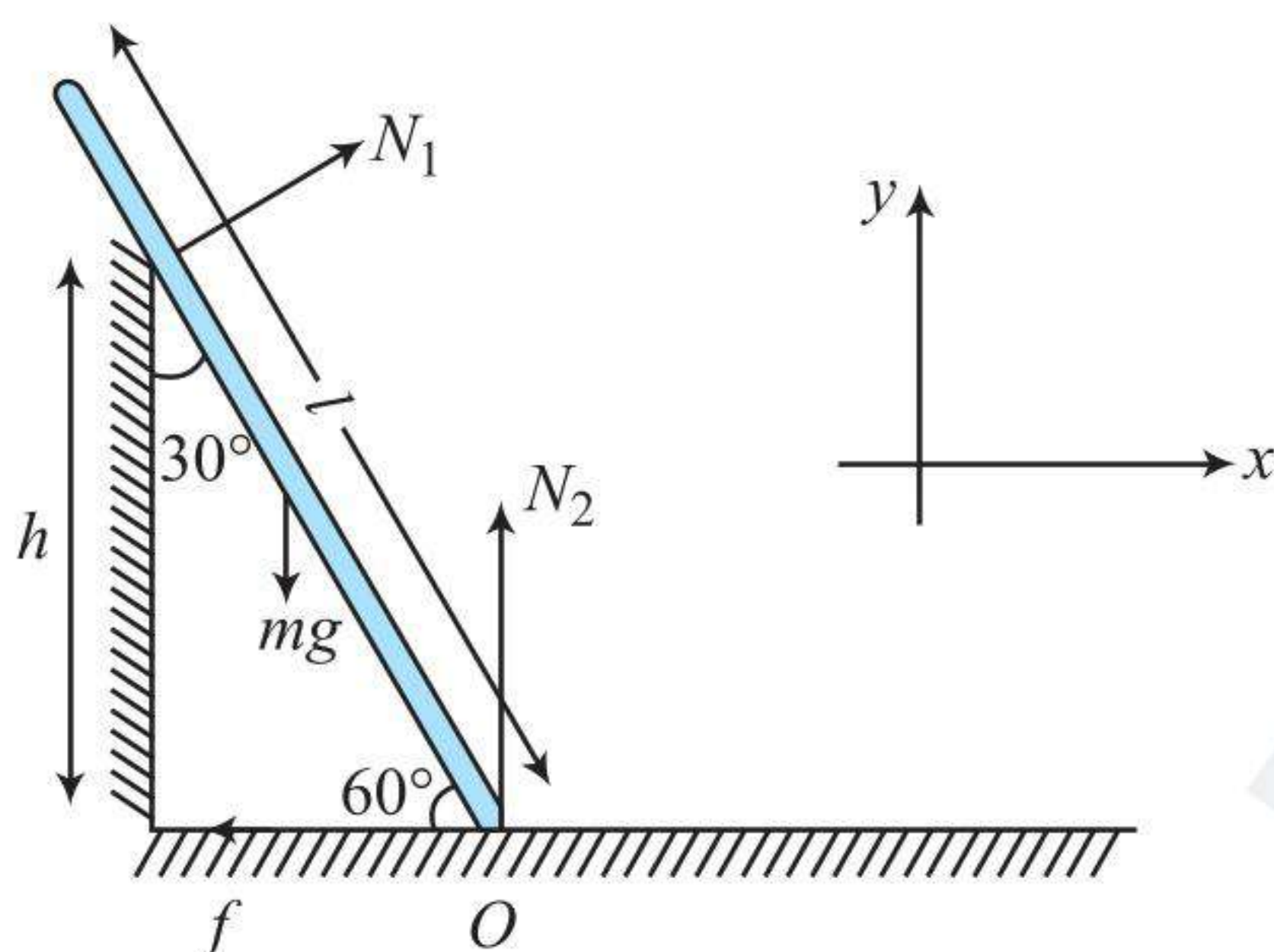
Hence, the graph $|\tau|$ and t is a straight line passing through the (0, 0). Option (2) represents correct plot.

3. (1)



$$v_r = |2v \sin \theta| = |2v \sin \omega t|$$

4. (4)



Force equation in x-direction,

$$N_1 \cos 30^\circ - f = 0 \quad \dots(i)$$

Force equation in y-direction,

$$N_1 \sin 30^\circ + N_2 - mg = 0 \quad \dots(ii)$$

Torque equation about O,

$$mg \frac{l}{2} \cos 60^\circ - N_1 \frac{h}{\cos 30^\circ} = 0 \quad \dots(iii)$$

$$\text{Also, given } N_1 = N_2 \quad \dots(iv)$$

[Note taking reaction from floor as normal reaction only]

Solving (i), (ii), (iii) and (iv) we have

$$\frac{h}{l} = \frac{3\sqrt{3}}{16} \text{ and } f = \frac{16\sqrt{3}}{3}$$

5. (2) As the roller starts rolling without slipping

$$V_{\text{centre}} = \omega R \quad (R \text{ is radius of roller})$$

$$\therefore \text{Velocity of scale} = (V_{\text{centre}} + \omega r) \quad [r \text{ is radius of axle}]$$

$$\text{Given, } V_{\text{centre}} \cdot t = 50 \text{ cm}$$

$$\therefore \text{Distance moved by scale} = (V_{\text{centre}} + \omega r)t$$

$$= \left(V_{\text{centre}} + \frac{V_{\text{centre}} r}{R} \right) t = \frac{3V_{\text{centre}}}{2} \cdot t = 75 \text{ cm}$$

It means the relative displacement (with respect to centre of roller) is (75 - 50) cm = 25 cm.

Multiple Correct Answers Type

1. (2),(3)

$$V_A = 0, V_B = \omega r, V_C = 2\omega r$$

$$\vec{V}_C - \vec{V}_A \text{ is towards the right and } \vec{V}_B - \vec{V}_C \text{ is towards left,}$$

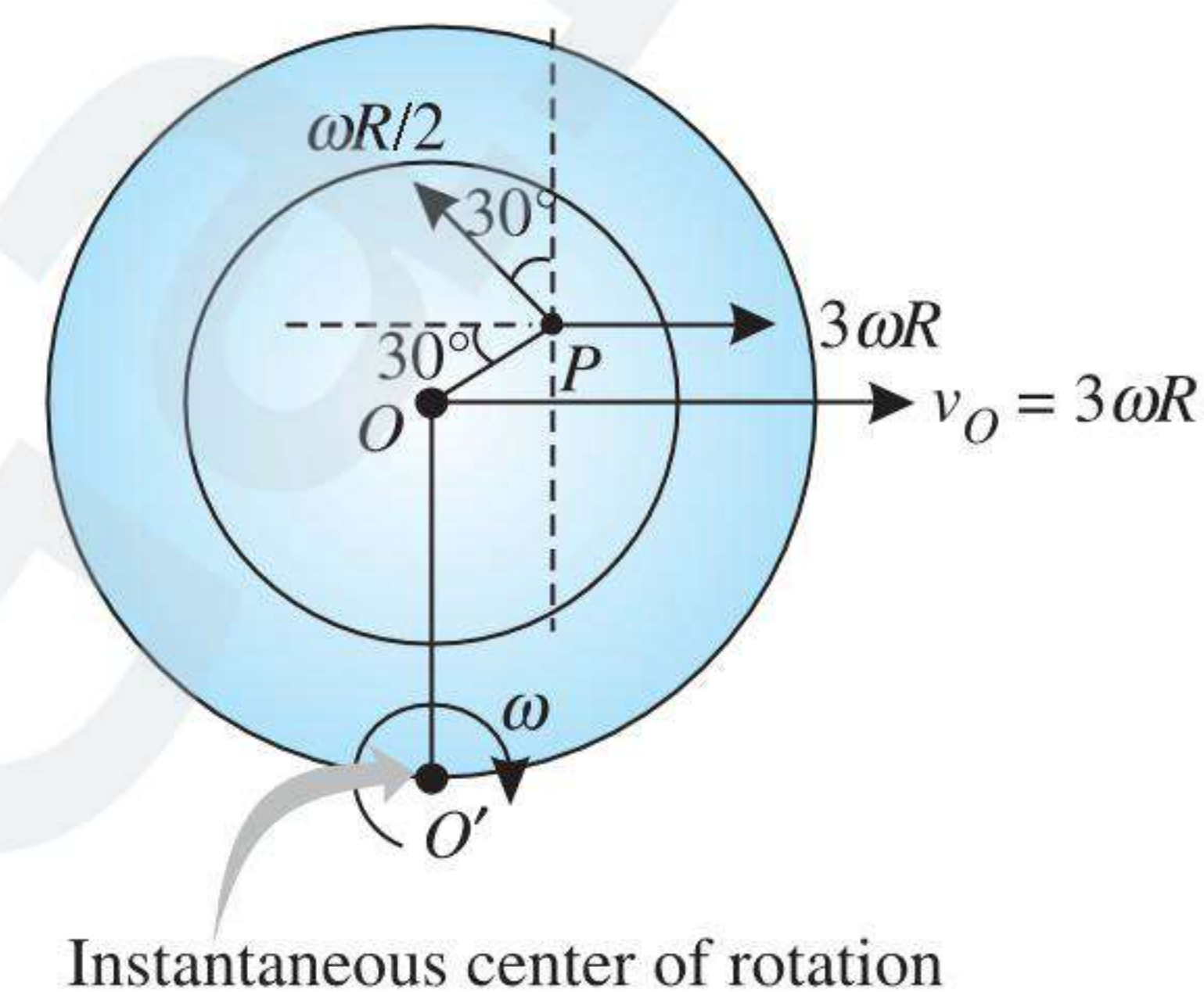
Hence, option (1) is incorrect.

But in magnitude, option (3) is correct.

$$\vec{V}_C - \vec{V}_A = 2\omega r = 2\vec{V}_B$$

2. (1),(2)

As ring is rolling the point of contact of ring with ground (O') will be instantaneous center of rotation. Hence for pure rolling $v_O = \omega(3R) = 3R\omega$ and its direction is along positive x.



The velocity at the point P

$$v_P = 3R\omega \hat{i} - \frac{R\omega}{2} \sin 30^\circ \hat{i} + \frac{R\omega}{2} \cos 30^\circ \hat{k}$$

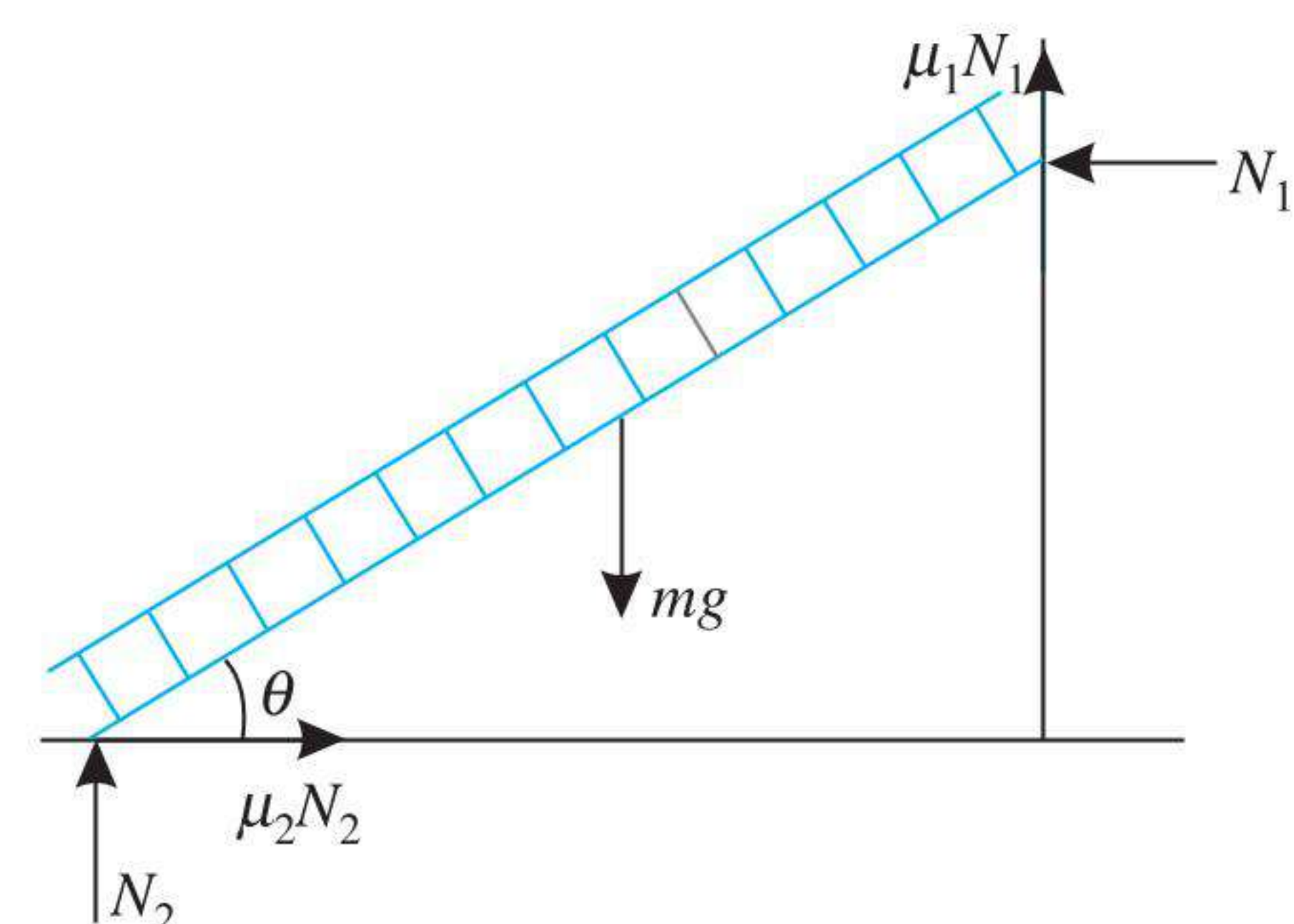
$$v_P = \frac{11}{4} R\omega \hat{i} + \frac{\sqrt{3}}{4} R\omega \hat{k}$$

3. (3),(4)

Consider a translational equilibrium

$$N_1 = \mu_2 N_2 \quad \dots(i)$$

$$N_2 + \mu_1 N_1 = Mg \quad \dots(ii)$$



$$\text{Solving } N, N_2 = \frac{mg}{1 + \mu_1 \mu_2}$$

$$N_1 = \frac{\mu_2 mg}{1 + \mu_1 \mu_2}$$

Applying torque equation about corner (left) point on the floor

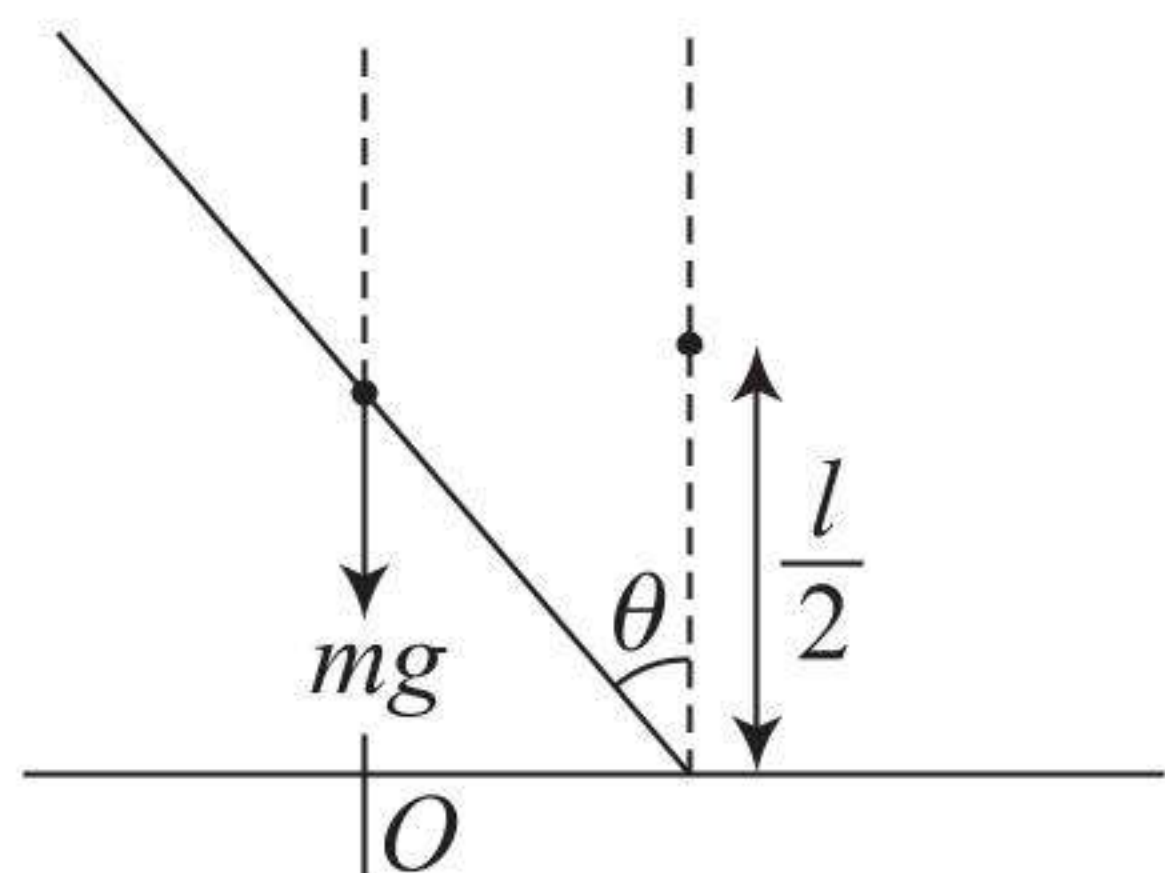
$$mg \frac{l}{2} \cos \theta = N_1 l \sin \theta + \mu_1 N_1 l \cos \theta$$

$$\text{Solving } \tan \theta = \frac{1 - \mu_1 \mu_2}{2\mu_2}$$

4. (1),(2),(3)

Torque about O , at any instant is $mg \cdot \frac{l}{2} \sin \theta$

$\therefore$ Option (3) is correct.

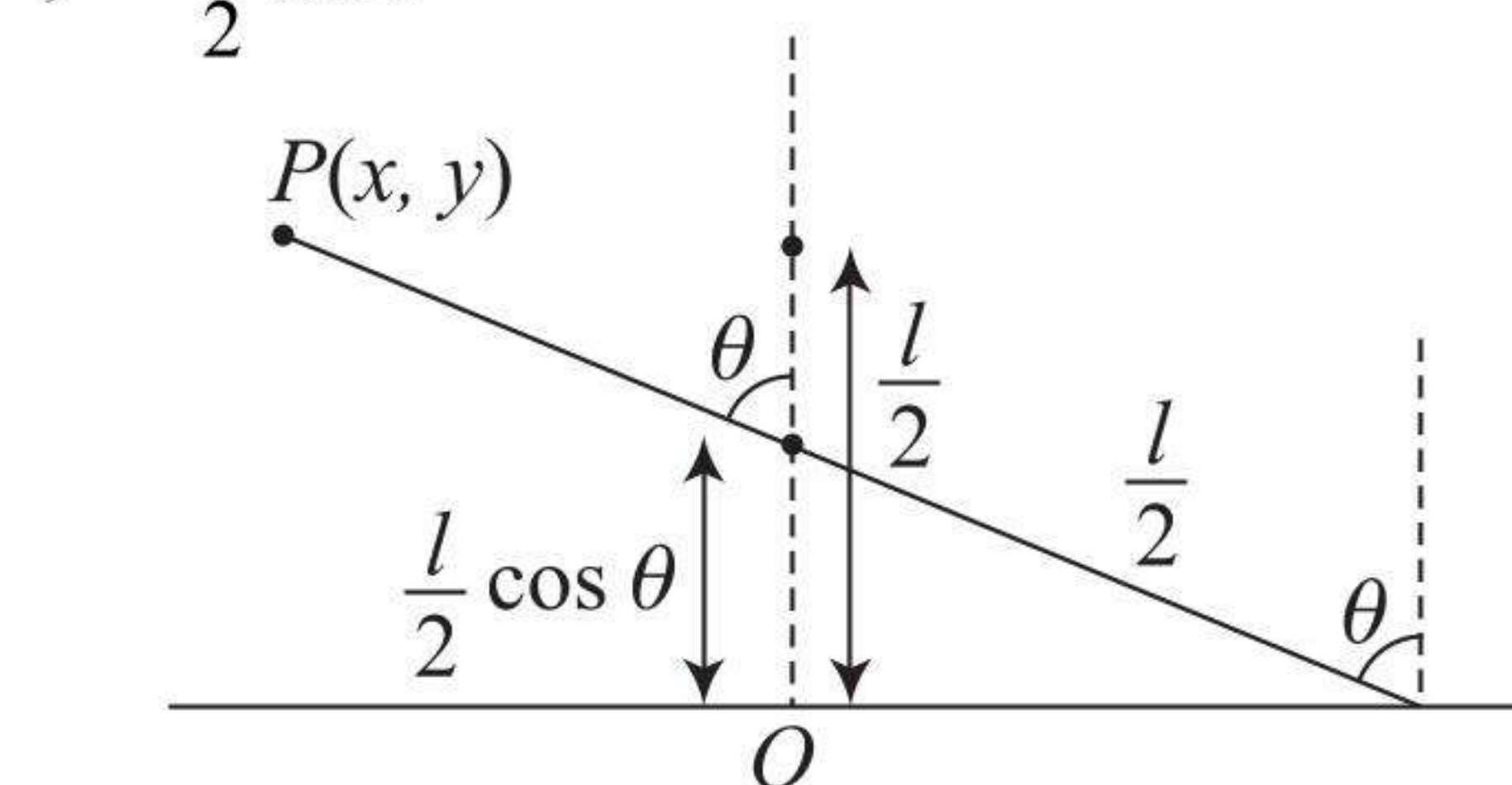


As no external force acts along x -axis, therefore centre of mass will fall vertically downward.

$\therefore$ Option (2) is correct.

When the bar makes an angle θ ; the height of its COM (mid-point) is $\frac{l}{2} \cos \theta$

Displacement of centre of mass along y -axis



$$= \frac{l}{2} - \frac{l}{2} \cos \theta = \frac{l}{2} (1 - \cos \theta)$$

$\therefore$ Option (1) is correct.

$$x = -\frac{l}{2} \sin \theta, y = l \cos \theta$$

$$\Rightarrow \left(-\frac{2x}{l}\right)^2 + \frac{y^2}{l^2} = 1 \text{ or } \left(\frac{x}{l/2}\right)^2 + \frac{y^2}{l^2} = 1$$

$\therefore$ Trajectory is not parabola. Path of A is an ellipse.

5. (1), (3) Here we are given

$$\vec{F} = (\alpha t)\hat{i} + \beta\hat{j} \quad [\text{At } t=0, v=0, \vec{r}=\vec{0}]$$

Substituting, we get $\alpha = 1, \beta = 1 \Rightarrow \vec{F} = t\hat{i} + \hat{j}$

$$\text{or } m \frac{d\vec{v}}{dt} = t\hat{i} + \hat{j} \Rightarrow m d\vec{v} = (t\hat{i} + \hat{j}) dt$$

$$\text{On integrating; } m \int_0^v d\vec{v} = \int_0^t t dt \hat{i} + \int_0^1 dt \hat{j}$$

$$\vec{v} = \frac{t^2}{2} \hat{i} + t \hat{j} \quad [\text{as } m = 1 \text{ kg}]$$

$$\frac{d\vec{r}}{dt} = \frac{t^2}{2} \hat{i} + t \hat{j} \quad [\vec{r} = \vec{0} \text{ at } t = 0]$$

$$d\vec{r} = \frac{t^2}{2} dt \hat{i} + t dt \hat{j}$$

$$\text{On integrating, we get } \Rightarrow \vec{r} = \frac{t^3}{6} \hat{i} + \frac{t^2}{2} \hat{j}$$

At $t = 0$ s

$$\vec{\tau} = (\vec{r} \times \vec{F}) = \left(\frac{1}{6} \hat{i} + \frac{1}{2} \hat{j}\right) \times (\hat{i} + \hat{j})$$

$$= \frac{1}{6} (\hat{i} \times \hat{j}) + \frac{1}{2} (\hat{j} \times \hat{i}) = \frac{1}{6} \hat{k} - \frac{1}{2} \hat{k} \text{ or } \vec{\tau} = -\frac{1}{3} \hat{k}$$

$$\Rightarrow |\vec{\tau}| = \frac{1}{3} \text{ Nm}$$

Hence option (1) is correct.

$$\vec{v} = \frac{t^2}{2} \hat{i} + t \hat{j}$$

At $t = 0$ s

$$\vec{v} = \left(\frac{1}{2} \hat{i} + \hat{j}\right) = \frac{1}{2} (\hat{i} + 2\hat{j}) \text{ m/s}$$

Hence option (3) is correct.

$$\text{At } t = 1 \text{ s, } \vec{s} = \vec{r}_1 - \vec{r}_0$$

$$\vec{s} = -\left[\frac{1}{6} \hat{i} + \frac{1}{2} \hat{j}\right] - [0] \text{ or } \vec{s} = -\frac{1}{6} \hat{i} - \frac{1}{2} \hat{j}$$

$$|\vec{s}| = \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{10}}{6} \text{ m}$$

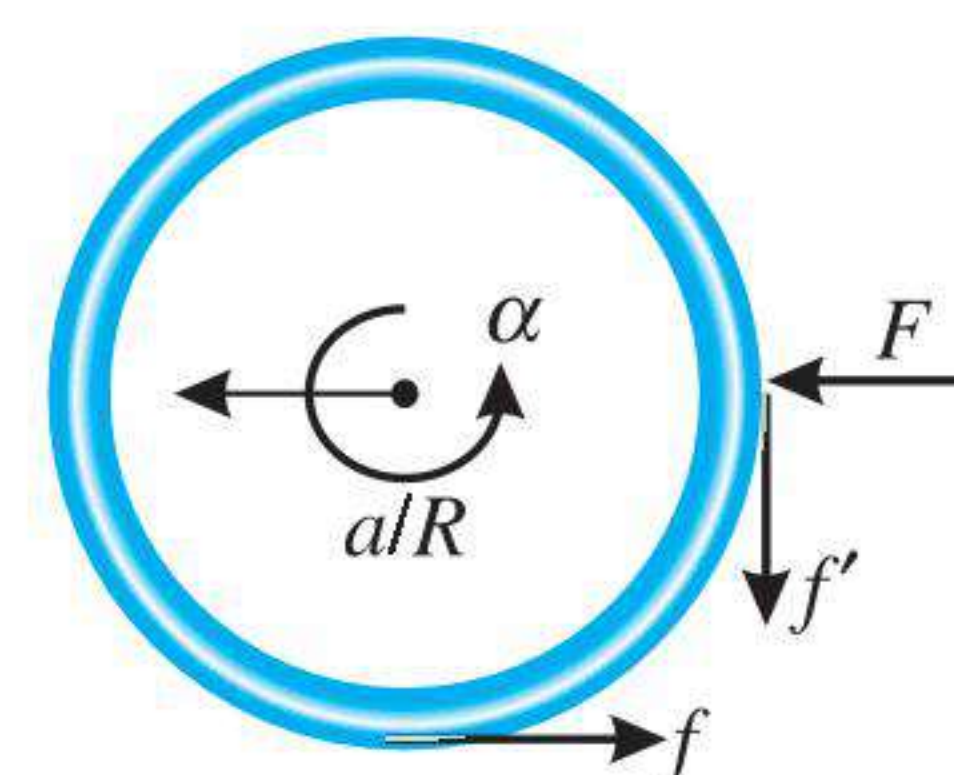
Linked Comprehension Type

- (4) Angular speed about the instantaneous axis passing through centre of mass is ω for both the cases.
- (1) Instantaneous axis passing through cm is vertical in both the cases.

Numerical Value Type

- (4) There are two possibilities for applied force by string.
 - If net force applied by the rod is considered to be 2 N.
 - If only normal reaction applied by the rod is considered to be 2 N.

Let us consider 1st possibility that if net force applied by the rod is considered to be 2 N.



Hence, net force applied by the rod

$$\sqrt{f'^2 + F^2} = 2 \quad \dots(i)$$

Applying torque equation about point of contact

$$FR - f'R = 2mR^2 \frac{a}{R}$$

$$F - f' = 2ma = 1.2 \quad \dots(ii)$$

From (i) and (ii), $(1.2 + f')^2 + f'^2 = 2^2$

$$2f'^2 + 2.4f' + 1.44 = 4$$

$$f'^2 + 1.2f' + 0.72 - 2 = 0$$

$$f'^2 + 1.2f' - 1.28 = 0$$

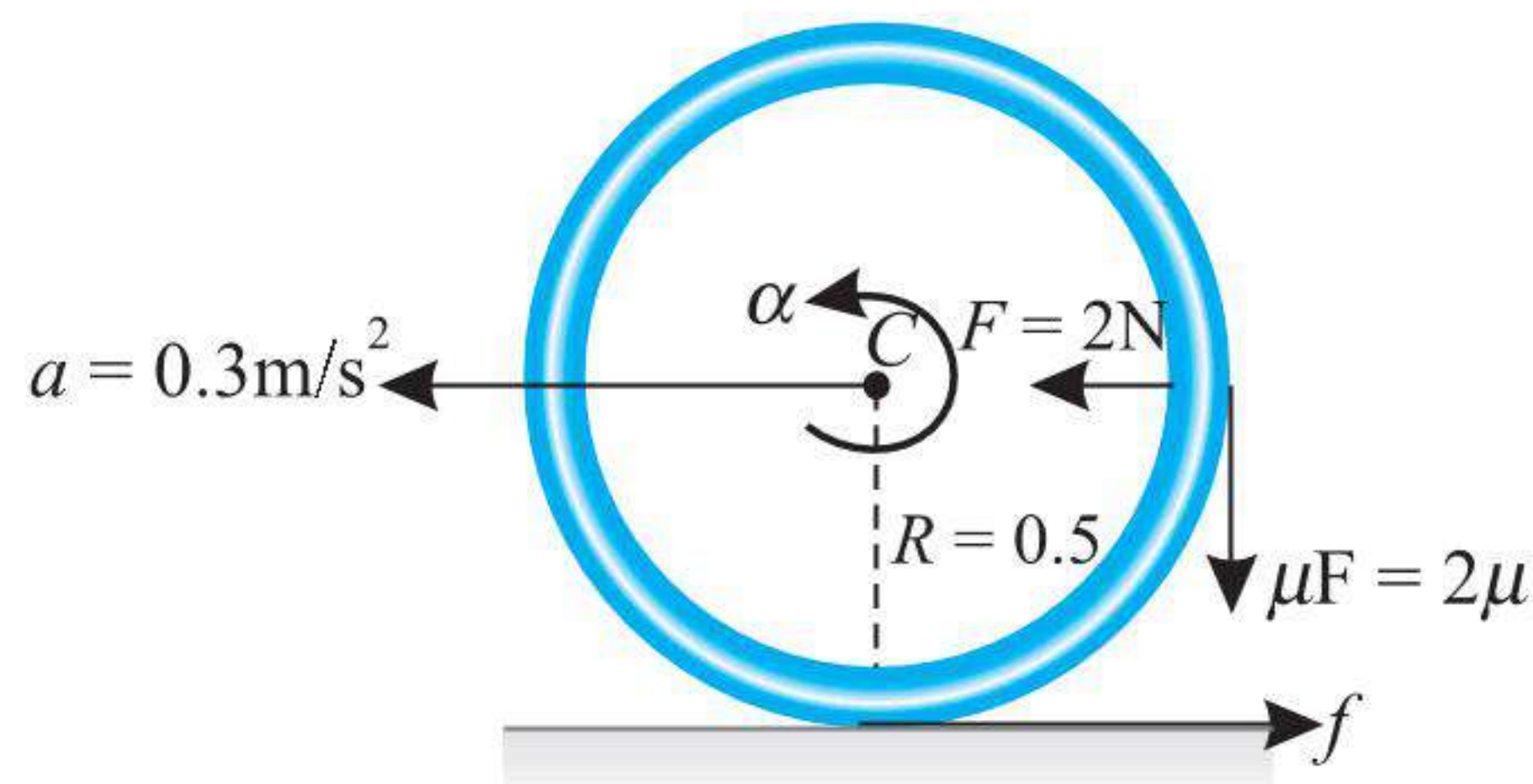
$$f' = \frac{-1.2 \pm \sqrt{1.44 + 4 \times (1.28)}}{2}$$

$$= 0.6 \pm \sqrt{0.36 + 1.28} = -0.6 \pm \sqrt{1.64} = 0.68$$

From Eq. (ii), $F = 1.88$

$$\mu = \frac{0.68}{1.88} = \frac{P}{10} \Rightarrow P = 3.16 \approx 4$$

Now let us consider the possibility that if only normal reaction applied by the rod is considered to be 2 N.



From FBD of ring, writing equation of motion of ring

$$2 - f = 2(0.3) \Rightarrow f = 2 - 0.6 = 1.4 \text{ N} \quad \dots(i)$$

As disc is rolling, $A = R\alpha \Rightarrow 0.3 = \alpha(0.5)$

$$\alpha = \frac{3}{5} \text{ rad/s} \quad \dots(ii)$$

Now torque equation, $\tau_c = I_c \alpha$

$$fR - 2\mu R = mR^2 \alpha$$

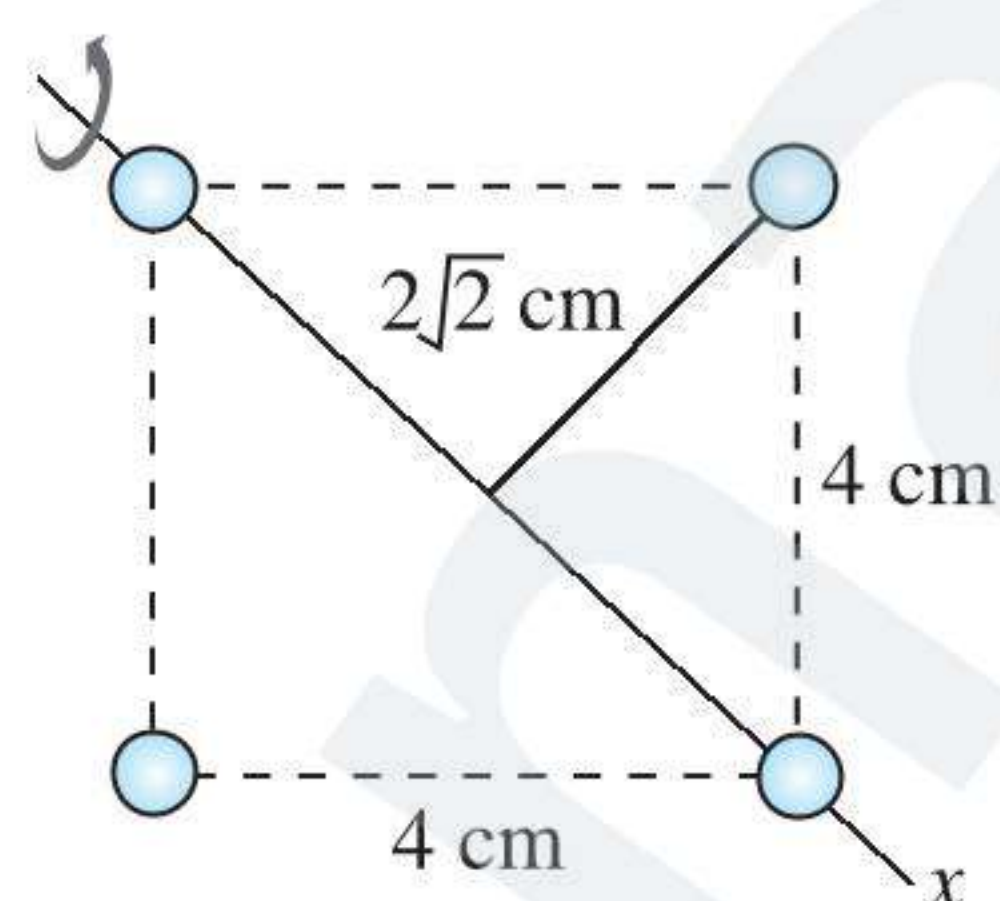
$$f - 2\mu = mR\alpha$$

$$1.4 - 2\mu = \frac{2}{2} \left(\frac{3}{5} \right)$$

$$0.8 = 2\mu \Rightarrow \mu = 0.4 = \frac{P}{10} \therefore P = 4$$

2. (9) The moment of inertia of the system about the diagonal of the square

$$I = \left(\frac{2}{5} MR^2 \right) 2 + \left(\frac{2}{5} MR^2 + Mx^2 \right) 2$$



$$= \left(\frac{2}{5} MR^2 \right) 2 + \left(\frac{2}{5} MR^2 \right) 2 = (Mx^2) 2$$

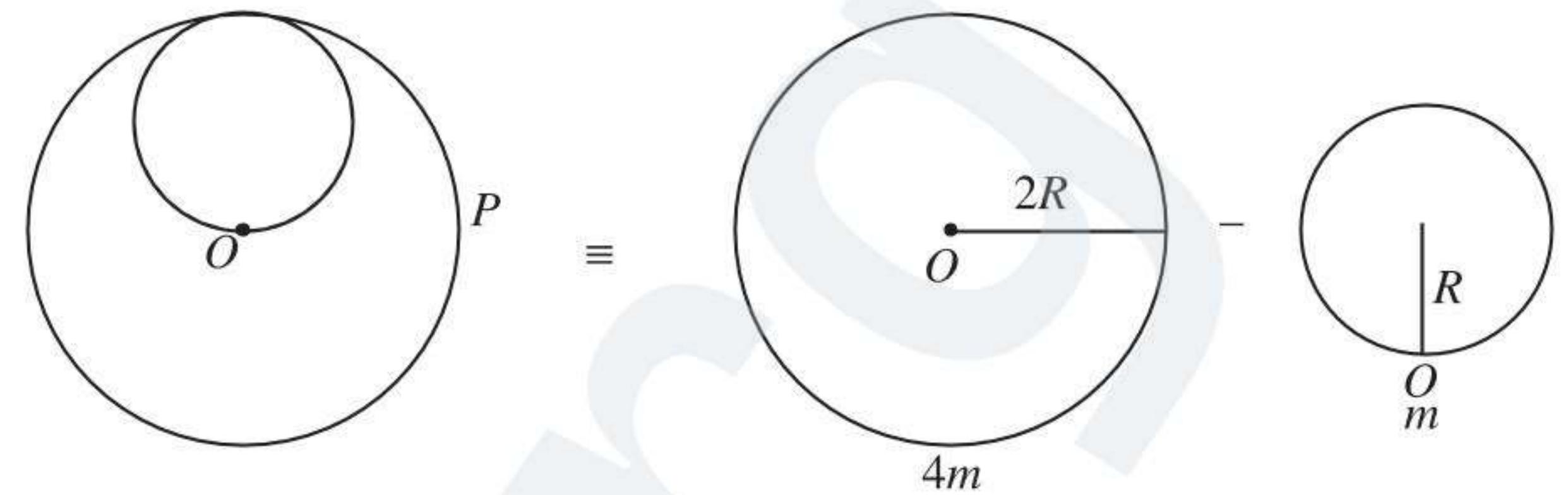
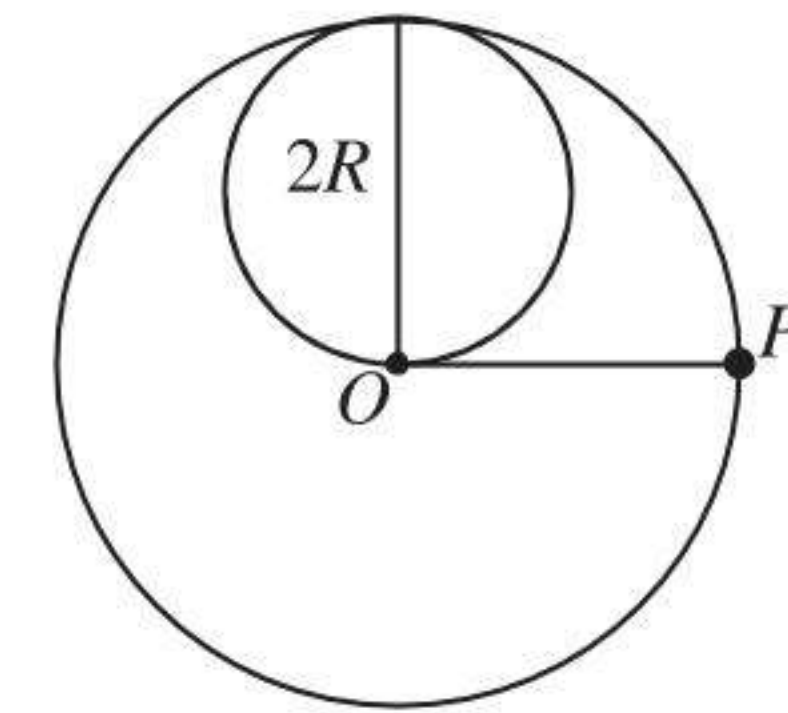
$$= 4 \left(\frac{2}{5} MR^2 \right) + 2Mx^2 = \frac{8}{5} MR^2 + 2mx^2$$

$$= \left[\frac{8}{5} \times 0.5 \times \left(\frac{\sqrt{5}}{2} \right)^2 + 2 \times (0.5) \times (4 \times 2) \right] 10^{-4}$$

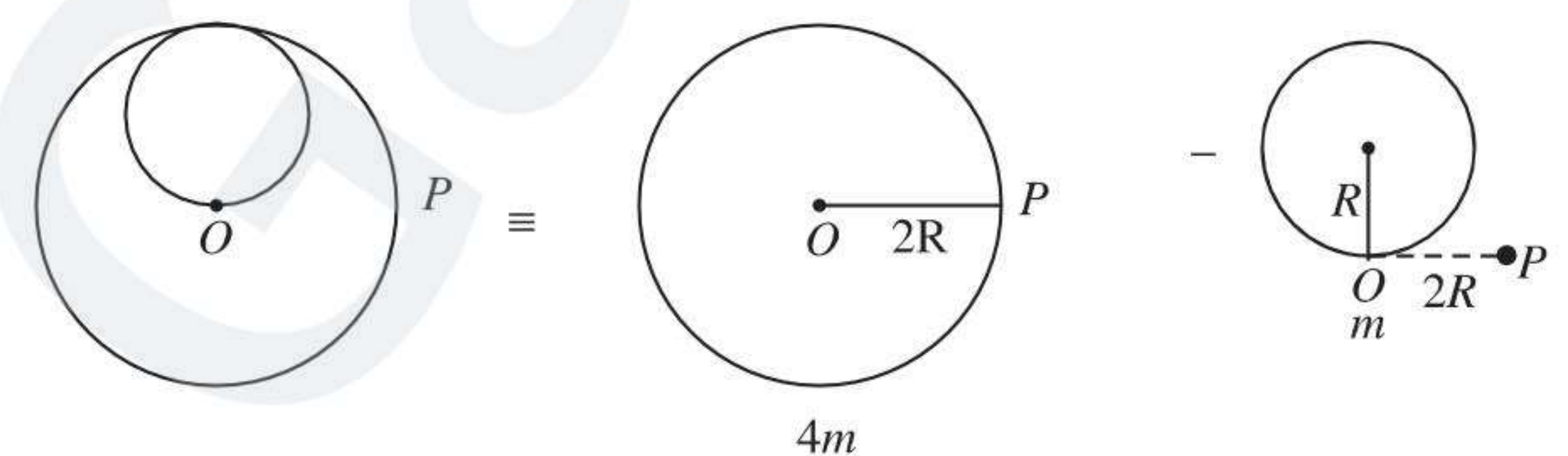
$$= \left[\frac{5}{5} + 8 \right] \times 10^{-4} = 9 \times 10^{-4} = N \times 10^{-4}$$

So, $N = 9$

3. (3)



$$I_0 = \frac{(4m)(2R)^2}{2} - \frac{3}{2} mR^2 = mR^2 \left[8 - \frac{3}{2} \right] = \frac{13}{2} mR^2$$

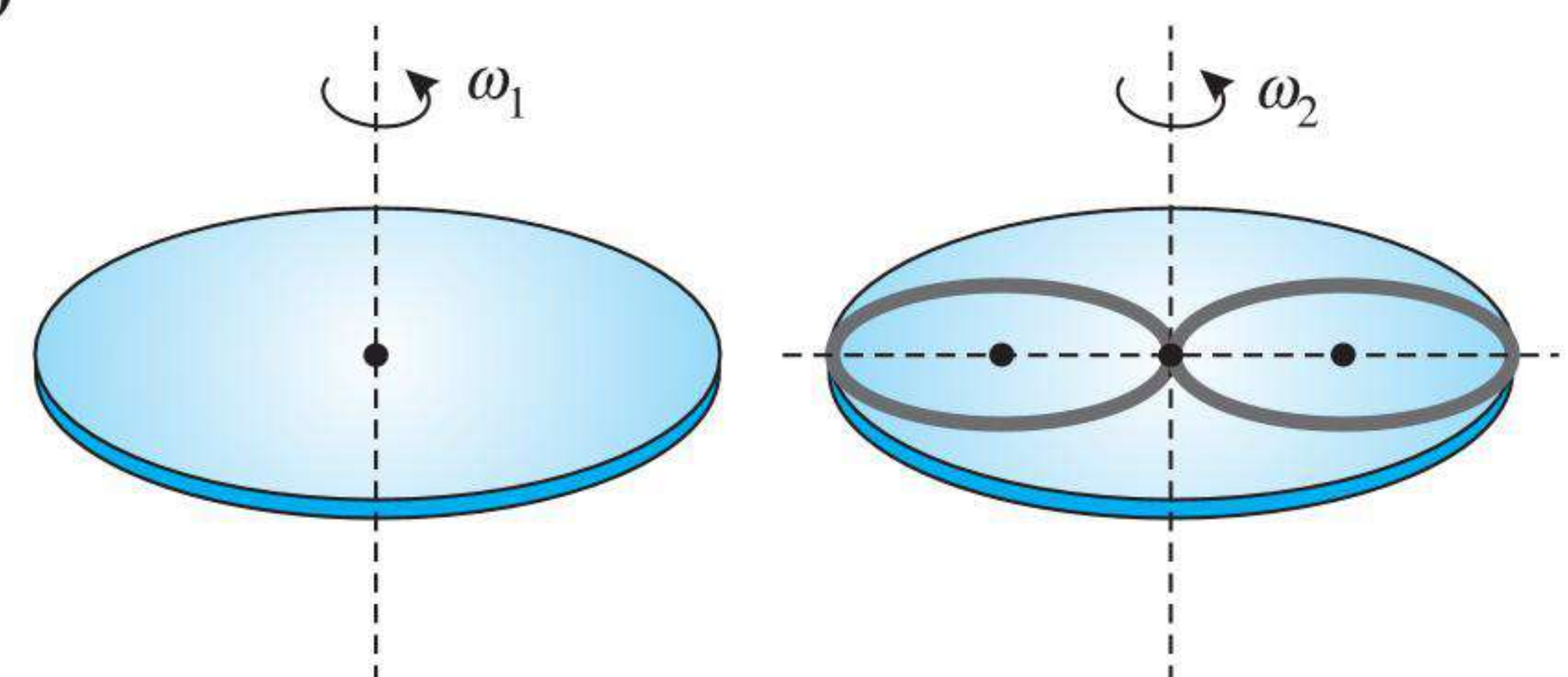


$$I_P = \frac{3}{2} (4m)(2R)^2 - \left[\frac{mR^2}{2} + m \{ (2R)^2 + R^2 \} \right]$$

$$= 24mR^2 - \frac{11}{2} mR^2 = \frac{37}{2} mR^2$$

$$\frac{I_P}{I_O} = \frac{\frac{37}{2}}{\frac{13}{2}} = \frac{37}{13} \approx 3$$

4. (8)



(a) Only disc is rotating

(b) Rotation of disc with ring

Moment of inertia of the disc about its axis

$$I_1 = \frac{1}{2} MR^2 = \frac{1}{2} \times 50 \text{ kg} \times (0.4 \text{ m})^2 = 4 \text{ kg m}^2$$

$\therefore$ Initial angular momentum of the disc is $L_1 = I_1 \omega_1$

When two uniform circular rings are placed symmetrically on the disc such that they are touching each other along the axis of the disc and are horizontal as shown in Fig. (b). Then the moment of inertia of the system about the given axis

$$I_2 = \frac{1}{2} MR^2 + 2mr^2 + 2mr^2 = \frac{1}{2} MR^2 + 4mr^2$$

$$= \frac{1}{2} \times 50 \text{ kg} \times (0.4 \text{ m})^2 + 4 \times 6.25 \text{ kg} \times (0.2 \text{ m})^2$$

$$= 4 \text{ kg m}^2 + 1 \text{ kg m}^2 = 5 \text{ kg m}^2$$

Let ω_2 be the final angular speed of the system.

Final angular momentum of the system is $L_2 = I_2 \omega_2$

According to law of conservation of angular momentum, we get

$$L_1 = L_2 \quad \text{or} \quad I_1 \omega_1 = I_2 \omega_2$$

$$\omega_2 = \frac{I_1}{I_2} \omega_1 = \frac{4 \text{ kg m}^2}{5 \text{ kg m}^2} \times 10 \text{ rad s}^{-1} = 8 \text{ rad s}^{-1}$$

$$5. (2) \tau_{\text{net}} = 3\tau_1 = 3Fr \sin 30^\circ = 3(0.5)(0.5)\frac{1}{2} = \frac{3}{8} \text{ N-m}$$

$$I = \frac{1.5(0.5)^2}{2} = \frac{3}{16}$$

$$\alpha = \frac{\tau}{I} = 2 \text{ rad/s}^2$$

$$\omega = \alpha t = 2 \text{ rad/s}$$

